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Apex Height Control of a Four-Link Hopping Robot

Frank B. Mathis[1] and Ranjan Mukherjee[2]

***Abstract*— Bipedal robots have the potential to provide robust locomotion over uneven terrain and its dynamic stability has been shown to be analogous to that of a spring loaded inverted pendulum (SLIP). The SLIP model is fundamentally limited in its ability to accurately represent legged locomotion since it does not take into account the impulsive dynamics of foot ground interaction. In this paper we investigate the control of a four-link hopping robot based on the complete dynamics of the system. Using partial feedback linearization to control the configuration in continuous time, and discrete parameter variations the control object of apex height control of the robot is achieved. Simulation results are presented to show the efficacy of the control scheme.**

I. Introduction

Compared to wheeled robots, biped robots are better suited to locomotion over uneven terrain, and therefore, dynamic stability of biped robots has been an active area of research. The locomotion of biped robots involve transition of support from one leg to another, and therefore, walking and running actions have similarities with hopping of a monoped. The analysis of hopping motion of a single legged robot began with the work by Raibert [10], [17], where a mass on a hydraulic piston was used to experimentally demonstrate a stable hopping gait. The controller of the hydraulic monoped utilized a spring-like motion to produce vertical and lateral translation in addition to a stabilizing torque at the hip. The spring loaded inverted pendulum (SLIP) model was shown to be an accurate representation of running and hopping in biological systems [1], [4]. The validity of the SLIP model was also established by Koditscheck and Bruehler [13]. Schwind [20] and Saranli et al. [18] used the symetric SLIP model to design a controller for a four-link monoped and Hyon et al. [11] provided validity to the theoretical results in the literature with experimentation on a 3-link monoped.

Once it was established that the SLIP model can be used to achieve dynamic stability of a multi-link monoped, analysis of the SLIP model increased in popularity. Altendorfer et al. [2], [3] studied the return map of the non-integrable SLIP dynamics and investigated the stability of the map. Ghigliazza et al. [8], [7] studied the stability and bifurcation characteristics of the SLIP model using different spring dynamics. Seipel and Holmes [21], [22] investigated the stability of the SLIP model for a three-dimensional system, and Poulakakis and Grizzle [15], [16] investigated the dynamics and control of a legged robot with a more accurate asymmetric SLIP model. Hamed and Grizzle [9] later proposed a robust event-based control method to improve stability of the controlled legged system. Although the analysis of the SLIP model is useful and provides the basis for the design of effective controllers, it is incomplete since it does not account for the impulsive dynamics associated with foot-ground interaction.

In this paper we investigate the hybrid dynamics of a four-link hopping robot and propose a control method which takes into account the complete dynamics of the system. The hybrid dynamics is comprised of the flight phase, the impact phase, and the contact phase. Partial feedback linearization is used during the flight and contact phases to continuously control the configuration of the hopper, and "chaos control" [19] is used to stabilize the zero dynamics and control the apex height based on the hopping map. The validity of the method is shown through a simulation and a video animation. Our paper is structured as follows. In Section II we present the dynamics for the flight phase, impact phase, and contact phase. In Section III we design the continuous controllers for the flight and contact phases. In Section IV the first return map between consecutive hops is determined and a chaos controller is used to guarantee asymptotic stability of a periodic configuration. Simulations are presented in Section V and concluding remarks are presented in Section VI.

II. Dynamics

Consider the four link monoped hopping robot shown in Fig.1. Let x and y be the Cartesian coordinates of the base of the foot of the robot (point O) relative to the fixed ground reference. For $i = 1, 2, 3, 4$, the mass, moment of inertia, and length of each link are denoted by m_i, I_i, and l_i respectively. The angular displacement of the ith link is denoted by θ_i and the distance to its center of mass is denoted by d_i - see Fig.1. The states are defined as

$$q = \begin{bmatrix} x & y & \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix}^T \quad (1)$$

