An Algebraic Criterion for Global Exponential Stability of Cohen-Grossberg Neural Networks with Time-varying Delays

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***Abstract*—In this paper, by constructing an appropriate Lyapunov functional, sufficient criteria independent of the delays for global exponential stability of the network are derived. The algebra criteria are applicable for other neural network models. This results are less conservative and restrictive than previously known results and can be easily verified. And the result has overcome the obvious drawback that previous works neglect the signs of the connecting weights, and thus, do not distinguish the differences between excitatory and inhibitory connections. It is believed that the results are significant and useful for the design and applications of the Cohen-Grossberg model.**

***Keywords*—Novel Criterion, Global Exponential Stability, Cohen-Grossberg Neural Network, Time-varying Delay**

I. INTRODUCTION

A large class of neural networks, which can function as stable content addressable memories or CAMs, are proposed by Cohen and Grossberg [1] and [2]. These Cohen-Grossberg networks were designed to include additive neural networks [3], [4], [7], [9], [10], [11], [16], [17], [21], [22], [23]and [24], later studied by Hopfield [3] and [4], and shunting neural networks. In general, the Cohen-Grossberg neural network model is described by the set of ordinary differential equations:

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) + J_i \right] \qquad (1)$$

where i = 1, 2, … , n, J_i, denote the constant inputs from outside of the system and w_{ij} represent the connection weights. $a_i(x_i)$ and $b_i(x_i)$, are the amplification functions and the self-signal functions, respectively, while $f_j(x_j)$, j = 1,2, … , n, are the activation functions. In the original analysis, Cohen and Grossberg assumed that the weight matrix $W=(w_{ij})_{n\times n}$ was symmetric and the amplification functions $a_i(x_i)>0$. Meanwhile, the activation functions f_j are assumed to be continuous, differentiable, monotonically increasing and bounded, such as the sigmoid-type function. For simplicity, the explanation of the lowcase i bound is omitted in the following equations.

In reality, time delays inevitably exist in biological and artificial neural networks due to the finite signal switching and transmission speed in a network. It is also important to incorporate time delay into the model equations of the network such as delayed cellular neural network, which can be used to solve problems like the processing of moving images [14] and [15]. Discrete delays were introduced into system (1) by considering the following system [5] and [8]:

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} v_{ij}^{(k)} f_j(x_j(t-\tau_k)) + J_i \right] \qquad (2)$$

where the matrix $V_k = (v_{ij}^{(k)})_{n\times n}$ represents the interconnections associated with delay τ_k of which τ_k, k = 0, 1, … , K, are arranged such that $0 = \tau_0 < \tau_1 <\ldots< \tau_k$. Furthermore, their global limit property needs to satisfy the requirements that the connection should possess certain amount of symmetry and the discrete delays are sufficiently small. Hence, the work of Ye et al. [8] cannot tell what would happen when the delays increase. We have mentioned above that large delay could destroy the stability of the equilibrium in a network. Even if the delay does not change the stability, it could affect the basin of attraction of the stable equilibrium. For this topic, interested readers may refer to the recent work [18], [29], [30] and [31]. Recently, for the delayed Hopfield networks [7], [9], [10] and [11], cellular neural networks [14] and [15] as well as BAM networks [12] and [13], some delay-independent criteria for the global asymptotic stability are established without assuming the monotonicity and the differentiability of the activation functions, nor the symmetry of the connections, [7] and [9]. Wang and Zou [5][6] studied the following system

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} v_{ij} f_j(x_j(t-\tau_{ij})) + J_i \right] \qquad (3)$$

where $v_{ij} \geq 0$ are delays caused during the switching and transmission processes. Some criteria for the exponential stability of a unique equilibrium are obtained.

Liao [19] studied a modified Cohen-Grossberg model with discrete delays described by the differential difference equation of the form

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} v_{ij} f_j(x_j(t-\tau_{ij})) + J_i\right] \quad (4)$$

In this paper, we consider a modified Cohen-Grossberg model with multiple time-varying delays described by the differential difference equation of the form

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} v_{ij}^{(k)} f_j(x_j(t-\tau_{ij}^{(k)}(t))) + J_i\right] \quad (5)$$

Unlike most of the previous authors, we did not confine to the symmetric connections. Moreover, we did not assume the monotonicity and differentiability of the activation functions. In this paper, the amplification functions only require to be continuous and positive. Moreover, the self-signal functions are not assumed to be differentiable. Thus a much broader connection topology of the network is allowed. The global exponential stability criteria of the Cohen-Grossberg neural network with time-varying delays were also obtained by constructing appropriate Lyapunov functions. And the result has overcome the obvious drawback that it neglects the signs of the connecting weights, and doesn't distinguish the differences between excitatory and inhibitory connections.

