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Simultaneous state and dead-zone parameter estimation using high-gain observers

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***Abstract*— In this paper we propose a new methodology of estimation of nonlinear systems subject to unknown hard-input nonlinearities. Using the fact that the dead-zone input nonlinearity can be modelled as a line input function with a bounded disturbance term then, by appropriate selection of the system input, we show that the simultaneous estimation of the dead-zone parameters with the unmeasured system states is possible under some mild conditions. The proposed observation strategy can be applied to both symmetric and non-symmetric dead-zone input with possible extension to time-varying dead-zone characteristics. Several cases are considered with illustrative examples in order to show the effectiveness of high-gain observers in identification of the dead-zone parameters.**

***Index Terms*— Dead-zone inputs; Nonlinear Observer Design; System Theory; Uncertain Systems; Identification; Mechatronics.**

I. Introduction

SYSTEMS with hard input nonlinearities are ubiquitous in various electrical and mechatronics devices like ultrasonic motors, valves, smart actuators and sensors. Hysteresis and dead-zone are some of the common hard nonlinearities that are usually encountered in newly smart actuators giving rise to inherent difficulties in control and observation due to the presence of singularities in the input channels. As it has been reported in many references, the dead-zone input nonlinearity is a non-differentiable function that characterizes a non-sensitivity for small excitation inputs. Therefore, the presence of such nonlinearity in feedback or observation systems may cause severe deterioration of the system or the observer performances. To cope with this inherent problem, adaptive control techniques may be applied to design controllers. The study of adaptive control for systems subject to dead-zone actuators was initiated in [1] [2], [3], [4], and the extensions may referred to [5], [6], [7]. Fuzzy-logic and neural network approaches were further explored to give different looks [8], [9], [10]. Robust stabilization of unknown sandwich systems with known uncertainties bounds was discussed in the references [11], [12].

Numerous existing adaptive procedures use inverse dead-zone nonlinearities to handle the effects of non-smooth nonlinearities [3], [13]. However, the available techniques that dealt with computation of dead-zone inverses lead, in general, to some static error that cannot be completely eliminated. As an alternative, a robust adaptive control scheme was developed in [14] without constructing the dead-zone inverse, where the dead-zone input nonlinearity is modeled as a combination of a line and a disturbance-like term. However, this scheme requires symmetric dead-zones inputs. Extension of the results to non-symmetric dead-zone-input systems is given in [7]. Based on this result, we focus on identification of dead-zone inputs associated to nonlinear systems written is uniformly-observable canonical form. We disclose that it is possible to identify the parameters of non-symmetric dead-zone inputs when these non-affine input nonlinearities are associated to a class of bounded-states nonlinear systems. By using high-gain observers we show that it is possible to reconstruct both the dead-zone parameters and the true inverse of the dead-zone characteristic. The second objective of this paper is to show that it is possible to reconstruct asymptotically the system states by switching between two nonlinear-observer schemes. To avoid the peaking of the estimates and excessive high-gain design, a new adaptive nonlinear observer with non-symmetric saturation function is analyzed.

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II. Observation of nonlinear systems subject to partially known dead-zone inputs

Since in most practical situations the dead-zone parameters are poorly or partially known then, it is not possible to design an observer as a copy of the system dynamics with extra output correction. Therefore, systems having dead-zone input nonlinearities are seen as unknown-input systems where the unknown input is not necessarily bounded. Due to the fact that this hard-input nonlinearity is not always symmetric, not accessible for measurements, and its slopes may be constants or time-varying then, it is quite difficult to represent the nonlinearity as a smooth parameterized function. For this reason, we propose a new observation techniques to identify the unknown dead-zone parameters by considering positive and negative excitation inputs. Here, we distinguish two independent cases.

