2013 IEEE/RSJ International Conference on
Intelligent Robots and Systems (IROS)
November 3-7, 2013. Tokyo, Japan

Identifying the Singularity Conditions of Canadarm2 Based on Elementary Jacobian Transformation*

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Abstract—**The Canadarm2, also named Space Station Remote Manipulator System (SSRMS), is a 7-joint redundant manipulator. Without spherical wrists, the singularity analysis and avoidance of these manipulators are very difficult. In this paper, a method is presented to analytically identify its singular configurations based on the elementary transformation of Jacobian matrix. Firstly, we constructed a general kinematics model to describe them in a united manner. Correspondingly, the differential kinematics equation and the modified form are derived. Secondly, the singularity conditions are isolated and collected in a 3×4 sub-matrix by several times row transformation of the modified Jacobian matrix, which is partitioned into a block-triangle matrix. Finally, all the singularity configurations are determined by analyzing the rank degeneracy conditions of the 3×4 sub-matrix. The proposed method isolates the singularity conditions, and collects them in a 3×4 sub-matrix, largely reducing the computation workload.**

I. Introduction

The Canadarm2 (or SSRMS) is a 7-DOF redundant manipulator. The three axes of the shoulder/wrist do not intersect in a common point, i.e. they have not spherical wrists/shoulders. Thus, it is very difficult to solve the inverse kinematics problem [1] and determine the singular configurations analytically of the SSRMS.

Recently, Nokleby and Podhorodeski [2] successfully used the reciprocity-based methodology to identify the singularities of CSA/ISE STEAR test-bed manipulator, a ground-based manipulator with link lengths and offsets. Nokleby [3] further analyzed the singularities of the SSRMS using the similar method. Dupuis [4] presented the singular vector algorithm, which is actually a reformulation of the reciprocity-based methodology using linear algebra terms instead of the reciprocal screw. Kong and Gosselin [5] proposed a dependent-screw suppression approach for the singularity analysis of Canadarm2. Nokleby and Podhorodeski [6] considered the modified Canadarm2 (incorporating two additional link lengths not found in the Canadarm2), and identified the complete set of singular configurations for the Canadarm2 and its modified version using reciprocity-based methodology.

Up to now, the reciprocity-based methodology is thought to be the most effective method for the singularity analysis of a redundant manipulator [5]. One key issue of this approach is to find a general expression of a reciprocal screw for a group of six linearly dependent screws. It is usually not easy to find it, which most depends on the personal experience. In order to preliminarily determine the possible singularity conditions, the determinant of a 6×6 Jacobian sub-matrix must be first derived analytically. This also increases the difficulty of using this method. Furthermore, the singularity avoidance problem is not decomposed into lower dimension sub-problems to reduce the computation workload, similar with previous methods based on “workspace decomposition” [7-8].

In this paper, a method based on the elementary transformation of Jacobian matrix is presented to analyze the singularity configurations of SSRMS manipulator. The produced matrix has a block-triangle form, similar with that of a redundant manipulator with spherical wrist [7, 9]. It is partitioned into four sub-matrices----two 3×4 sub-matrices (the upper-left and lower-left sub-matrices), a 3×3 sub-matrix (the upper-right sub-matrix) and a 3×3 null matrix (the lower-right sub-matrix). The upper 3×7 block (composed of the upper-left 3×4 and the upper-right 3×3 sub-matrices) is proved to be full rank. Therefore, the singularity conditions are isolated and collected in the other 3×4 sub-matrix. Then, the singularity determination of the 6×7 Jacobian matrix is reduced to identify the rank degeneracy condition of the 3×4 sub-matrix. Compared with previous methods, the singularity analysis is greatly simplified.

II. The Model of The Canadarm2

In the construction and maintenance of the International Space Station, the Canadarm2 has been playing important roles. It was also used to support the on-orbit experiments, extra-vehicular activity (EVA) and scientific activities onboard the ISS. As Ref. [6], a general kinematics model is constructed by adding two additional links offsets to the practical Canadarm2. Its DH (Denavit-Hartenberg) frames are defined in Fig. 1 (when the joint angles are all zeros), where z_{i-1} is the rotation direction of the i^{th} joint (denoted by J_i). Frames $\{x_0y_0z_0\}$ and $\{x_7y_7z_7\}$ are respectively the base frame and the end-effector frame of the manipulator. The corresponding DH parameters of each link are listed in Table I. Parameters a_2 and a_5 (denoted as b and g in Ref. [6]) are the additional link offsets, not found in the actual Canadarm2. That is to say, by setting a_2 and a_5 to zero, the results obtained for the modified Canadarm2 can be extended to the actual Canadarm2.

*This work was supported in part by the National Natural Science Foundation of China (61175098), State Key Laboratory of Robotics and System (HIT) under Grant SKLRS-2012-MS-05, and Aerospace Science and Technology Innovation Fund of China (CASC201102).