The equations of motion of the hopper are given by

$$M(q)\ddot{q} + N(q, \dot{q}) = AT + F_{ext} \quad (2)$$

where $M(q)$ is the mass matrix, $N(q, \dot{q})$ is the vector of Coriolis, centrifugal, and gravitational forces, $A \in \mathbb{R}^{6\times3}$ is the matrix give bellow:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, \quad (3)$$

T is the vector of input torques

$$T = \begin{bmatrix} \tau_1 & \tau_2 & \tau_3 \end{bmatrix}^T, \quad (4)$$

[1]F. Mathis is with the Dept. of Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA mathisfr@msu.edu

[2]R. Mukherjee is with the Dept. of Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA mukherji@msu.edu

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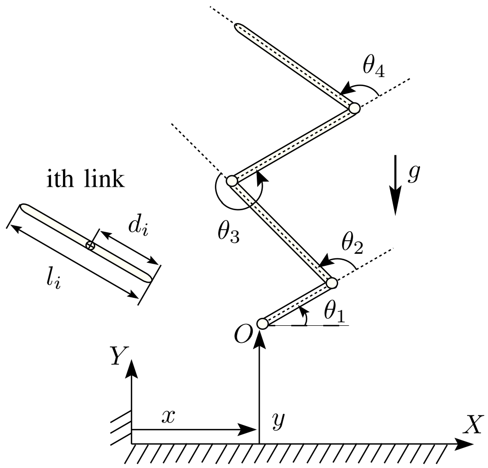

Fig. 1. Four-link hopping robot

and F_{ext} it the force applied by the ground on the robot given by

$$F_{ext} = \left[\begin{array}{cccccc} F_x & F_y & 0 & 0 & 0 & 0 \end{array} \right]^T \tag{5}$$

In Eq.(5), F_x and F_y denote the x and y components of the force applied to the robot by the ground at point O. The dynamics of the hopper may be separated into three phases: the flight phase for which $y > 0$, the impact phase which occurs at the instant $y = 0$, and the contact phase which occurs for the duration in which the foot remains in contact with the ground, $y \equiv 0$.

A. Flight Phase

During the flight phase $F_{ext} = 0$. Furthermore, the dynamics in Eq.(2) results in the non-holonomic constraint due to conservation of angular momentum about the center of mass of the hopper.

B. Impact

At the time of impact we assume:

Assumption 1: The applied torques T are not impulsive.

Assumption 1 does not imply that the torques T cannot be discontinuous; it simply implies that the torques cannot produce discrete jumps in the states.

Assumption 2: The hopper's foot comes in contact with the ground only at point O.

Assumption 2 can be enforced through proper choice of control gains.

Assumption 3: At the instant the foot contacts the ground ($y = 0$), the ground applies an impulsive force that results in $\dot{x} = \dot{y} = 0$.

Assumption 3 simply implies inelastic impact.

Taking the integral over the infinitesimal period of time in which the impact occurs we have

$$\int_{t_0}^{t_0+\epsilon} \ddot{q} dt = \int_{t_0}^{t_0+\epsilon} M^{-1}(q) \left[AT - N(q,\dot{q}) + F_{ext} \right] dt \tag{6}$$

$$\dot{q}^+ = \dot{q}^- + M^{-1}(q) F_{ext} \tag{7}$$

where $\dot{q}^+$ and $\dot{q}^-$ are the right and left limits in time of $\dot{q}$. This follows form our earlier work [6]. Partitioning q according to

$$q = [x \;\; y \mid \theta]^T \tag{8}$$

where θ is given by

$$\theta = [\theta_1 \; \theta_2 \; \theta_3 \; \theta_4]^T \tag{9}$$

results in the corresponding partition of $M^{-1}(q)$ given by

$$M^{-1}(q) = \left[\begin{array}{c|c} (M^{-1})_{11} & (M^{-1})_{12} \\ \hline (M^{-1})_{21} & (M^{-1})_{22} \end{array} \right] \tag{10}$$

Solving Eq.(7) results in the following change in the state variables

$$\begin{aligned} q^+ &= q^- \\ \dot{x}^+ &= 0 \\ \dot{y}^+ &= 0 \\ \dot{\theta}^+ &= \dot{\theta}^- - (M^{-1})_{21} [(M^{-1})_{11}]^{-1} \left[\begin{array}{c} \dot{x}^- \\ \dot{y}^- \end{array} \right] \end{aligned} \tag{11}$$

C. Contact Phase

During the contact phase F_{ext} is such that $\ddot{y} = 0$.