The rest of this paper is organized as follows. In Section 2, some preliminary analyses are given. By constructing a new Lyapunov functional, we studied the Cohen-Grossberg models with time-varying delays and gained some global exponential stability criteria in Section 3. Some comparisons and an example are given in Section 4 to demonstrate our main results. Finally, conclusions are drawn in Section 5.

II. Some Preliminaries

Let R denotes the set of real numbers and if $x \in R^n$, then $x^T = (x_1, x_2, \dots, x_n)$ denotes the transpose of x. Let $R^{n\times n}$ denotes the set of $n \times n$ real matrices. The initial conditions associated with (5) are in the form:

$$x_i(s) = \phi_i(s) \in C([-\tau, 0], R), s \in [-\tau, 0], i = 1,2,\cdots,n,$$

$$\tau = \max_{1\le i,j\le n, 0\le k\le K}\{\tau_{ij}^{(k)}(t)\}.$$

For functions $a_i(x_i)$ and $b_i(x_i)$, $i = 1,2, \dots, n$, the following assumptions are used.

(H_1) $a_i(u)$ are continuous and there exist positive constants $\underline{a}_i$ and $\overline{a}_i$ such that $0 < \underline{a}_i \le a_i(u) \le \overline{a}_i$, for all $u \in R$, $i = 1,2, \dots, n$.

(H_2) b_i and b_i^{-1}, $i = 1,2, \dots, n$, are locally Lipschiz continuous and there exist $\gamma_i > 0$, $i = 1,2, \dots, n$, such that

$u[b_i(u+x)-b_i(x)] \ge \gamma_i u^2$

Remark 1. In [5], it is necessary that $b_i'(x) > 0$. However, our condition (H_2) does not require that $b_i(x)$, $i = 1,2, \dots, n$, to be continuous and differentiable.Moreover, our results are less conservative and restrictive than those given in [5].

Usually, the assumed activation functions are continuous, differentiable, and monotonically increasing and bounded, such as the sigmoid-type functions. However, as pointed out in [10], [11], [12], [13], [14], [15], [16] and [17], for certain purposes, non-monotonic and not necessarily smooth functions might be better candidates for the neuron activation functions in designing and implementing an artificial neural network. Note that in many electronic circuits, amplifiers possess neither monotonically increasing nor continuously differentiable input-output functions are frequently adopted.

Moreover, we assume that the activation functions f_i, $i = 1,2, \dots, n$, satisfy either (H_3) or (H_3'):

(H_3) (i) f_i, $i = 1,2, \dots, n$, are bounded in R;

(ii) $0 < \dfrac{f_i(x+u) - f_i(x)}{u} \le L_i, i = 1,2,\cdots,n.$

(H_3') (i) f_i, $i = 1,2, \dots, n$, are bounded in R;

(ii) $f_i(x+u) - f_i(x) \le L_i|u|, i = 1,2,\cdots,n.$

(H_4) $\tau_{ij}^{(k)}(t)$:$[0,+\infty] \to [0,+\infty]$ is continuous, differentiable, and $0 \le \tau_{ij}^{(k)}(t) \le \bar{\tau}$, $\dot{\tau}_{ij}^{(k)}(t) \le q<1$;

Hence, the hypothesis (H_4) ensures that $t-\tau_{ij}^{(k)}(t)$ has differential inverse function denoted by $\varphi_{ij}^{(k)}(t)$ and $\dot{\varphi}_{ij}^{(k)}(t) > 0$.

Remark 2. Note that unlike the requirement stated in[8], $V = \sum_{k=0}^{K} V_k$ is not required to be symmetric in this paper. This means that our results are applicable to networks with a much broader connection structure.

Before stating the main results, we first need the following definitions and lemmas.For convenience, we introduce some notations. For a symmetric matrix, A>0 (<0) means that A is positive definite (negative definite).

Definition 1. For any continuous function h: $R \to R$, Dini's time-derivative of $h(t)$ is defined as

$$D^+h(t) = \limsup_{\theta\to 0^+} \frac{h(t+\theta) - h(t)}{\theta}.$$

It is easy to see that if $h(t)$ is locally Lipschitz, then $D^+h(t) < \infty$.

Definition 2. If there exist k>0 and $\gamma(k) > 1$, such that

$$\|x(t)\| \le \gamma(k)e^{-kt} \sup_{-\tau\le\theta\le 0}\|x(\theta)\|, \forall t > 0,$$

system (5) is considered as exponential stable, and k is called the degree of exponential stability.