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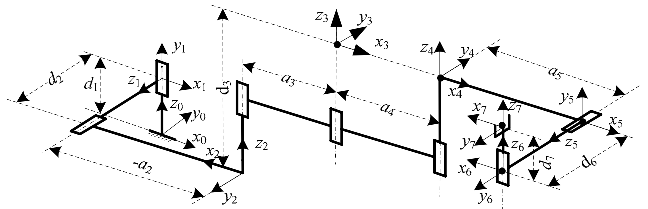

Fig. 1 The DH Frames of the General Kinematics Structure (Zeros-Displacement Configuration)

TABLE I THE DH PARAMETERS OF THE 7-DOF MANIPULATOR

Link i	θ_i /°	α_i /°	a_i / m	d_i / m
1	0	90	0	d_1
2	180	90	$-a_2$	d_2
3	180	0	a_3	d_3
4	0	0	a_4	0
5	0	90	a_5	0
6	180	90	0	d_6
7	0	0	0	d_7

III. MODIFIED DIFFERENTIAL KINEMATICS EQUATION

A. *The Differential Kinematics Equation*

The differential kinematics equation of a manipulator can be written as:

$$\begin{bmatrix} \boldsymbol{\omega}_e \\ \boldsymbol{v}_e \end{bmatrix} = \boldsymbol{J}(\boldsymbol{\Theta})\dot{\boldsymbol{\Theta}} \tag{1}$$

where, $\boldsymbol{\omega}_e \in \mathbf{R}^3$ and $\boldsymbol{v}_e \in \mathbf{R}^3$ respectively denote the angular and linear velocities of the manipulator's end-effector; $\boldsymbol{\Theta} = [\theta_1, \theta_2, \cdots, \theta_7] \in \mathbf{R}^7$ is a vector formed by all joint angles of the manipulator; $\boldsymbol{J}(\boldsymbol{\Theta}) \in \mathbf{R}^{6\times7}$ is the Jacobian matrix, establishing the relationship between the joint rates and end-effector velocities.

For a manipulator with n DOF(in this paper, n=7), the Jacobian matrix can be described as:

$$\boldsymbol{J}(\boldsymbol{\Theta}) = [\boldsymbol{J}_1, \ \boldsymbol{J}_2, \cdots, \boldsymbol{J}_n] \tag{2}$$

According to the definition, the i^{th} column of it is calculated by the following equation:

$$\boldsymbol{J}_i = \begin{bmatrix} \boldsymbol{z}_{i\text{-}1} \\ \boldsymbol{z}_{i\text{-}1} \times \boldsymbol{r}_{i\text{-}1\to e} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_{i\text{-}1} \\ \boldsymbol{z}_{i\text{-}1} \times (\boldsymbol{p}_e - \boldsymbol{r}_{i\text{-}1}) \end{bmatrix} \tag{3}$$

where, $\boldsymbol{z}_{i\text{-}1}$ denotes the unit vector of the i^{th} joint's rotation axe; $\boldsymbol{r}_{i\text{-}1}$ is the position vector of the $(i\text{-}1)^{\text{th}}$ frame's origin; $\boldsymbol{r}_{i\text{-}1\to e}$ is the vector from the origin of the $(i\text{-}1)^{\text{th}}$ to that of the end-effector frame; $\boldsymbol{p}_e = \boldsymbol{r}_7$.

B. *Modified Joint Screws and Jacobian Matrix*

The right part of (3) can be factored as:

$$\boldsymbol{J}_i = \begin{bmatrix} \boldsymbol{z}_{i\text{-}1} \\ \boldsymbol{z}_{i\text{-}1} \times (\boldsymbol{p}_e - \boldsymbol{r}_{i\text{-}1}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{3\times3} & \boldsymbol{O}_{3\times3} \\ -\boldsymbol{p}_e^{\times} & -\boldsymbol{I}_{3\times3} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_{i\text{-}1} \\ \boldsymbol{z}_{i\text{-}1} \times \boldsymbol{r}_{i\text{-}1} \end{bmatrix} \tag{4}$$

where, $\boldsymbol{I}_{3\times3}$ is a 3×3 identity matrix; $\boldsymbol{p}_e^{\times}$ is a skew-symmetric matrix which is given by:

$$\boldsymbol{p}_e^{\times} = \begin{bmatrix} 0 & -p_{ez} & p_{ey} \\ p_{ez} & 0 & -p_{ex} \\ -p_{ey} & p_{ex} & 0 \end{bmatrix} \tag{5}$$

If matrix $\boldsymbol{M}$ and vector $\boldsymbol{S}_i$ are defined as follows:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I}_{3\times3} & \boldsymbol{O}_{3\times3} \\ -\boldsymbol{p}_e^{\times} & -\boldsymbol{I}_{3\times3} \end{bmatrix} \tag{6}$$

$$\boldsymbol{S}_i = \begin{bmatrix} \boldsymbol{z}_{i\text{-}1} \\ \boldsymbol{z}_{i\text{-}1} \times \boldsymbol{r}_{i\text{-}1} \end{bmatrix} \tag{7}$$

Equation (4) can be written as:

$$\boldsymbol{J}_i = \boldsymbol{M} \bullet \boldsymbol{S}_i \tag{8}$$

Vector $\boldsymbol{S}_i$ is a 6×1 vector similar with the joint screw [6], whose last three rows is actually ${}^{ref}\boldsymbol{z}_{i\text{-}1} \times {}^{ref}\boldsymbol{r}_{i\text{-}1\to ref}$. Substituting (8) into (2) yields:

$$\boldsymbol{J}(\boldsymbol{\Theta}) = \boldsymbol{M} \bullet \boldsymbol{S}(\boldsymbol{\Theta}) \tag{9}$$

In (9), $\boldsymbol{S}(\boldsymbol{\Theta})$ is a matrix formed as $\boldsymbol{S}(\boldsymbol{\Theta}) = [\boldsymbol{S}_1, \ \boldsymbol{S}_2, \ \cdots, \ \boldsymbol{S}_7]$. For the convenience of discussion, $\boldsymbol{S}_i$ is called the modified joint screw of joint i, and $\boldsymbol{S}(\boldsymbol{\Theta})$ is called the modified Jacobian matrix. Known from (6), the matrix $\boldsymbol{M}$ is invertible. Then, the matrices $\boldsymbol{J}(\boldsymbol{\Theta})$ and $\boldsymbol{S}(\boldsymbol{\Theta})$ have the same singular conditions, according to (9). As the expression of $\boldsymbol{S}_i$ is generally simpler than that of $\boldsymbol{J}_i$, we use $\boldsymbol{S}(\boldsymbol{\Theta})$ instead of $\boldsymbol{J}(\boldsymbol{\Theta})$ to analyzed the singularity configurations of a redundant manipulator.