Assumption 4: The force F_x is always sufficiently large such that $\ddot{x} = 0$ during the contact phase.

During the contact phase the dynamics of the hopper is given by

$$DM(q)D^T D\ddot{q} + DN(q,\dot{q}) = DAT \tag{12}$$

where D is the matrix

$$D = \left[\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \tag{13}$$

The contact phase transitions to the flight phase when

$$F_y = 0 \qquad \dot{F}_y < 0 \tag{14}$$

III. Continuous Control

For control of the position of the center of mass of the hopper, we define r to be the vector from the base of the foot to the center of mass of the hopper. If r_x and r_y denote the horizontal and vertical components of r, we can write

$$r = \left[\begin{array}{c} r_x \\ r_y \end{array} \right] = \left[\begin{array}{c} f_x(q) \\ f_y(q) \end{array} \right] \tag{15}$$

where $f_x(q)$ and $f_y(q)$ are given by

$$\begin{aligned} f_x(q) = \; & a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) + \\ & a_3 \cos(\theta_1 + \theta_2 + \theta_3) + \\ & a_4 \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) \end{aligned} \tag{16}$$

$$\begin{aligned} f_y(q) = \; & a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) + \\ & a_3 \sin(\theta_1 + \theta_2 + \theta_3) + \\ & a_4 \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) \end{aligned} \tag{17}$$

in Eq.(16) and (17), the constants have the expressions

$$\begin{aligned} a_1 &= \frac{m_1 d_1 + (m_2 + m_3 + m_4) l_1}{\overline{m}} \\ a_2 &= \frac{m_2 d_2 + (m_3 + m_4) l_2}{\overline{m}} \\ a_3 &= \frac{m_3 d_3 + m_3 l_3}{\overline{m}} \\ a_4 &= \frac{m_4 d_4}{\overline{m}} \\ \overline{m} &= m_1 + m_2 + m_3 + m_4 \end{aligned} \quad (18)$$

Differentiating with respect to time gives

$$\dot{r} = \begin{bmatrix} \dot{r}_x \\ \dot{r}_y \end{bmatrix} = \begin{bmatrix} J_x(q) \\ J_y(q) \end{bmatrix} D\dot{q} \quad (19)$$

where $J_x(q)$ and $J_y(q)$ are Jacobian matrices.

In addition to control of the center of mass position r, we wish to control the angle of the first link, θ_1. To this end we define the desired equilibrium point of the system as follows

$$(r_x, r_y, \theta_1, \dot{r}_x, \dot{r}_y, \dot{\theta}_1) = (0, y_d, \theta_d, 0, 0, 0) \quad (20)$$

A. Contact Phase

To achieve the goal of a vertical hop, we control the position of the center of mass, the angle of the foot, and the angular momentum of the system. During the contact phase the system dynamics are described by Eq.(12). To transform this dynamics to normal form [12], we use the transformations in Eqs.(15),(19), and

$$\begin{aligned} \eta &= \psi_1(q) \\ \zeta_1 &= \psi_2(q, \dot{q}) = -\frac{1}{\overline{m}g} CDM(q)D^T D\dot{q} \end{aligned} \quad (21)$$

where the matrix C is given by

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \quad (22)$$

In addition, ζ_i, $i \in [2, 7]$, are defined as

$$\begin{array}{lll} \zeta_2 = r_x & \zeta_3 = r_y - y_d & \zeta_4 = \theta_1 - \theta_d \\ \zeta_5 = \dot{r}_x & \zeta_6 = \dot{r}_y & \zeta_7 = \dot{\theta}_1 \end{array} \quad (23)$$