We note that x^* is an equilibrium of system (5) if and only if $x^* = [x_1^*, x_2^*, \cdots, x_n^*]^T \in R$ is a solution of the following equations

$$b_i(x_i^*) - \sum_{j=1}^{n} w_{ij} f_j(x_j^*) - \sum_{k=0}^{K}\sum_{j=1}^{n} v_{ij}^{(k)} f_j(x_j^*) + J_i \tag{6}$$

Similar to the discussion in [5], we can easily obtain the following results.

Lemma 1. *If* (H_1)-(H_2) *and* (H_3) *or* (H_3') *hold, then for every input J, there exists an equilibrium point for system* (5).

Let x^* be an equilibrium of system (5) and $z(t) = x(t) - x^*$. Substituting $x(t) = z(t) + x^*$ into system (5) leads to

$$\dot{z}_i(t) = -a_i(z_i(t)+x_i^*)\Bigg[b_i(z_i(t)+x_i^*) - \sum_{j=1}^{n} w_{ij} f_j(z_j(t)+x_j^*) - - \sum_{k=0}^{K}\sum_{j=1}^{n} v_{ij}^{(k)} f_j(z_j(t-\tau_{ij}^{(k)}(t))+x_j^*) + J_i\Bigg] \tag{7}$$

By (7), system (5) can be rewritten as

$$\dot{z}_i(t) = -\alpha_i(z_i(t))\Bigg[\beta_i(z_i(t)) - \sum_{j=1}^{n} w_{ij} g_j(z_j(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} v_{ij}^{(k)} g_j(z_j(t-\tau_{ij}^{(k)}(t)))\Bigg] \tag{8}$$

where

$$\begin{aligned} \alpha_i(z_i(t)) &= a_i(z_i(t)+x_i^*) \\ \beta_i(z_i(t)) &= b_i(z_i(t)+x_i^*) - b_i(x_i^*) \\ g_i(z_i(t)) &= f_i(z_i(t)+x_i^*) - f_i(x_i^*) \end{aligned} \tag{9}$$

It is obvious that x^* is global exponential stable for system (5) if and only if the trivial solution $z = 0$ of system (7) is global exponential stable. Moreover, the uniqueness of the equilibrium of (5) follows from its global exponential stability.

From assumption H_3', we have

(H_5) $|g_i(z_i)| \le L_i |z_i|$, $g_i(0) = 0$, $i = 1,2,\cdots,n$.

Then, it is easy to see that

$z_i g_i(z_i) \le L_i z_i^2$.

Definition 3. For any scalar $a \in R$, a^+ is defined as

$$a^+ = \max\{0, a\}. \tag{10}$$

or

$$a^+ = |a|. \tag{11}$$

Lemma 2. Any scalar $a \in R$, there exist scalar $a+$, we hold the inequality $a \le a^+$.

Proof. Any scalar $a \in R$, we have

$a \le \max\{0, a\}$, and $a \le |a|$.

It is easy to see that $a \le a^+$.

III. Main Results

Theorem 1. *Consider the delayed system* (5) *and assume that conditions* (H_1)-(H_4) *are satisfied. If there exist* ψ_{ij}, ρ_{ij}, ζ_{ij}, $\eta_{ij} \in [0, 1]$, *and positive constants* r_1, r_2, d_i, $i, j = 1,2, \ldots, n$, *and* $\sigma > 0$, *and the following condition holds:*

$$\Delta = \max_{1\le i\le n}\Bigg\{\Bigg[d_i\Big(\gamma_i - \frac{\sigma}{\underline{\alpha}_i}\Big)\Bigg]^{-1}\Bigg[\sum_{j=1}^{n}\Big(r_1 d_i (w_{ij}^+)^{2\psi_{ij}} L_j^{2\rho_{ij}} + \frac{1}{r_1} d_j (w_{ji}^+)^{2(1-\psi_{ji})} L_i^{2(1-\rho_{ji})}\Big) + \sum_{k=0}^{K}\sum_{j=1}^{n}\Big(r_2 d_i ((v_{ij}^{(k)})^+)^{2\zeta_{ij}} L_j^{2\eta_{ij}} (1-q)^{-1} e^{2\sigma\tau_{ij}^{(k)}(t)} + \frac{1}{r_2} d_j ((v_{ji}^{(k)})^+)^{2(1-\zeta_{ji})} L_i^{2(1-\eta_{ji})}\Big)\Bigg]\Bigg\} < 2 \tag{12}$$