Substituting (9) into (1), the following equation is obtained:

$$\begin{bmatrix} \boldsymbol{\omega}_e \\ \boldsymbol{v}_e \end{bmatrix} = \boldsymbol{M} \bullet \boldsymbol{S}(\boldsymbol{\Theta}) \bullet \dot{\boldsymbol{\Theta}} \tag{10}$$

Equation (10) can be written as the following expression:

$$\boldsymbol{M}^{-1} \begin{bmatrix} \boldsymbol{\omega}_e \\ \boldsymbol{v}_e \end{bmatrix} = \boldsymbol{S}(\boldsymbol{\Theta}) \bullet \dot{\boldsymbol{\Theta}} \tag{11}$$

The term on the left-hand side of (11) can be defined as follows:

$$\begin{bmatrix} \tilde{\boldsymbol{\omega}}_e \\ \tilde{\boldsymbol{v}}_e \end{bmatrix} = \boldsymbol{M}^{-1} \bullet \begin{bmatrix} \boldsymbol{\omega}_e \\ \boldsymbol{v}_e \end{bmatrix} \tag{12}$$

From (11) and (12), the modified differential kinematics equation is obtained:

$$\begin{bmatrix} \tilde{\boldsymbol{\omega}}_e \\ \tilde{\boldsymbol{v}}_e \end{bmatrix} = \boldsymbol{S}(\boldsymbol{\Theta}) \bullet \dot{\boldsymbol{\Theta}} \tag{13}$$

Different reference coordinate system does not change the singular conditions of a manipulator. Choosing the 5th frame as the reference frame [6], the modified Jacobian matrix has a concise form. Correspondingly, the modified joint screws can be derived as follows:

$${}^5\boldsymbol{S}_1 = \begin{bmatrix} s_2 c_{345} \\ -c_2 \\ s_2 s_{345} \\ d_3 s_2 s_{345} + c_2(-a_2 s_{345} - d_2 c_{345} + a_3 s_{45} + a_4 s_5) \\ -s_2(d_2 + a_3 s_3 + a_4 s_{34} + a_5 s_{345}) \\ -d_3 s_2 c_{345} - c_2(d_2 s_{345} - a_2 c_{345} + a_3 c_{45} + a_4 c_5 + a_5) \end{bmatrix} \tag{14}$$

$$\begin{aligned} {}^5\boldsymbol{S}_2 = [&s_{345}, 0, -c_{345}, -d_3 c_{345}, \\ &-a_2 + a_3 c_3 + a_4 c_{34} + a_5 c_{345}, -d_3 s_{345}]^{\mathrm{T}} \end{aligned} \tag{15}$$

$$ {}^{5}\boldsymbol{S}_3 = \begin{bmatrix} 0 & 1 & 0 & -a_3 s_{45} - a_4 s_5 & 0 & a_3 c_{45} + a_4 c_5 + a_5 \end{bmatrix}^T \quad (16) $$

$$ {}^{5}\boldsymbol{S}_4 = \begin{bmatrix} 0 & 1 & 0 & -a_4 s_5 & 0 & a_4 c_5 + a_5 \end{bmatrix}^T \quad (17) $$

$$ {}^{5}\boldsymbol{S}_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & a_5 \end{bmatrix}^T \quad (18) $$

$$ {}^{5}\boldsymbol{S}_6 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T \quad (19) $$

$$ {}^{5}\boldsymbol{S}_7 = \begin{bmatrix} s_6 & -c_6 & 0 & -d_6 c_6 & -d_6 s_6 & 0 \end{bmatrix}^T \quad (20) $$

where, $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, $s_{ij} = \sin(\theta_i + \theta_j)$, $c_{ij} = \cos(\theta_i + \theta_j)$, $s_{ijk} = \sin(\theta_i + \theta_j + \theta_k)$, $c_{ijk} = \cos(\theta_i + \theta_j + \theta_k)$.

IV. Singular Analysis Based on the Elementary Transformation

According to (14) ~ (20), the modified Jacobian matrix is as follows:

$$ {}^{5}\boldsymbol{S}(\boldsymbol{\Theta}) = \begin{bmatrix} {}^{5}\boldsymbol{S}_1 & {}^{5}\boldsymbol{S}_2 & {}^{5}\boldsymbol{S}_3 & {}^{5}\boldsymbol{S}_4 & {}^{5}\boldsymbol{S}_5 & {}^{5}\boldsymbol{S}_6 & {}^{5}\boldsymbol{S}_7 \end{bmatrix} = \begin{bmatrix} s_2 c_{345} & s_{345} & 0 & 0 & 0 & 0 & s_6 \\ -c_2 & 0 & 1 & 1 & 1 & 0 & -c_6 \\ s_2 s_{345} & -c_{345} & 0 & 0 & 0 & 1 & 0 \\ J_{41} & -d_3 c_{345} & J_{43} & -a_4 s_5 & 0 & 0 & -d_6 c_6 \\ J_{51} & J_{52} & 0 & 0 & 0 & 0 & -d_6 s_6 \\ J_{61} & -d_3 s_{345} & J_{63} & J_{64} & a_5 & 0 & 0 \end{bmatrix} \quad (21) $$