A. Unknown width dead-zone inputs

Consider the nonlinear system

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) + f_1(x_1(t)),\\
\dot{x}_2(t) &= x_3(t) + f_2(x_1(t), x_2(t)),\\
&\vdots\\
\dot{x}_i(t) &= x_{i+1}(t) + f_i(x_1(t), \cdots, x_i(t)),\\
&\vdots\\
\dot{x}_n(t) &= f_n(x_1(t), x_2(t), \cdots, x_n(t)) + \mathscr{D}(u),\\
y(t) &= x_1(t),
\end{aligned} \tag{1}$$

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where $x(t) \in \mathbb{R}^n$ is the state vector, $f(x)$ is a smooth nonlinearity with $f(0) = 0$, $y(t) \in \mathbb{R}$ is the only measured output and $\mathscr{D}(u) \in \mathbb{R}$ is the dead-zone input nonlinearity represented in Fig. 1. This nonlinear input characteristic may be symmetric or non-symmetric as represented in Fig. 1. In matrix notation, the aforementioned system is given by:

$$\begin{aligned} \dot{x} &= A\,x + f(x) + B\,\mathscr{D}(u), \\ y &= C\,x \end{aligned} \tag{2}$$

where

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$B \triangleq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n,\ C \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}' \in \mathbb{R}^n, \tag{3}$$

$$f(x(t)) \triangleq \begin{bmatrix} f_1(x_1(t)) \\ f_2(x_1(t), x_2(t)) \\ \vdots \\ f_n(x_1(t), x_2(t), \cdots, x_n(t)) \end{bmatrix} \in \mathbb{R}^n.$$

The non-symmetric dead-zone represented in Fig. 1 is defined as

$$\mathscr{D}(u) \triangleq \begin{cases} \theta_1(u + d_1) & \text{if, } u \leq -d_1, \\ \theta_2(u - d_2) & \text{if, } u \geq d_2, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

When the slopes of the dead-zone characteristic are the same and the breakpoints are symmetric then the dead-zone input nonlinearity becomes symmetric and consequently,

$$\mathscr{D}_s(u) \triangleq \begin{cases} \theta(u + d) & \text{if, } u \leq -d_1, \\ \theta(u - d) & \text{if, } u \geq d_2, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

In this subsection, we show how to estimate asymptotically the unknown breakpoints of the non-symmetric dead-zone under the assumption that the slopes θ_1 and θ_2 are known. This result is summarized in the following statement.

Theorem 1: Consider the nonlinear system (2) subject to the non-symmetric dead-zone input (4). Assume that for a given bounded input $u(t) > d_2$ and some initial condition $x_0 \in \mathbb{R}^n$ the system states $(x_i)_{1\leq i\leq n}$ are globally bounded such that all the trajectories of system (2) are well-defined and contained in Ω defined as

$$\Omega^+ \triangleq \left\{ x \in \mathscr{M} \subset \mathbb{R}^n \middle| -\underline{\omega}_i^+ \leq x_i(t) \leq +\bar{\omega}_i^+,\ 1 \leq i \leq n \right\}, \tag{6}$$

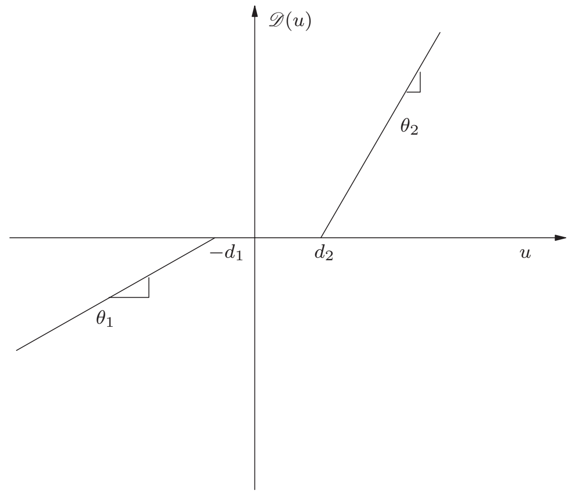

Fig. 1. The non-symmetric dead-zone

where $(\underline{\omega}_i^+)_{1\leq i\leq n}$, $(\bar{\omega}_i^+)_{1\leq i\leq n}$ are positive constants. Define the nonlinear observer as