then, for every input J, the equilibrium x for system* (5) *is global exponential stable. This implies that there exists constants* $\eta \ge 1$ *such that every solution of* (5) *satisfies*

$$\sum_{i=1}^{n}(x_i(t)-x_i^*)^2 \le \eta e^{-2\sigma t}\sum_{i=1}^{n}\sup_{s\in[-\tau,0]}(\phi_i(s)-x_i^*)^2, t > 0 \tag{13}$$

where

$$\eta \le \max_{1\le i\le n}\Bigg\{\frac{d_i}{\underline{\alpha}_i}\Bigg(1 + \frac{\overline{\alpha}_i}{2\sigma}(1-e^{-2\sigma\tau})\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})} L_j^{2(1-\eta_{ij})}\Bigg)\Bigg\} \Bigg/ \min_{1\le i\le n}\Bigg(\frac{d_i}{\overline{\alpha}_i}\Bigg).$$

Proof. We construct the following nonnegative functional as the Lyapunov functional candidate:

$$V(z(t)) = \sum_{i=1}^{n} d_i\Bigg\{e^{2\sigma t}\int_0^{z_i(t)}\frac{s}{\alpha_i(s)}ds + 2\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})} L_j^{2(1-\eta_{ij})}\int_{t-\tau_{ij}^{(k)}(t)}^{t} z_j^2(s) e^{2\sigma s} ds\Bigg\} \tag{14}$$

By computing its the Dini's time-derivative of $V(z(t))$ along the trajectories of (8) and making use of the Cauchy inequality (i.e, $ra^2 + b^2/r\ 2ab$, $r > 0$), we have

$$\begin{aligned} D^+V(z(t)) = \sum_{i=1}^{n} d_i\Bigg\{ & 2e^{2\sigma t}\frac{z_i(t)}{\alpha_i(z_i(t))}\dot{z}_i(t) + 4\sigma e^{2\sigma t}\int_0^{z_i(t)}\frac{s}{\alpha_i(s)}ds + \\ & \sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})} L_j^{2(1-\eta_{ij})} z_j^2(t) e^{2\sigma t} \\ & - \sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})} L_j^{2(1-\eta_{ij})}(1-\dot{\tau}_{ij}^{(k)}(t)) z_j^2(t-\tau_{ij}^{(k)}(t)) e^{2\sigma(t-\tau_{ij}^{(k)}(t))}\Bigg\} \\ = \sum_{i=1}^{n} d_i\Bigg\{ & -2e^{2\sigma t} z_i(t)\Bigg[\beta_i(z_i(t)) - \sum_{j=1}^{n} w_{ij} g_j(z_j(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} v_{ij}^{(k)} g_j(z_j(t-\tau_{ij}^{(k)}(t)))\Bigg] \\ & + 4\sigma e^{2\sigma t}\int_0^{z_i(t)}\frac{s}{\alpha_i(s)}ds + \sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})} L_j^{2(1-\eta_{ij})} z_j^2(t) e^{2\sigma t} \end{aligned}$$

$$-\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}(1-\dot{\tau}_{ij}^{(k)}(t))z_j^2(t-\tau_{ij}^{(k)}(t))e^{2\sigma(t-\tau_{ij}^{(k)}(t))}\Bigg\}$$

$$\leq\sum_{i=1}^{n}d_ie^{2\sigma}\Bigg\{-2z_i(t)\beta_i(z_i(t))+2\sum_{j=1}^{n}w_{ij}z_i(t)g_j(z_j(t))+2\sum_{k=0}^{K}\sum_{j=1}^{n}v_{ij}^{(k)}z_i(t)$$

$$g_j(z_j(t-\tau_{ij}^{(k)}(t)))+\frac{2\sigma}{\underline{\alpha}_i}z_i^2(t)+\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}z_j^2(t)-$$

$$\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}(1-q)z_j^2(t-\tau_{ij}^{(k)}(t))e^{-2\sigma\tau_{ij}^{(k)}(t)}\Bigg\}$$

$$\leq\sum_{i=1}^{n}d_ie^{2\sigma}\Bigg\{-2\gamma_iz_i^2(t)+2\sum_{j=1}^{n}w_{ij}^+L_j|z_i(t)||z_j(t)|+2\sum_{k=0}^{K}\sum_{j=1}^{n}(v_{ij}^{(k)})^+L_j|z_i(t)|$$

$$|z_j(t-\tau_{ij}^{(k)}(t))|+\frac{2\sigma}{\underline{\alpha}_i}z_i^2(t)+\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}z_j^2(t)$$