where,

$$ \begin{cases} J_{41} = d_3 s_2 s_{345} + c_2 \left(-a_2 s_{345} - d_2 c_{345} + a_3 s_{45} + a_4 s_5\right) \\ J_{43} = -a_3 s_{45} - a_4 s_5 \\ J_{51} = -s_2 \left(d_2 + a_3 s_3 + a_4 s_{34} + a_5 s_{345}\right) \\ J_{52} = -a_2 + a_3 c_3 + a_4 c_{34} + a_5 c_{345} \\ J_{61} = -d_3 s_2 c_{345} \\ \qquad - c_2 \left(d_2 s_{345} - a_2 c_{345} + a_3 c_{45} + a_4 c_5 + a_5\right) \\ J_{63} = a_3 c_{45} + a_4 c_5 + a_5 \\ J_{64} = a_4 c_5 + a_5 \end{cases} $$

Observing the matrix ${}^{5}\boldsymbol{S}$ given in (21), we can simplify the forms of the 4th ~ 7th columns by some elementary transformations, since they have relative simple expressions. Firstly, the 6th row of ${}^{5}\boldsymbol{S}$ is replaced by the sum of it and $-a_5$ times of the 2nd row. The row operation is represented as $R_6 - a_5 R_2 \to R_6$, where R_i denotes the i^{th} row. The corresponding elementary matrix is given as follows:

$$ \boldsymbol{P}_1 = ET\left(R_6 - a_5 R_2 \to R_6\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -a_5 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (22) $$

where, ET is the abbreviation of "elementary transformation"; P_1 denotes the elementary matrix corresponding to 1st elementary row transformation. The matrix produced by this transformation of ${}^{5}\boldsymbol{S}$ is then obtained as:

$$ \boldsymbol{P}_1 \bullet {}^{5}\boldsymbol{S}(\boldsymbol{\Theta}) = \begin{bmatrix} s_2 c_{345} & s_{345} & 0 & 0 & 0 & 0 & s_6 \\ -c_2 & 0 & 1 & 1 & 1 & 0 & -c_6 \\ s_2 s_{345} & -c_{345} & 0 & 0 & 0 & 1 & 0 \\ J_{41} & -d_3 c_{345} & J_{43} & -a_4 s_5 & 0 & 0 & -d_6 c_6 \\ J_{51} & J_{52} & 0 & 0 & 0 & 0 & -d_6 s_6 \\ J_{61} + a_5 c_2 & -d_3 s_{345} & J_{63} - a_5 & J_{64} - a_5 & 0 & 0 & a_5 c_6 \end{bmatrix} \quad (23) $$

According to (23), the last three elements of the 1st row and the 4th~ 6th rows have similar forms: two elements are zeros and the last element is a multiple of s_6 or c_6. Then two more elementary row transformations can further simplify the matrix given in (23). They are respectively $R_5 + d_6 R_1 \to R_5$ and $R_6 + (a_5 / d_6) R_4 \to R_6$, i.e. adding d_6 times of the 1st row to the 5th row, and adding (a_5 / d_6) times of the 4th row to the 6th row. The elementary matrices corresponding to the two operations are denoted by $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$. They are written as follows:

$$ \boldsymbol{P}_2 = ET\left(R_5 + d_6 R_1 \to R_5\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \mathrm{d}_6 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (24) $$

$$ \boldsymbol{P}_3 = ET\left(R_6 + \frac{a_5}{d_6} R_4 \to R_6\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{a_5}{d_6} & 0 & 1 \end{bmatrix} \quad (25) $$

The resulting matrix after the above transformations is:

$$ {}^{5}\overline{\boldsymbol{S}}(\boldsymbol{\Theta}) = \boldsymbol{P}_3 \boldsymbol{P}_2 \boldsymbol{P}_1 \bullet {}^{5}\boldsymbol{S}(\boldsymbol{\Theta}) = \begin{bmatrix} & 0 & 0 & s_6 \\ & 1 & 0 & -c_6 \\ * & 0 & 1 & 0 \\ & 0 & 0 & -d_6 c_6 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{bmatrix} \quad (26) $$

where, "*" denotes the 1~4th columns of ${}^{5}\overline{\boldsymbol{S}}(\boldsymbol{\Theta})$. The last three elements of the 5th ~ 6th rows of the matrix shown as (26) are all zeros. The sub-matrix formed by the last three elements of the 1st and the 4th rows is written as:

$$ \boldsymbol{Q} = {}^{5}\overline{\boldsymbol{S}}_{1-4,5-7} = \begin{bmatrix} 0 & 0 & s_6 \\ 1 & 0 & -c_6 \\ 0 & 1 & 0 \\ 0 & 0 & -d_6 c_6 \end{bmatrix} \quad (27) $$

Since s_6 and c_6 can not simultaneously equal to zeros, the rank of the matrix $\boldsymbol{Q}$ is always 3, i.e. Rank($\boldsymbol{Q}$) = 3. If s_6=0 (c_6=±1), all the elements of the 1st row of $\boldsymbol{Q}$ will be zero; else, one of the 1st and 4th rows can be transformed to be zero after an elementary row transformation. In the following contents, the singularity configurations of the SSRMS-type manipulator will be completely identified by analyzing the two cases: s_6=0 and $s_6 \neq 0$.