It can be shown

$$\begin{aligned} \dot{\eta} &= \frac{\partial \psi_1(q)}{\partial q} \dot{q} = f(\eta, \zeta) \\ \dot{\zeta}_1 &= \frac{\partial \psi_2(q, \dot{q})}{\partial q} \dot{q} + \frac{\partial \psi_2(q, \dot{q})}{\dot{q}} \ddot{q} \\ &= \frac{-CD}{\overline{m}g} \left[AT - N(q, \dot{q}) + \dot{M}(q) D^T D\dot{q} \right] = \zeta_2 \end{aligned} \quad (24)$$

and

$$\begin{aligned} \left[\dot{\zeta}_2,\ \dot{\zeta}_3,\ \dot{\zeta}_4 \right]^T &= [\zeta_5,\ \zeta_6,\ \zeta_7]^T = J(q) D\dot{q} \quad (25) \\ \left[\dot{\zeta}_5,\ \dot{\zeta}_6,\ \dot{\zeta}_7 \right]^T &= J(q) D\ddot{q} + \dot{J}(q) D\dot{q} \quad (26) \end{aligned}$$

where $J(q)$ is given by

$$J(q) = \begin{bmatrix} J_x(q) \\ J_y(q) \\ C \end{bmatrix} \quad (27)$$

The expression $\dot{\zeta}_1 = \zeta_2$ follows intuitively from $\overline{m}g\zeta_1$ being the angular momentum of the hopper about its foot, and $\overline{m}g\zeta_2$ being the resulting torque about the foot due to gravity.

The dynamics in Eq.(12) are described by Eqs.(24), and (26) in the region where the transformations in Eq.(21) and (23) are diffeomorphic. Substituting Eq.(12) into (26) gives

$$\begin{bmatrix} \dot{\zeta}_5 \\ \dot{\zeta}_6 \\ \dot{\zeta}_7 \end{bmatrix} = J(q)(DMD^T)^{-1} D \left[AT - N(q, \dot{q}) \right] + \dot{J}(q) D\dot{q} \quad (28)$$

Defining the torques T to be

$$\begin{aligned} T = {} & [J(q)(DM(q)D^T)^{-1} DA]^{\#} [v_g \\ & + J(q)(DM(q)D^T)^{-1} DN(q, \dot{q}) - \dot{J}(q) D\dot{q}] \end{aligned} \quad (29)$$

where $(\cdot)^{\#}$ is the right pseudoinverse of $(\cdot)$, results in

$$\left[\dot{\zeta}_5,\ \dot{\zeta}_6,\ \dot{\zeta}_7 \right]^T = v_g \quad (30)$$

We choose v_g to the be given by

$$v_g = \begin{bmatrix} -K_1\zeta_1 - K_2\zeta_2 - K_5\zeta_5 \\ -K_3\zeta_3 - \alpha K_6 \zeta_6 \\ -K_4\zeta_4 - K_7\zeta_7 \end{bmatrix} \quad (31)$$

with α defined as

$$\alpha = \begin{cases} 1 & \zeta_6 \leq 0 \\ \nu & \zeta_6 > 0 \end{cases} \quad (32)$$

and the gains K_i chosen such that $K_i > 0\ \forall i$ and

$$K_6 < 2\sqrt{K_3} \quad (33)$$

This ensures asymptotic convergence of trajectories to the manifold

$$\mathcal{M} = \{\zeta \in \mathbb{R}^7 | (\zeta_1, \zeta_2, \zeta_4, \zeta_5, \zeta_7) = (0, 0, 0, 0, 0)\} \quad (34)$$

On $\mathcal{M}$, the trajectories of the system obey

$$\begin{bmatrix} \dot{\zeta}_3 \\ \dot{\zeta}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_3 & -\alpha K_6 \end{bmatrix} \begin{bmatrix} \zeta_3 \\ \zeta_6 \end{bmatrix} \quad (35)$$

which represents a "mass spring damper" whose damping is positive or negative based on the value of ν. By modulating ν we will increase or decrease the energy of the system and achieve apex height control.