$$-\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}(1-q)z_j^2(t-\tau_{ij}^{(k)}(t))e^{-2\sigma\tau_{ij}^{(k)}(t)}\Bigg\}$$

$$=\sum_{i=1}^{n}d_ie^{2\sigma}\Bigg\{-2\left(\gamma_i-\frac{\sigma}{\underline{\alpha}_i}\right)z_i^2(t)+2\sum_{j=1}^{n}((w_{ij}^+)^{\psi_{ij}}L_j^{\rho_{ij}}|z_i(t)|)((w_{ij}^+)^{1-\psi_{ij}}L_j^{1-\rho_{ij}}|z_j(t)|)+$$

$$2\sum_{k=0}^{K}\sum_{j=1}^{n}\left[((v_{ij}^{(k)})^+)^{\zeta_{ij}}L_j^{\eta_{ij}}(1-q)^{-\frac{1}{2}}e^{\sigma\tau_{ij}^{(k)}(t)}|z_i(t)|\right]$$

$$\left[(v_{ij}^{(k)})^+)^{1-\zeta_{ij}}L_j^{1-\eta_{ij}}(1-q)^{\frac{1}{2}}|z_j(t-\tau_{ij}^{(k)}(t))|e^{-\sigma\tau_{ij}^{(k)}(t)}\right]+$$

$$\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}z_j^2(t)-$$

$$\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}(1-q)z_j^2(t-\tau_{ij}^{(k)}(t))e^{-2\sigma\tau_{ij}^{(k)}(t)}\Bigg\}$$

$$\leq\sum_{i=1}^{n}d_ie^{2\sigma}\Bigg\{-2\left(\gamma_i-\frac{\sigma}{\underline{\alpha}_i}\right)z_i^2(t)+\sum_{j=1}^{n}r_1(w_{ij}^+)^{2\psi_{ij}}L_j^{2\rho_{ij}}z_i^2(t)+$$

$$\sum_{j=1}^{n}\frac{1}{r_1}(w_{ij}^+)^{2(1-\psi_{ij})}L_j^{2(1-\rho_{ij})}z_j^2(t)+\sum_{k=0}^{K}\sum_{j=1}^{n}r_2((v_{ij}^{(k)})^+)^{2\zeta_{ij}}L_j^{2\eta_{ij}}(1-q)^{-1}e^{2\sigma\tau_{ij}^{(k)}(t)}z_i^2(t)$$

$$+\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}(1-q)z_j^2(t-\tau_{ij}^{(k)}(t))e^{-2\sigma\tau_{ij}^{(k)}(t)}$$

$$+\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}z_j^2(t)-$$

$$\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}(1-q)z_j^2(t-\tau_{ij}^{(k)}(t))e^{-2\sigma\tau_{ij}^{(k)}(t)}\Bigg\}$$

$$=\sum_{i=1}^{n}d_i\left(\gamma_i-\frac{\sigma}{\underline{\alpha}_i}\right)e^{2\sigma}\Bigg\{-2+\left[d_i\left(\gamma_i-\frac{\sigma}{\underline{\alpha}_i}\right)\right]^{-1}\Bigg[\sum_{j=1}^{n}\Big(r_1d_i(w_{ij}^+)^{2\psi_{ij}}L_j^{2\rho_{ij}}$$

$$\frac{1}{r_1}d_j(w_{ji}^+)^{2(1-\psi_{ji})}L_i^{2(1-\rho_{ji})}\Big)+\sum_{k=0}^{K}\sum_{j=1}^{n}\Big(r_2d_i((v_{ij}^{(k)})^+)^{2\zeta_{ij}}L_j^{2\eta_{ij}}(1-q)^{-1}e^{2\sigma\tau_{ij}^{(k)}(t)}$$

$$+\frac{1}{r_2}d_j((v_{ji}^{(k)})^+)^{2(1-\zeta_{ji})}L_i^{2(1-\eta_{ji})}\Big)\Bigg]\Bigg\}z_i^2(t)$$

$$=\max_{1\leq i\leq n}\left\{d_i\left(\gamma_i-\frac{\sigma}{\underline{\alpha}_i}\right)(-2+\Delta)\right\}\sum_{i=1}^{n}z_i^2(t)e^{2\sigma t}=-B\sum_{i=1}^{n}z_i^2(t)e^{2\sigma t}<0 \quad .(15)$$