A. Singular Analysis with $s_6 = 0$

1) Singularity Condition Separation

When $s_6 = 0$, c_6=±1, the matrix ${}^5\tilde{\boldsymbol{S}}(\boldsymbol{\Theta})$, shown in (26), can be written as (28). Observing the matrix given in (28), the last three elements of the 1st , 5th and 6th rows are all zeros. If the 4th row within the matrix is switched with 1st, a 3×3 null sub-matrix is formed. This row operation is denoted as $R_1 \leftrightarrow R_4$, and the corresponding elementary transformation matrix is given as(29).

$$
{}^5\tilde{\boldsymbol{S}} = \left[\begin{array}{cc} & \boxed{\begin{matrix}0 & 0 & 0\end{matrix}} \\ & \begin{matrix}1 & 0 & \mp 1\end{matrix} \\ * & \begin{matrix}0 & 1 & 0\end{matrix} \\ & \begin{matrix}0 & 0 & \mp d_6\end{matrix} \\ & \boxed{\begin{matrix}0 & 0 & 0 \\ 0 & 0 & 0\end{matrix}} \end{array}\right] \tag{28}
$$

$$
\hat{\boldsymbol{P}}_4 = ET\left(R_1 \leftrightarrow R_4\right) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}
$$

Then, the matrix obtained by exchanging the 1st row and 4th row of ${}^5\tilde{\boldsymbol{S}}$ has the following forms:

$$
{}^5\hat{\boldsymbol{S}} = \hat{\boldsymbol{P}}_4 \bullet {}^5\tilde{\boldsymbol{S}} = \left[\begin{array}{c:c} \hat{\boldsymbol{S}}_{11} & \hat{\boldsymbol{S}}_{12} \\ \hdashline \hat{\boldsymbol{S}}_{21} & \boldsymbol{O}_{3\times 3} \end{array}\right] \tag{30}
$$

where, $\boldsymbol{O}_{3\times 3}$ is a 3×3 zero matrix; the others sub-matrices are as follows:

$$
\hat{\boldsymbol{S}}_{11} = \begin{bmatrix} J_{41} & -d_3 c_{345} & J_{43} & -a_4 s_5 \\ -c_2 & 0 & 1 & 1 \\ s_2 s_{345} & -c_{345} & 0 & 0 \end{bmatrix} \tag{31}
$$

$$
\hat{\boldsymbol{S}}_{12} = \begin{bmatrix} 0 & 0 & \pm d_6 \\ 1 & 0 & \pm 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{32}
$$

$$
\hat{\boldsymbol{S}}_{21} = \begin{bmatrix} s_2 c_{345} & J_{51} + d_6 s_2 c_{345} & J_{61} + a_5 c_2 + \left(a_5 / d_6\right) J_{41} \\ s_{345} & J_{52} + d_6 s_{345} & -d_3 s_{345} - \left(a_5 / d_6\right) d_3 c_{345} \\ 0 & 0 & J_{63} - a_5 + \left(a_5 / d_6\right) J_{43} \\ 0 & 0 & J_{64} - a_5 - \left(a_5 / d_6\right) a_4 s_5 \end{bmatrix}^{\mathrm{T}} \tag{33}
$$

Equation (30) shows that the modified Jacobian matrix ${}^5\boldsymbol{S}(\boldsymbol{\Theta})$, given in (21), can be transformed to a block-triangle matrix, after 4 times elementary transformations.

According to the characteristics of matrix rank, the following inequality holds:

$$
\operatorname{rank}\left(\hat{\boldsymbol{S}}_{12}\right) \le \operatorname{rank}\left(\left[\hat{\boldsymbol{S}}_{11} \;\vdots\; \hat{\boldsymbol{S}}_{12}\right]\right) \le 3 \tag{34}
$$

On the other hand, the determinate of the sub-matrix, given in (32), can be calculated as:

$$
\det\left(\hat{\boldsymbol{S}}_{12}\right) = \left\| \begin{bmatrix} 0 & 0 & \pm d_6 \\ 1 & 0 & \pm 1 \\ 0 & 1 & 0 \end{bmatrix} \right\| = \pm d_6 \neq 0 \tag{35}
$$

From (35), $\hat{\boldsymbol{S}}_{12}$ is full rank, i.e. $\operatorname{rank}\left(\hat{\boldsymbol{S}}_{12}\right) = 3$. The rank of the block-triangle matrix ${}^5\hat{\boldsymbol{S}}$, given in (30), can be computed according to the following equation [10]:

$$
\operatorname{rank}\left({}^5\hat{\boldsymbol{S}}\right) = \operatorname{rank}\left(\hat{\boldsymbol{S}}_{12}\right) + \operatorname{rank}\left(\hat{\boldsymbol{S}}_{21}\right) \tag{36}
$$

If ${}^5\hat{\boldsymbol{S}}$ is singular, $\operatorname{rank}\left({}^5\hat{\boldsymbol{S}}\right) < 6$. According to (36), the equivalent condition:

$$
\operatorname{rank}\left(\hat{\boldsymbol{S}}_{21}\right) < 3 \tag{37}
$$

Then, the singularity configurations for $s_6 = 0$ can be identified by analyzing the rank deficient conditions of a 3×4 sub-matrix $\hat{\boldsymbol{S}}_{21}$. In order word, the singularity conditions originally distributed in the 6×7 matrix ${}^5\boldsymbol{S}(\boldsymbol{\Theta})$ with $s_6 = 0$ are collected in a 3×4 sub-matrix. The above process is called "singularity condition isolation".