$$\begin{aligned} \dot{\hat{x}} &= A\,\hat{x} + f(\text{sat}^+(\hat{x}), u) + B\,(\theta_2 u + \hat{d}_2) \\ &\qquad + \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_n \end{bmatrix} (y - C\,\hat{x}), \\ \dot{\hat{d}}_2 &= \ell_{n+1}(y - C\hat{x}), \\ \dot{\gamma} &= \begin{cases} \gamma_0(y - C\hat{x})^2 & \text{If, } y - C\hat{x} \neq 0,\ \gamma_0 > 0,\ \gamma(0) > 1, \\ 0 & \text{If, } y - C\hat{x} = 0, \end{cases} \end{aligned} \tag{7}$$

where the observer gain is issued from the solution of the following Algebraic Riccati Equation:

$$\begin{aligned} &\tilde{P}\tilde{A}' + \tilde{A}\,\tilde{P} - \tilde{P}\tilde{C}'\tilde{C}\tilde{P} + \tilde{Q} = 0,\ \ \tilde{C} = [C\ \ 0], \\ &\tilde{D}\tilde{Q} + \tilde{Q}\tilde{D} > 0,\ \tilde{D} = \text{diag}(1, 2, \cdots, n+1), \tilde{Q} = \tilde{Q}', \\ &P(\gamma)\tilde{A}' + \tilde{A}\,P(\gamma) - P(\gamma)\tilde{C}'\tilde{C}P(\gamma) + Q(\gamma) = 0, \\ &Q(\gamma) = \gamma^2\,D(\gamma)\tilde{Q}D(\gamma),\ \ D(\gamma) = \text{diag}(1, \gamma, \gamma^2, \cdots, \gamma^n), \\ &L = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_n \\ \ell_{n+1} \end{bmatrix} = P(\gamma)\tilde{C}'. \end{aligned} \tag{8}$$

where $\tilde{A}$ and $\text{sat}(x)$ are defined as

$$\tilde{A} \triangleq \begin{bmatrix} A & B \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix},\ \text{sat}^+(x) \triangleq \begin{bmatrix} \text{sat}_1^+(x_1) \\ \text{sat}_2^+(x_2) \\ \vdots \\ \text{sat}_n^+(x_n) \end{bmatrix},$$

$$\text{sat}_i^+(w) \triangleq \begin{cases} w & \text{if, } -\underline{\omega}_i^+ \leq w \leq \bar{\omega}_i^+ \\ \bar{\omega}_i^+ & \text{if, } w \geq \bar{\omega}_i^+, \\ -\underline{\omega}_i^+ & \text{if, } w \leq \underline{\omega}_i^+. \end{cases} \tag{9}$$

Then,

$$\lim_{t\to\infty}(x_i-\hat{x}_i)=0,\ 1\le i\le n,\quad \lim_{t\to\infty}\frac{\hat{d}_2}{\theta_2}=-d_2. \quad (10)$$

Furthermore, if for given input $u\le -d_1$, the system states are contained in the set

$$\Omega^- \triangleq \left\{x\in\mathscr{M}\subset \mathbb{R}^n \middle| -\underline{\omega}_i^- \le x_i(t)\le +\bar{\omega}_i^-,\ 1\le i\le n\right\} \quad (11)$$

then, by using the same observer gain, the following nonlinear saturated-state observer

$$\begin{aligned}\dot{\hat{x}} &= A\,\hat{x}+f(\mathrm{sat}^-(\hat{x}),u)+B\,(\theta_1\,u+\hat{d}_1)\\ &\qquad + \begin{bmatrix}\ell_1\\ \vdots\\ \ell_n\end{bmatrix}(y-C\,\hat{x}),\\ \dot{\hat{d}}_1 &= \ell_{n+1}(y-C\hat{x}),\\ \dot{\gamma} &= \begin{cases}\gamma_0(y-C\hat{x})^2 & \text{If, } y-C\hat{x}\neq 0,\ \gamma_0>0,\ \gamma(0)>1,\\ 0 & \text{If, } y-C\hat{x}=0.\end{cases}\end{aligned} \quad (12)$$