This implies that $V(z(t))\leq V(z(0))$ for t>0, and

$$V(z(0))=\sum_{i=1}^{n}d_i\Bigg\{2\int_0^{z_i(0)}\frac{s}{\alpha_i(s)}ds+\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}$$

$$L_j^{2(1-\eta_{ij})}\int_{-\tau_{ij}^{(k)}(0)}^{0}z_j^2(s)e^{2\sigma s}ds\Bigg\}$$

$$\leq\sum_{i=1}^{n}d_i\Bigg\{\frac{1}{\underline{\alpha}_i}z_i^2(0)+\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}$$

$$\sup_{s\in[-\tau,0]}\{z_j^2(s)\}\int_{-\tau_{ij}^{(k)}(0)}^{0}e^{2\sigma s}ds\Bigg\}$$

$$\leq\max_{1\leq i\leq n}\Bigg\{\frac{d_i}{\underline{\alpha}_i}\Bigg(1+\frac{\underline{\alpha}_i}{2\sigma}(1-e^{-2\sigma\tau})$$

$$\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{1}{r_2}((v_{ij}^{(k)})^+)^{2(1-\zeta_{ij})}L_j^{2(1-\eta_{ij})}\Bigg)\Bigg\}\sum_{i=1}^{n}\sup_{s\in[-\tau,0]}(\phi_i(s)-x_i^*)^2$$

By (14), we can easily obtain

$$V(z(0))\geq e^{2\sigma}\sum_{i=1}^{n}2d_i\int_0^{z_i(t)}\frac{s}{\alpha_i(s)}ds\geq\min_{1\leq i\leq n}\left(\frac{d_i}{\overline{\alpha}_i}\right)e^{2\sigma}\sum_{i=1}^{n}z_i^2(t) \quad (16)$$

Hence, we have the following results

$$\sum_{i=1}^{n}(x_i(t)-x_i^*)^2\leq\eta e^{-2\sigma}\sum_{i=1}^{n}\sup_{s\in[-\tau,0]}(\phi_i(s)-x_i^*)^2,t>0 \quad (17)$$

Now, by a standard Lyapunov-type theorem for functional differential equation, the trivial solution of (8) with (9) is global exponential stable for Δ<2. Therefore, x^* is global exponential stable for system (5). This completes the proof of Theorem 1.

Remark 3. In [19][26][27], the author presented a sufficient condition for global exponential stability of (2) and (4). However, compared with Theorem 1 above the previous result has an obvious drawback: it neglects the signs of the connecting weights, and thus, does not distinguish the differences between excitatory ($w_{ij}>0$) and inhibitory ($w_{ij}<0$) connections [25].

If we set K =0, in Theorem 1, then the following corollary is immediate.

Corollary 1. *Consider the delayed system* (5) *and assume that conditions* (H_1)-(H_4) *are satisfied. If there exist* ψ_{ij}, ρ_{ij}, ζ_{ij}, η_{ij} ∈ [0, 1], *and positive constants* r_1, r_2, d_i, $i, j = 1,2, \dots, n$, *and* $\sigma>0$, *and the following condition holds:*

$$\Delta=\max_{1\leq i\leq n}\Bigg\{\left[d_i\left(\gamma_i-\frac{\sigma}{\underline{\alpha}_i}\right)\right]^{-1}\Bigg[\sum_{j=1}^{n}\Big(r_1d_i(w_{ij}^+)^{2\psi_{ij}}L_j^{2\rho_{ij}}+\frac{1}{r_1}d_j$$

$$(w_{ji}^+)^{2(1-\psi_{ji})}L_i^{2(1-\rho_{ji})}\Big)+\sum_{j=1}^{n}\Big(r_2d_i((v_{ij})^+)^{2\zeta_{ij}}L_j^{2\eta_{ij}}(1-q)^{-1}$$

$$e^{2\sigma\tau_{ij}^{(k)}(t)}+\frac{1}{r_2}d_j((v_{ji})^+)^{2(1-\zeta_{ji})}L_i^{2(1-\eta_{ji})}\Big)\Bigg]\Bigg\}<2 \quad (18)$$

then, for every input J, the equilibrium x^* *for system* (5) *is global exponential stable.*

If we set $w_{ij} = 0$, $i, j = 1,2, \ldots, n$, in Theorem 1, then the following corollary is immediate.