2) Singular Configuration Identification

As discussed above, the singularity conditions are separated and collected in the 3×4 sub-matrix $\hat{\boldsymbol{S}}_{21}$. Since $\operatorname{rank}\left(\hat{\boldsymbol{S}}_{21}\right) = \operatorname{rank}\left(\hat{\boldsymbol{S}}_{21}\hat{\boldsymbol{S}}_{21}^{\mathrm{T}}\right)$, where $\hat{\boldsymbol{S}}_{21}\hat{\boldsymbol{S}}_{21}^{\mathrm{T}}$ is a square matrix, the inequality (37) holds if and only if:

$$
\det\left(\hat{\boldsymbol{S}}_{21} \bullet \hat{\boldsymbol{S}}_{21}^{\mathrm{T}}\right) = 0 \tag{38}
$$

By using Cauchy-Binet formula [11], the determinant of $\hat{\boldsymbol{S}}_{21} \bullet \hat{\boldsymbol{S}}_{21}^{\mathrm{T}}$ can be expressed as

$$
\det\left(\hat{\boldsymbol{S}}_{21} \bullet \hat{\boldsymbol{S}}_{21}^{\mathrm{T}}\right) = \sum_{i=1}^{4} \hat{M}_i^2 \tag{39}
$$

where, $\hat{M}_i$ is defined as the distinct, order-3 minor of the sub-matrix $\hat{\boldsymbol{S}}_{21}$ for i=1,..,4. According to (38) and (39), the singularity conditions of $\hat{\boldsymbol{S}}_{21}$ are determined by the following equations:

$$
\hat{M}_i = 0 \quad \left(i = 1, \cdots, 4\right) \tag{40}
$$

Using symbol $\hat{\boldsymbol{S}}_{21,k}$ to denote the kth column vector of $\hat{\boldsymbol{S}}_{21}$, $\hat{M}_i$ can be calculated as follows:

$$
\begin{aligned} \hat{M}_1 &= \det\left(\left[\hat{\boldsymbol{S}}_{21,1} \quad \hat{\boldsymbol{S}}_{21,2} \quad \hat{\boldsymbol{S}}_{21,3}\right]\right) \\ &= -\left(s_2\left[a_3 s_4 \eta_1 + \left(a_3 c_4 + a_4\right)\eta_2\right]\eta_3\right) / d_6 \end{aligned} \tag{41}
$$

$$
\hat{M}_2 = \det\left(\left[\hat{\boldsymbol{S}}_{21,2} \quad \hat{\boldsymbol{S}}_{21,3} \quad \hat{\boldsymbol{S}}_{21,4}\right]\right) = 0 \tag{42}
$$

$$
\hat{M}_3 = \det\left(\left[\hat{\boldsymbol{S}}_{21,3} \quad \hat{\boldsymbol{S}}_{21,4} \quad \hat{\boldsymbol{S}}_{21,1}\right]\right) = 0 \tag{43}
$$

$$
\hat{M}_4 = \det\left(\left[\hat{\boldsymbol{S}}_{21,1} \quad \hat{\boldsymbol{S}}_{21,2} \quad \hat{\boldsymbol{S}}_{21,4}\right]\right) = -a_4 s_2 \eta_2 \eta_3 / d_6 \tag{44}
$$

where,

$$
\eta_1 = a_5 c_5 + d_6 s_5 = \frac{1}{\sqrt{a_5^2 + d_6^2}} \sin\left(\theta_5 + \varphi\right)
$$

$$\eta_2 = a_5 s_5 - d_6 c_5 = -\frac{1}{\sqrt{a_5^2 + d_6^2}} \cos(\theta_5 + \varphi)$$

$$\eta_3 = d_2 s_{345} - a_2 c_{345} + a_3 c_{45} + a_4 c_5 + a_5$$

The angle φ in the expressions of η_1 and η_2 is a constant defined as $\varphi = \text{atan2}(a_5, d_6)$. We can get the conditions which make all the minors $(\hat{M}_1, \hat{M}_2, \hat{M}_3, \hat{M}_4)$ to equal zero. All the singularity conditions for $s_6 = 0$ are listed as Table II.

TABLE II THE SINGULAR CONDITIONS FOR THE REDUNDANT MANIPULATOR FOR THE CASE s_6=0

Condition Index	The Expressions of Singular Conditions
$\hat{k}_{s1}$	$s_6 = 0$ and $s_2 = 0$
$\hat{k}_{s2}$	$s_6 = 0$ and $\eta_3 = 0$
$\hat{k}_{s3}$	$s_6 = 0$, $s_4 = 0$ and $\eta_2 = a_5 s_5 - d_6 c_5 = 0$

B. Singular Analysis with $s_6 \neq 0$

1) Singularity Condition Separation

See (26), when $s_6 \neq 0$, the last three elements of the 4th row of the matrix ${}^5\overline{\boldsymbol{S}}(\boldsymbol{\Theta})$ will become zeros by the row operation: $(d_6 c_6 / s_6) R_1 \to R_4$, i.e. adding $d_6 c_6 / s_6$ times of the 1st row to the 4th row. The corresponding elementary matrix is given as (45), and the matrix produced by this transformation from the matrix ${}^5\overline{\boldsymbol{S}}(\boldsymbol{\Theta})$ is written as (46).