B. Flight Phase

During the flight phase, the position of the foot relative to the center of mass is controlled in order to achieve a desired foot placement at the time of touchdown. In this phase the system has additional dynamics of x, $\dot{x}$, y, and $\dot{y}$, which were not present in the contact phase. We define the states

$$\begin{array}{ll} d = x + r_x & h = y + r_y \\ \dot{d} = \dot{x} + \dot{r}_x & \dot{h} = \dot{y} + \dot{r}_y \end{array} \quad (36)$$

where d and h represent the horizontal and vertical component of the center of mass in the inertial frame of reference. Using Eq.(2) we can show

$$\ddot{d} = 0, \qquad \ddot{h} = -g \quad (37)$$

Additionally, the angular momentum of the system about its center of mass is conserved. The angular momentum about the center of mass is given by

$$H_c = \sum_{i=1}^{4}\left[r_i \times m_i\dot{r}_i + I_i\sum_{j=1}^{i}\dot{\theta}_j\right] \tag{38}$$

Substituting Eqs.(21) and (23) into (38) gives

$$H_c = -\overline{m}[g\zeta_1 + (\zeta_3 + y_d)\zeta_5 - \zeta_2\zeta_6] \tag{39}$$

Sovling Eq.(39) for ζ_1 gives

$$\zeta_1 = \frac{1}{\overline{m}g}\left[\overline{m}\zeta_2\zeta_6 - \overline{m}(\zeta_3 + y_d)\zeta_5 - H_c\right] \tag{40}$$

This shows ζ_1 is related to ζ_2, ζ_3, ζ_5, and ζ_6 via an algebraic relationship. The dynamics of ζ_2, ζ_3, and ζ_4 are the same as in Eq.(25), whereas the dynamics of ζ_5, ζ_6, and ζ_7 can be obtained by substituting Eq.(2) into Eq.(26):

$$\begin{bmatrix} \dot{\zeta}_5 \\ \dot{\zeta}_6 \\ \dot{\zeta}_7 \end{bmatrix} = J(q)DM^{-1}\left[AT - N(q,\dot{q})\right] + \dot{J}(q)D\dot{q} \tag{41}$$

Defining the torques T to be

$$\begin{aligned} T \;\; = \;\; & [J(q)DM^{-1}(q)A]^{\#}[v_f \\ & + J(q)DM^{-1}(q)N(q,\dot{q}) - \dot{J}(q)D\dot{q}] \end{aligned} \tag{42}$$

results in

$$\left[\dot{\zeta}_5,\ \dot{\zeta}_6,\ \dot{\zeta}_7\right]^T = v_f \tag{43}$$

We choose v_f as follows:

$$v_f = \begin{bmatrix} -K_2\zeta_2 - K_5\zeta_5 \\ -K_3\zeta_3 - K_6\zeta_6 \\ -K_4\zeta_4 - K_7\zeta_7 \end{bmatrix} \tag{44}$$

where $K_i > 0\ \forall i \in [2,\ 7]$. This guarantees that the variables $\zeta_i\ i \in [2,\ 7]$ will asymptotically converge to zero. Additionally, if $H_c = 0$, ζ_1 will asymptotically converge to zero.

Since the angular moment H_c cannot be controlled in the flight phase, our objective is to bring it to zero during the contact phase.

C. Hybrid Dynamics of Closed-Loop System

The hybrid dynamics of the system over one hop is summarized as follows:

Flight Phase: The dynamics during flight phase is described by

$$\begin{bmatrix} \ddot{d} \\ \ddot{h} \\ \dot{\eta} \\ \dot{\zeta}_2 \\ \dot{\zeta}_3 \\ \dot{\zeta}_4 \\ \dot{\zeta}_5 \\ \dot{\zeta}_6 \\ \dot{\zeta}_7 \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ f(\eta,\zeta) \\ \zeta_5 \\ \zeta_6 \\ \zeta_7 \\ -K_2\zeta_2 - K_5\zeta_5 \\ -K_3(\zeta_3 - y_d) - K_6\zeta_6 \\ -K_4(\zeta_4 - \theta_d) - K_7\zeta_7 \end{bmatrix} \tag{45}$$

and the non-holonmic constraint given by Eq.(40) for ζ_1.