Corollary 2. *Consider the delayed system* (2) *and* (5) *with pure delays and assume that conditions* (H_1)-(H_4) *are satisfied. If there exist* $\psi_{ij}, \rho_{ij}, \zeta_{ij}, \eta_{ij} \in [0, 1]$, *and positive constants* r_1, r_2, d_i, $i, j = 1,2, \ldots, n$, *and* $\sigma > 0$, *and the following condition holds:*

$$\Delta = \max_{1\le i\le n}\left\{\left[d_i\left(\gamma_i - \frac{\sigma}{\underline{\alpha}_i}\right)\right]^{-1}\left[\sum_{k=0}^{K}\sum_{j=1}^{n}\left(r_2 d_i((v_{ij}^{(k)})^+)^{2\zeta_{ij}} L_j^{2\eta_{ij}} (1-q)^{-1} e^{2\sigma\tau_{ij}^{(k)}(t)} + \frac{1}{r_2} d_j((v_{ji}^{(k)})^+)^{2(1-\zeta_{ji})} L_i^{2(1-\eta_{ji})}\right)\right]\right\} < 2 \tag{19}$$

then, for every input J, the equilibrium x for system* (2) *and* (5) *is global exponential stable.*

When we set $a_i(x) = 1$, the model becomes the following neural networks with multiple time-varying delays:

$$\dot{x}_i(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) + \sum_{k=0}^{K}\sum_{j=1}^{n} v_{ij}^{(k)} f_j(x_j(t-\tau_{ij}^{(k)}(t))) + J_i \tag{20}$$

When we set $a_i(x) = 1$, $b_i(x) = \gamma_i x$ in system (5), where $\gamma_i > 0$, $i = 1,2, \ldots, n$, the model becomes the following cellular neural networks with multiple time-varying delays:

$$\dot{x}_i(t) = -\gamma_i x_i(t) + \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) + \sum_{k=0}^{K}\sum_{j=1}^{n} v_{ij}^{(k)} f_j(x_j(x_j(t-\tau_{ij}^{(k)}(t)))) + J_i \tag{21}$$

Hence, by virtue of Theorem 1, we can immediately obtain the following result:

Corollary 3. *Consider the delayed system* (20) *and* (21) *and assume that conditions* (H_1)-(H_4) *are satisfied. If there exist* $\psi_{ij}, \rho_{ij}, \zeta_{ij}, \eta_{ij} \in [0, 1]$, *and positive constants* r_1, r_2, d_i, $i, j = 1,2, \ldots, n$, *and* $\sigma > 0$, *and the following condition holds:*

$$\Delta = \max_{1\le i\le n}\left\{[d_i(\gamma_i - \sigma)]^{-1}\left[\sum_{j=1}^{n}\left(r_1 d_i (w_{ij}^+)^{2\psi_{ij}} L_j^{2\rho_{ij}} + \frac{1}{r_1} d_j (w_{ji}^+)^{2(1-\psi_{ji})} L_i^{2(1-\rho_{ji})}\right) + \sum_{k=0}^{K}\sum_{j=1}^{n}\left(r_2 d_i((v_{ij}^{(k)})^+)^{2\zeta_{ij}} L_j^{2\eta_{ij}} (1-q)^{-1} e^{2\sigma\tau_{ij}^{(k)}(t)} + \frac{1}{r_2} d_j((v_{ji}^{(k)})^+)^{2(1-\zeta_{ji})} L_i^{2(1-\eta_{ji})}\right)\right]\right\} < 2 \tag{22}$$

then, for every input J, the equilibrium x for system* (20) *and* (21) *is global exponential stable.*

IV. REMARKS AND AN EXAMPLE

In this Section, we will give an example to illustrate that the conditions given in this paper are less conservative and restrictive as than previously known results.

Example1 Consider the following system:

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = -\begin{pmatrix}4+\sin x_1 & 0\\ 0 & 2+\cos x_2\end{pmatrix}\left[\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} - \begin{pmatrix}-1 & 0.25\\ 0.5 & -0.85\end{pmatrix}\begin{pmatrix}\tanh(x_1(t-\tau_1))\\ \tanh(2x_2(t-\tau_2))\end{pmatrix} + \begin{pmatrix}J_1\\ J_2\end{pmatrix}\right] \tag{23}$$

In this example, $L_1 = 1, L_2 = 2, \gamma_1 = 2, \gamma_2 = 2, \overline{\alpha}_1 = 5, \underline{\alpha}_1 = 3$, $\overline{\alpha}_2 = 3, \underline{\alpha}_2 = 1$. By virtue of condition in [5], it is easy to compute that

$$\theta_1 = \frac{\underline{\alpha}_1}{\overline{\alpha}_1}\gamma_1 - L_1\sum_{k=0}^{1}\sum_{j=1}^{2}\left|v_{j1}^{(k)}\right| = -0.3 \quad , \quad \text{and}$$

$$\theta_2 = \frac{\underline{\alpha}_2}{\overline{\alpha}_2}\gamma_2 - L_2\sum_{k=0}^{1}\sum_{j=1}^{2}\left|v_{j2}^{(k)}\right| = -1.533$$

Therefore, we have $\theta = \min\{\theta_1, \theta_2\} = -1.533 < 0$, hence, the conditions in [5] are not satisfied.