$$\hat{\boldsymbol{P}}_4 = ET\left(\frac{d_6 c_6}{s_6} R_1 \to R_4\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{d_6 c_6}{s_6} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{45}$$

$${}^5\hat{\boldsymbol{S}} = \hat{\boldsymbol{P}}_4 \bullet {}^5\overline{\boldsymbol{S}} = \left[\begin{array}{c|c} \hat{\boldsymbol{S}}_{11} & \hat{\boldsymbol{S}}_{12} \\ \hline \hat{\boldsymbol{S}}_{21} & \boldsymbol{O}_{3\times 3} \end{array}\right] \tag{46}$$

The sub-matrices are respectively as:

$$\hat{\boldsymbol{S}}_{11} = \begin{bmatrix} s_2 c_{345} & s_{345} & 0 & 0 \\ -c_2 & 0 & 1 & 1 \\ s_2 s_{345} & -c_{345} & 0 & 0 \end{bmatrix} \tag{47}$$

$$\hat{\boldsymbol{S}}_{12} = \begin{bmatrix} 0 & 0 & s_6 \\ 1 & 0 & -c_6 \\ 0 & 1 & 0 \end{bmatrix} \tag{48}$$

$$\hat{\boldsymbol{S}}_{12} = \begin{bmatrix} J_{41} + d_6 \frac{c_6}{s_6} s_2 c_{345} & J_{51} + d_6 s_2 c_{345} & J_{61} + a_5 c_2 + \frac{a_5}{d_6} J_{41} \\ -d_3 c_{345} + d_6 \frac{c_6}{s_6} s_{345} & J_{52} + d_6 s_{345} & -d_3 s_{345} - \frac{a_5}{d_6} d_3 c_{345} \\ J_{43} & 0 & J_{63} - a_5 + \frac{a_5}{d_6} J_{43} \\ -a_4 s_5 & 0 & J_{64} - a_5 - \frac{a_5}{d_6} a_4 s_5 \end{bmatrix}^{\mathrm{T}} \tag{49}$$

Since $\det(\hat{\boldsymbol{S}}_{12}) = s_6 \neq 0$, $\hat{\boldsymbol{S}}_{12}$ is always non-singular for the case $s_6 \neq 0$, i.e. $\text{rank}(\hat{\boldsymbol{S}}_{12}) = 3$. Similarly, the singularity conditions of ${}^5\hat{\boldsymbol{S}}$ is the same as those of the sub-matrix $\hat{\boldsymbol{S}}_{21}$, i.e:

$$\text{rank}({}^5\hat{\boldsymbol{S}}) < 6 \quad \Leftrightarrow \quad \text{rank}(\hat{\boldsymbol{S}}_{21}) < 3 \tag{50}$$

The singularity conditions originally distributed in the 6×7 matrix ${}^5\boldsymbol{S}(\boldsymbol{\Theta})$ with $s_6 \neq 0$ are also separated and collected in the 3×4 sub-matrix $\hat{\boldsymbol{S}}_{21}$. Therefore, the singularity configurations of the redundant manipulator for the case $s_6 \neq 0$ can be identified by analyzing the rank deficient conditions of the 3×4 sub-matrix $\hat{\boldsymbol{S}}_{21}$.

2) Singular Configuration Identification

Similarly, the singularity conditions of $\hat{\boldsymbol{S}}_{21}$ are determined by setting all the order-3 minors of it to zero, i.e.:

$$\hat{M}_i = 0 \quad (i = 1, 2, 3, 4) \tag{51}$$

These order-3 minors are calculated as follows:

$$\hat{M}_1 = (A s_2 \eta_4 - B \eta_5) / s_6 \tag{52}$$

$$\hat{M}_2 = a_3 a_4 s_4 \eta_5 \tag{53}$$

$$\hat{M}_3 = a_3 a_4 s_2 s_4 \eta_4 \tag{54}$$

$$\hat{M}_4 = (\hat{A} s_2 \eta_4 - \hat{B} \eta_5) / s_6 \tag{55}$$

where,

$$\begin{cases} \eta_4 = -d_2 - a_3 s_3 - a_4 s_{34} - a_5 s_{345} + d_6 c_{345} \\ \eta_5 = -a_2 + a_3 c_3 + a_4 c_{34} + a_5 c_{345} + d_6 s_{345} \end{cases} \tag{56}$$

$$\begin{cases} A = a_4 (d_3 c_{34} s_6 + \eta_2 s_{345} c_6) \\ B = a_4 \left[(d_2 c_2 c_{34} + a_2 c_2 s_{34} - d_3 s_2 s_{34} - a_3 c_2 s_4) s_6 + \eta_2 s_2 c_{345} c_6 \right] \end{cases} \tag{57}$$

$$\begin{cases} \hat{A} = a_3 \left[d_3 c_3 s_6 + (a_5 s_{45} - d_6 c_{45}) s_{345} c_6 \right] + A \\ \hat{B} = a_3 (d_2 c_2 c_3 + a_2 c_2 s_3 - d_3 s_2 s_3) s_6 \\ \quad + a_3 (a_5 s_{45} - d_6 c_{45}) s_2 c_{345} c_6 + B \end{cases} \tag{58}$$

Then we can obtain the conditions causing all the order-3 minors to zeros. Then all the singularity conditions for the case $s_6 \neq 0$ are determined and listed in Table III.