Impact: The hopper makes contact with the ground when $y = 0$ or $h - \zeta_3 - y_d = 0$. The impulse due to impact is given by Eq.(11).

Contact Phase: From Eqs.(24), (26), (30) and (31), and $\ddot{x} = \ddot{y} = 0$ the dynamics during contact phase are given by

$$\begin{bmatrix} \dot{\eta} \\ \dot{\zeta}_1 \\ \dot{\zeta}_2 \\ \dot{\zeta}_3 \\ \dot{\zeta}_4 \\ \dot{\zeta}_5 \\ \dot{\zeta}_6 \\ \dot{\zeta}_7 \end{bmatrix} = \begin{bmatrix} f(\eta,\zeta) \\ \zeta_2 \\ \zeta_5 \\ \zeta_6 \\ \zeta_7 \\ -K_1\zeta_1 - K_2\zeta_2 - K_5\zeta_5 \\ -K_3(\zeta_3 - y_d) - \alpha K_6\zeta_6 \\ -K_4(\zeta_4 - \theta_d) - K_7\zeta_7 \end{bmatrix} \tag{46}$$

where

$$\alpha = \begin{cases} 1 & \zeta_6 \le 0 \\ \nu & \zeta_6 > 0 \end{cases} \tag{47}$$

and the states h and $\dot{d}$ are given by

$$h \equiv \zeta_3 + y_d, \qquad \dot{d} \equiv \zeta_5 \tag{48}$$

The contact phase ends at the instant F_y in Eq.(5) is equal to 0, that is

$$F_y = \overline{m}\dot{\zeta}_6 + \overline{m}g = 0 \Rightarrow \dot{\zeta}_6 = -g \tag{49}$$

IV. Periodic Behaviour

Hopping is described by consecutive sequences of flight phase, impact, and contact phase. To describe a single hop, we define the state χ

$$\chi \in \Omega, \qquad \Omega = \{(\theta,\dot{\theta}) \mid \dot{\zeta}_6(\theta,\dot{\theta}) + g = 0\} \tag{50}$$

which define the configuration of the hopper at the time of transition from the contact to the flight phase. The configuration χ does not include d since the objective of this paper is to control only the height. The first return map between the kth hop and the $(k+1)$th hop is defined as

$$\chi(k+1) = P(\chi(k)) \tag{51}$$

where $P(\chi(k))$ is the solution of the closed-loop hybrid dynamics.

A. Period One Orbits

For a period one orbit [5] we have

$$\chi(k) - P(\chi(k)) = 0 \tag{52}$$

Let χ^* to be any value of χ that satisfies Eq.(52). Note that χ lies in a 7 embedded manifold of $\mathbb{R}^8$. Let V to be the matrix of linearly independent unit vectors

$$V = [v_1, v_2, \ldots v_7] \qquad v_i \in \mathbb{R}^8,\ |v_i| = 1,\ i \in [1,7] \tag{53}$$

where

$$\text{span}(V) = \Omega \tag{54}$$

Linearization of Eq.(51) about the periodic point χ^* gives

$$\chi(k+1) \approx P(\chi^*) + \sum_{i=1}^{7} \left[\frac{\partial P(\chi)}{\partial v_i} v_i^T |_{\chi=\chi^*} (\chi(k) - \chi^*) \right] \tag{55}$$

where $\partial P(\chi)/\partial v_i$ is given by

$$\frac{\partial P(\chi)}{\partial v_i} = \lim_{h \to 0} \frac{P(\chi + h v_i) - P(\chi)}{h} \tag{56}$$

It follows that Eq.(55) is asymptotically stable iff

$$\rho \left(\sum_{i=1}^{7} \left[\frac{\partial P(\chi)}{\partial v_i} v_i^T |_{\chi=\chi^*} \right] \right) < 1 \tag{57}$$

where $\rho(\cdot)$ is the spectral radius.