We can easily calculate

$$\theta_1 = \frac{1}{\gamma_1}\sum_{k=0}^{1}\sum_{j=1}^{2}(L_j\left|v_{1j}^{(k)}\right| + L_1\left|v_{j1}^{(k)}\right|) = 1.5 \quad , \quad \text{and}$$

$$\theta_2 = \frac{1}{\gamma_2}\sum_{k=0}^{1}\sum_{j=1}^{2}(L_j\left|v_{2j}^{(k)}\right| + L_1\left|v_{j2}^{(k)}\right|) = 2.2$$

Therefore, we have $\theta = \max\{\theta_1, \theta_2\} = 2.2 > 2$, hence, the conditions of Theorem 1 in [20] are not satisfied.

We can easily calculate

$$\theta_1 = \underline{\alpha}_1\gamma_1 - L_1\sum_{k=0}^{1}\sum_{j=1}^{2}\underline{\alpha}_j\left|v_{j1}^{(k)}\right| = 2.5 \quad , \quad \text{and}$$

$$\theta_2 = \underline{\alpha}_2\gamma_2 - L_2\sum_{k=0}^{1}\sum_{j=1}^{2}\underline{\alpha}_j\left|v_{j2}^{(k)}\right| = -1.2$$

Therefore, we have $\theta = \min\{\theta_1, \theta_2\} = -1.2 < 0$, hence, the conditions in [28] are not satisfied.

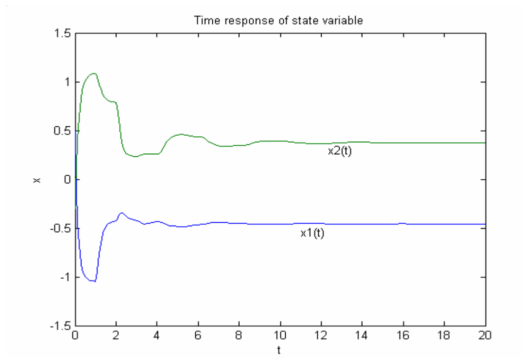

Figure 1. System (23) with initial state $x(0) = [0.5, -0.3]^T$.

However, if $\psi_{ij} = \rho_{ij} = \zeta_{ij} = \eta_{ij} = 1/2$, $w_{ij}^+ = \max\{w_{ij}, 0\}$, $d_1 = d_2 = r_1 = r_2 = 1$, $\sigma = 0.4624$ in Corollary 3, then we can calculate $\Delta_1 = 1.992$, $\Delta_2 = 1.145$. Therefore, we have $\Delta = \max\{\Delta_1, \Delta_2\} = 1.992 < 2$, hence, the conditions given in Corollary 3 are hold. For numerical simulation, System (23) with initial states $x(0) = [0.5, -0.3]^T$, inputs $J_1 = 1.5$, $J_2 = -1.5$, and delay parameters $\tau_1 = 1$ and $\tau_2 = 2$ are considered. The existence of an unique equilibrium point, $x^* = [-0.4568, 0.3732]^T$, and the global asymptotic stability of system (23) are guaranteed by means of computer simulations. Fig.1 depicted the time response of the state variables $x_1(t)$ and $x_2(t)$ for this case. Moreover, the results are independent of the delay parameters.

V. CONCLUSIONS

In this paper, the criteria for the global exponential stability of a class of Cohen-Grossberg neural networks with multiple time-varying delays have been derived. The results have been shown to be the generalization and improvement of existing results reported recently in the literature for the cases with delays. Analyses have also shown that the neuronal input-output activation function and the self-signal function only need to satisfy, respectively, conditions (H_1), (H_2), (H_3) and (H_4) given in this paper, but do not need to be continuous, differentiable, monotonically increasing and bounded, as usually required by other analyzing methods. Novel stability conditions are stated in simple algebraic forms so that their verification and applications are straightforward and convenient. The criteria are independent of the magnitudes of the delays. We only require $a_i(x)$ be continuous and positive. At the same time, we thinks over the signs of the entries of the connection matrices. Thus the differences between excitatory and inhibitory effects are considered. An example is given to demonstrate that our criteria are less conservative and restrictive than previously known results.

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