with

$$\mathrm{sat}^-(x)\triangleq \begin{bmatrix}\mathrm{sat}_1^-(x_1)\\ \mathrm{sat}_2^-(x_2)\\ \vdots\\ \mathrm{sat}_n^-(x_n)\end{bmatrix},$$
$$\mathrm{sat}_i^-(w)\triangleq\begin{cases} w & \text{if, } -\underline{\omega}_i^-\le w\le \bar{\omega}_i^-\\ \bar{\omega}_i^- & \text{if, } w\ge \bar{\omega}_i^-,\\ -\underline{\omega}_i^- & \text{if, } w\le \underline{\omega}_i^-.\end{cases} \quad (13)$$

guarantees the following convergence limits

$$\lim_{t\to\infty}(x_i-\hat{x}_i)=0,\ 1\le i\le n,\quad \lim_{t\to\infty}\frac{\hat{d}_1}{\theta_1}=d_1. \quad (14)$$

B. Unknown slope dead-zone inputs

In this subsection we show how to reproduce the dead-zone slopes when the breakpoints of the dead-zone characteristic are known. Based upon this crucial information, if we define $u_\Delta = u+d_1$, $u^\Delta = u-d_2$ then, for $u\ge d_2$ system (1) takes the form

$$\begin{bmatrix}\dot{x}\\ \dot{\theta}_2\end{bmatrix}=\begin{bmatrix}A & B\,u^\Delta\\ \mathbf{0}_{1\times n} & 0\end{bmatrix}\begin{bmatrix}x\\ \theta_2\end{bmatrix}+\begin{bmatrix}f(x)\\ 0\end{bmatrix} \quad (15)$$

Alternatively, for $u\le -d_1$, system (1) can be represented as

$$\begin{bmatrix}\dot{x}\\ \dot{\theta}_1\end{bmatrix}=\begin{bmatrix}A & B\,u_\Delta\\ \mathbf{0}_{1\times n} & 0\end{bmatrix}\begin{bmatrix}x\\ \theta_1\end{bmatrix}+\begin{bmatrix}f(x)\\ 0\end{bmatrix} \quad (16)$$

The idea of asymptotic estimating of the values of the slopes consists in designing two independent observers with positive and negative control inputs. In order to organize the analysis of the converging observers, we would rather present the following important result.

Lemma 1: Let

$$\begin{aligned}&\boldsymbol{A}(u)=\begin{bmatrix}A & B\,u\\ \mathbf{0}_{1\times n} & 0\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}C & 0\end{bmatrix},\\ &\boldsymbol{Q}(\gamma,u)=\gamma^2\boldsymbol{D}^{-1}(\gamma,u)\boldsymbol{Q}_1\boldsymbol{D}^{-1}(\gamma,u),\\ &\boldsymbol{D}(\gamma,u)=\mathrm{diag}\left(1,\frac{1}{\gamma},\frac{1}{\gamma^2},\cdots,\frac{1}{\gamma^{n-1}},\frac{u}{\gamma^n}\right),\\ &\boldsymbol{\Gamma}=\mathrm{diag}\left(1,2,3,\cdots,n+1\right),\\ &\boldsymbol{\Gamma}\,\boldsymbol{Q}_1+\boldsymbol{Q}_1\,\boldsymbol{\Gamma}>0,\quad \boldsymbol{Q}_1^{'}=\boldsymbol{Q}_1\in\mathbb{R}^{(n+1)\times(n+1)},\end{aligned} \quad (17)$$

where $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times 1}$ are defined as in (3). Then, the solution of the following input-dependent Algebraic Riccati Equation

$$\begin{aligned}\boldsymbol{P}(\gamma,u)\boldsymbol{A}'(u)+\boldsymbol{A}(u)\boldsymbol{P}(\gamma,u)-\boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}\boldsymbol{P}(\gamma,u)&\\ +\boldsymbol{Q}(\gamma,u)=0&\end{aligned} \quad (18)$$

has the following properties:

- $\boldsymbol{P}(\gamma,u)$ is symmetric and positive-definite $\forall\ \gamma>0$, $\forall u\in\mathbb{R}$ and admits a unique solution given by:
$$\boldsymbol{P}(\gamma,u)=\gamma\boldsymbol{D}^{-1}(\gamma,u)\tilde{\boldsymbol{P}}\boldsymbol{D}^{-1}(\gamma,u) \quad (19)$$
where $\tilde{\boldsymbol{P}}=\boldsymbol{P}(1,1)$;
- $\forall\gamma>0$, $\forall u\in\mathbb{R}$ $\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{P}^{-1}(\gamma,u)<0$, and $\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{P}(\gamma,u)>0$;
- For any lower triangular matrix $\mathscr{L}_{n\times n}\in\mathbb{R}^{n\times n}$ and for any $\gamma>1$ there exist three constants c_1, c_2 and c_3 independent of γ and u such that
$$\begin{aligned}&\left\|\begin{bmatrix}\mathscr{L}_{n\times n}^{'} & \mathbf{0}_{n\times 1}\\ \mathbf{0}_{1\times n} & 0\end{bmatrix}\boldsymbol{P}^{-1}(\gamma,u)\begin{bmatrix}\mathscr{L}_{n\times n} & \mathbf{0}_{n\times 1}\\ \mathbf{0}_{1\times n} & 0\end{bmatrix}\right\|_\infty\\ &\qquad\le\frac{c_1}{\gamma},\\ &\left\|\boldsymbol{D}(\gamma,u)\begin{bmatrix}\mathscr{L}_{n\times n} & \mathbf{0}_{n\times 1}\\ \mathbf{0}_{1\times n} & 0\end{bmatrix}\boldsymbol{D}^{-1}(\gamma,u)\right\|_\infty\\ &\qquad\le c_2+\frac{c_3}{\gamma}.\end{aligned} \quad (20)$$

Proof: i) By denoting $\boldsymbol{A}_1=\boldsymbol{A}(u)\Big|_{u=1}$, $\boldsymbol{Q}_1=\boldsymbol{Q}(\gamma,u)\Big|_{\substack{u=1\\ \gamma=1}}$ then, the matrix $\tilde{\boldsymbol{P}}$ verifies the following algebraic Riccati equation:

$$\tilde{\boldsymbol{P}}\boldsymbol{A}_1'+\boldsymbol{A}_1\tilde{\boldsymbol{P}}-\tilde{\boldsymbol{P}}\boldsymbol{C}'\boldsymbol{C}\tilde{\boldsymbol{P}}+\boldsymbol{Q}_1=0 \quad (21)$$

Pre-and post multiplying by the matrix $\gamma\boldsymbol{D}^{-1}(\gamma,u)$, we have

$$\begin{aligned}\gamma^2\boldsymbol{D}^{-1}(\gamma,u)\Big(\tilde{\boldsymbol{P}}\boldsymbol{A}_1'+\boldsymbol{A}_1\tilde{\boldsymbol{P}}-\tilde{\boldsymbol{P}}\boldsymbol{C}'\boldsymbol{C}\tilde{\boldsymbol{P}}+\boldsymbol{Q}_1\Big)\boldsymbol{D}^{-1}(\gamma,u)&\\ =0&\end{aligned} \quad (22)$$