B. Chaos Control

A periodic orbit defined by χ^* may not be stable for a given set of system parameters. However, we note that $P(\chi)$ is dependant on the variable ν. By defining ν^* to be the value of ν that satisfies Eq.(52) for $\chi = \chi^*$, we will vary the value of ν to ensure asymptotic stability of χ^*.

To design the input $\nu(k)$, first define the vector $\overline{v}$ as

$$\overline{v} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad |\overline{v}| = 1,\ a \in \mathbb{R}^8,\ b \in \mathbb{R} - \{0\} \tag{58}$$

where a satisfies Eq.(49). Defining the error E and the input u as

$$\begin{aligned} E(k) &= \chi(k) - \chi^* \\ u(k) &= \nu(k) - \nu^* \end{aligned} \tag{59}$$

we have the linearized equation of Eqn. (52):

$$E(k+1) = AE(k) + Bu(k) \tag{60}$$

where A and B are given by

$$\begin{aligned} A &= \sum_{i=1}^{7} \left[\frac{\partial P(\chi)}{\partial v_i} v_i^T \right] + \frac{\partial P(\chi(k))}{\partial \overline{v}} a^T \\ B &= \frac{\partial P(\chi(k))}{\partial \overline{v}} b \end{aligned} \tag{61}$$

and where the directional derivatives $\partial P(\chi)/\partial v_i$ and $\partial P(\chi)/\partial \overline{v}$ are evaluated at (χ^*, ν^*). For asymptotic stability, $u(k)$ is designed as

$$u(k) = GE(k) \tag{62}$$

where G is chosen such that

$$\rho(A + BG) < 1 \tag{63}$$

V. Simulations

For the four-link hopper, the masses are assumed to be

$$m_1 = 2.5\ kg,\ m_2 = 5\ kg,\ m_3 = 10\ kg,\ m_4 = 20\ kg \tag{64}$$

The length of the links of the hopper are assumed to be

$$l_1 = 0.1\ m, \quad l_2 = l_3 = l_4 = 0.3\ m \tag{65}$$

and the distance to the center of mass of each link - see Fig.1 are assumed to be

$$d_1 = 0.05\ m, \quad d_2 = d_3 = d_4 = 0.15\ m \tag{66}$$

The moment of inertia of each link is computed as

$$I_i = \frac{1}{12} m_i l_i^2 \qquad \forall i \in [1, 4] \tag{67}$$

The gains used for the continuous control are

$$\begin{array}{llll} K_1 = 12000 & K_2 = 8000 & K_3 = 300 & K_4 = 300 \\ K_5 = 120 & K_6 = 10 & K_7 = 8 & \end{array} \tag{68}$$

and the set points y_d and θ_d for the continuous control are

$$y_d = 0.4967\ m \qquad \theta_d = \frac{\pi}{2} - 0.1 \tag{69}$$

The value of y_d is $0.14\ m$ below the maximum height of the center of mass relative to the foot.

We choose the desired apex height of the center of mass to be 0.65 meters and compute the periodic point, (χ^*, ν^*) to be given by

$$\begin{array}{lll} \chi_1^* = 1.579 & \chi_2^* = -0.407 & \chi_3^* = 1.135 \\ \chi_4^* = -1.178 & \chi_5^* = -6.696 & \chi_6^* = 11.659 \\ \chi_7^* = -10.360 & \chi_8^* = 8.960 & \nu^* = -0.607 \end{array} \tag{70}$$

with all values given in rad and rad/s where appropriate. The stabilizing control gains G for the periodic point in Eq.(70) are by solving the discrete LQR problem:

$$\begin{array}{lll} G_1 = -0.023 & G_2 = 0.025 & G_3 = -0.005 \\ G_4 = 0.011 & G_5 = -0.051 & G_6 = -0.030 \\ G_7 = 0.005 & G_8 = -0.027 & \end{array} \tag{71}$$