TABLE III THE SINGULAR CONDITIONS FOR $\hat{\boldsymbol{S}}_{21}$ WITH $s_6 \neq 0$

Condition number	singular conditions
$\hat{k}_{s1}$	$s_2 = 0$ and $\eta_5 = 0$
$\hat{k}_{s2}$	$\eta_4 = 0$ and $\eta_5 = 0$
$\hat{k}_{s3}$	$s_4 = 0$ and $A s_2 \eta_4 - B \eta_5 = 0$ $(s_6 \neq 0)$

C. Summary of the Singular Conditions

The singularity conditions of the SSRMS-type manipulator are found by considering the two cases: $s_6 = 0$ and $s_6 \neq 0$. There are six conditions in total, three for each case. Actually, the conditions $\hat{k}_{s3}$ and $\hat{k}_{s3}$, determined for $s_6 = 0$ and $s_6 \neq 0$ respectively can be further combined. Observing the coefficients A, B, given in (57), we get the following logic relationship:

$$\begin{cases} s_6 = 0 \\ \eta_2 = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 0 \end{cases} \Rightarrow As_2\eta_4 - B\eta_5 = 0 \qquad (59)$$

Therefore, a new condition expression $k_{s1} = \hat{k}_{s3} \cup \overset{\frown}{k}_{s3}$ is used to represent the condition "$\hat{k}_{s3}$ or $\overset{\frown}{k}_{s3}$", the result is as follows:

$$k_{s1} = \hat{k}_{s3} \cup \overset{\frown}{k}_{s3} = \begin{cases} s_4 = 0 \\ As_2\eta_4 - B\eta_5 = 0 \end{cases} \qquad (60)$$

Finally, five singularity conditions of the SSRMS-type manipulator are obtained. They are denoted as k_{s1}, k_{s2}, …, k_{s5} respectively, and listed in Table IV. Since Nokleby and Podhorodeski [6] had identified the complete set of the singularity conditions of the Canadarm and its modified version, we can compare the results obtained using the proposed method with those reported in Ref. [6]. Although different frames (we use the classical D-H notation; and Nokleby and Podhorodeski [6] used the modified D-H notation) are defined and different symbols are used, the following relationships exist:

$$\begin{cases} d_2 = a,\ d_3 = c + f, d_6 = h,\ a_2 = b, a_3 = d \\ a_4 = e, a_5 = g,\ \theta_1 = \theta_{s1}, \theta_2 = \theta_{s2} + \pi,\ \theta_3 = \theta_{s3} + \pi \\ \theta_4 = \theta_{s4},\ \theta_5 = \theta_{s5}, \theta_6 = \theta_{6s} + \pi,\ \theta_7 = \theta_{s7} \end{cases} \qquad (61)$$

where, d_i, a_i and θ_i are the D-H parameters used in this paper and listed in Table I. The symbols *a*, *b*, …, *h* denoted the link lengths in Ref. [6], and θ_{si} is the joint variable corresponding the frames defined in Ref. [6]. Substituting (61) to the singularity conditions listed in Table IV, we get the same conditions as those of Ref. [6]. The corresponding expanded expressions of the singularities for the Canadarm2 are shown in Table V.

TABLE IV SINGULAR CONDITIONS FOR MODIFIED CANADARM2

Condition Index	Singular conditions	Remark
k_{s1}	$s_4 = 0$ and $As_2\eta_4 - B\eta_5 = 0$	$\hat{k}_{s3} \cup \overset{\frown}{k}_{s3}$
k_{s2}	$s_2 = 0$ and $s_6 = 0$	original $\hat{k}_{s1}$
k_{s3}	$s_2 = 0$ and $\eta_5 = 0$	original $\overset{\frown}{k}_{s1}$
k_{s4}	$s_6 = 0$ and $\eta_3 = 0$	original $\hat{k}_{s2}$
k_{s5}	$\eta_4 = 0$ and $\eta_5 = 0$	original $\overset{\frown}{k}_{s2}$

TABLE V SINGULAR CONDITIONS FOR CANADARM2

Method	The proposed method in this paper
1	$s_4 = 0$ and $-d_2c_2c_{34}s_6 + d_3s_2s_{34}s_6 + s_2N_a\left(d_6c_{345} - a_4s_{34} - a_3s_3 - d_2\right) + d_6s_2c_{345}c_5c_6 = 0$ where $N_a = \frac{d_3c_{34}s_6 - d_6s_{345}c_5c_6}{d_6s_{345} + a_4c_{34} + a_3c_3}$
2	$s_2 = 0$ and $s_6 = 0$
3	$s_2 = 0$ and $d_6s_{345} + a_4c_{34} + a_3c_3 = 0$
4	$s_6 = 0$ and $d_2s_{345} + a_3c_{45} + a_4c_5 = 0$
5	$d_6s_{345} + a_4c_{34} + a_3c_3 = 0$ and $d_6c_{345} - a_4s_{34} - a_3s_3 - d_2 = 0$

V. CONCLUSION

Compared with a 6-DOF manipulator, a redundant manipulator has great advantages in obstacle avoidance, joint torque optimization, manipulability enhancements, singularity handling, and so on. In the construction and maintenance of International Space Station, the Canadarm2 has been playing important roles. Due to lack of a spherical wrist, it is very difficult to solve the inverse kinematics problem and determine the singular configurations analytically of the Canadarm2. In this paper, we proposed a method to isolate the singularity condition and decompose the workspace. This method is based on the elementary transformation. By only several times row transformation, the Jacobian matrix is transformed to a block-triangle matrix, which is partitioned into four sub-matrices. Since the upper 3×7 non-zero block is proved to be full rank, the singularity conditions are isolated and collected in a 3×4 sub-matrix. Then, the singularity conditions of the 6×7 Jacobian matrix are identified by determining the rank degeneracy conditions of the 3×4 sub-matrix. Compared with previous methods, the singularity analysis is greatly simplified, and the computation workload is greatly reduced.

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