We choose the initial configuration of the system is assumed to be

$$\begin{aligned} (x(0), \dot{x}(0), y(0), \dot{y}(0)) &= (0.00, 0.00, 0.03, 0.00) \\ \theta(0) &= [1.65,\ -0.50,\ 1.07,\ -0.94]^T \\ \dot{\theta}(0) &= [0.00,\ 0.00,\ 0.00,\ 0.00] \end{aligned} \tag{72}$$

where the units are in meters, rad, and rad/sec. These initial conditions were chosen such that the center of mass of the hopper lies vertically above the point of support. The initial value of the discrete control input is chosen to be

$$u(0) = 0 \Rightarrow \nu(0) = \nu^* \tag{73}$$

Figure 2 shows the height of the center of mass as a function of time. It can be seen that the center of mass converges to the desired height in approximately 4 hops. Figure 3 displays the inputs torques τ_1, τ_2, and τ_3. The sharp peaks indicate the discontinuous jumps in the torques immediately following impact.

A video animation of the simulation, running for a longer duration of time than that presented in Figs.2 and 3, has been uploaded as supplementary material.

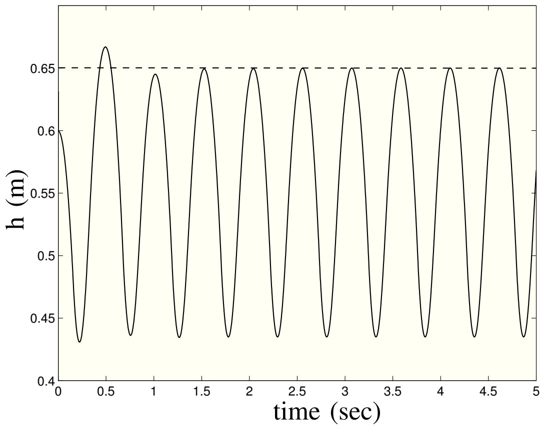

Fig. 2. Center of mass height above the ground for the four-link hopper

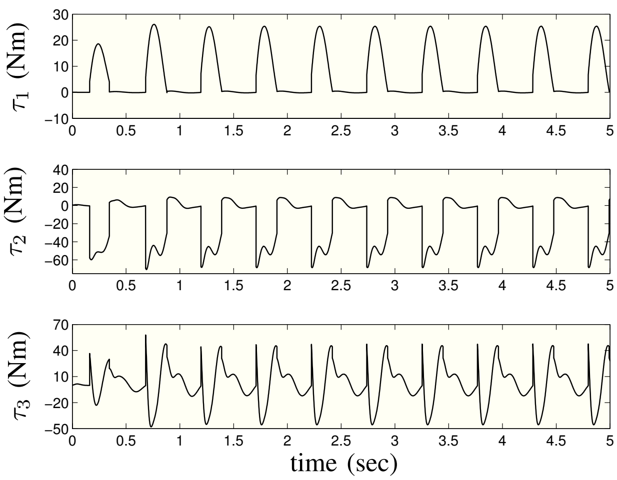

Fig. 3. Input torques for the four-link hopper

VI. CONCLUSION

A method for controlling the apex height of a four-link hopping robot was presented. The dynamics of the hopping robot was separated into the flight phase, impact phase, and contact phase; and partial feedback linearization was used to control the configuration of the robot during the flight and contact phases. A discrete chaos-controller was used to converge the first return map of consecutive hops to a periodic point utilizing variations of a specific parameter of the continuous controller. Simulation results were presented to show that the controller is indeed capable of converging the apex height of the robot to a desired value. Future work will be aimed at experimental verification, as well as generalization of the results to an N-link hopping robot. Based on preliminary investigations, it appears that the method presented in this paper will be applicable to hopping robots with revolute joints (like the one presented in this paper) for $N \geq 3$. The method will not be applicable for a two-link revolute joint hopping robot since it is not possible to decouple the X and Y motions. For a two-link prismatic joint robot, however, the dynamics in the X and Y directions are naturally decoupled and it is possible to design an apex height controller, as shown in our earlier work [14].

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