Using the following properties

$$\begin{aligned}&\boldsymbol{A}_1\boldsymbol{D}^{-1}(\gamma,u)=\frac{1}{\gamma}\boldsymbol{D}^{-1}(\gamma,u)\boldsymbol{A}'(u),\\ &\boldsymbol{D}^{-1}(\gamma,u)\boldsymbol{C}'=\boldsymbol{C}'\end{aligned} \quad (23)$$

then, the last ARE can be rewritten as

$$\Big[\gamma \boldsymbol{D}^{-1}(\gamma,u)\tilde{\boldsymbol{P}}\boldsymbol{D}^{-1}(\gamma,u)\Big]\boldsymbol{A}'(u) + \boldsymbol{A}(u)\Big[\gamma \boldsymbol{D}^{-1}(\gamma,u)\tilde{\boldsymbol{P}}\boldsymbol{D}^{-1}(\gamma,u)\Big] - \Big[\gamma \boldsymbol{D}^{-1}(\gamma,u)\tilde{\boldsymbol{P}}\boldsymbol{D}^{-1}(\gamma,u)\Big]\boldsymbol{C}'\boldsymbol{C}\Big[\gamma \boldsymbol{D}^{-1}(\gamma,u)\tilde{\boldsymbol{P}}\boldsymbol{D}^{-1}(\gamma,u)\Big] + \gamma^2\boldsymbol{D}^{-1}(\gamma,u)\boldsymbol{Q}_1\boldsymbol{D}^{-1}(\gamma,u) = 0. \tag{24}$$

By comparison of (24) with (18), we conclude that $\boldsymbol{P}(\gamma,u) = \gamma\boldsymbol{D}^{-1}(\gamma,u)\tilde{\boldsymbol{P}}\boldsymbol{D}^{-1}(\gamma,u)$ is a solution of (18). Since $\boldsymbol{Q}_1$ is symmetric and positive definite then $\boldsymbol{Q}(\gamma,u) = \gamma^2\boldsymbol{D}^{-1}(\gamma,u)\boldsymbol{Q}_1\boldsymbol{D}^{-1}(\gamma,u) > 0\ \forall\gamma, u \neq 0$ which implies the positivity of the solution $\boldsymbol{P}(\gamma,u)$ for any non-null γ and u.

ii) The monotonicity-increasing property of the matrix $\boldsymbol{P}(\gamma,u)$ depends on the matrix $\boldsymbol{Q}(\gamma,u)$. This can be seen by differentiating the ARE (18) with respect to γ which yields

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{P}(\gamma,u)\boldsymbol{A}'(u) + \boldsymbol{A}(u)\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{P}(\gamma,u) - \frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}\boldsymbol{P}(\gamma,u) - \boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{P}(\gamma,u) = -\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{Q}(\gamma,u), \tag{25}$$

or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{P}(\gamma,u)\Big(\boldsymbol{A}(u) - \boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}\Big)' + \Big(\boldsymbol{A}(u) - \boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}\Big)\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{P}(\gamma,u) = -\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{Q}(\gamma,u). \tag{26}$$

The matrix $\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{P}(\gamma,u) > 0$ if and only if the time-varying matrix $\boldsymbol{A}(u) - \boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}$ is Hurwitz and $\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{Q}(\gamma,u) > 0$. By decomposition of the matrix derivative as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{Q}(\gamma,u) &= \frac{\mathrm{d}}{\mathrm{d}\gamma}\Big[\gamma\boldsymbol{D}^{-1}(\gamma,u)\Big]\boldsymbol{Q}_1\Big[\gamma\boldsymbol{D}^{-1}(\gamma,u)\Big] + \Big[\gamma\boldsymbol{D}^{-1}(\gamma,u)\Big]\boldsymbol{Q}_1\frac{\mathrm{d}}{\mathrm{d}\gamma}\Big[\gamma\boldsymbol{D}^{-1}(\gamma,u)\Big] \\ &= \boldsymbol{D}^{-1}(\gamma,u)\Big(\boldsymbol{\Gamma}\boldsymbol{Q}_1 + \boldsymbol{Q}_1\boldsymbol{\Gamma}\Big)\boldsymbol{D}^{-1}(\gamma,u).\end{aligned} \tag{27}$$

Since the matrix $\boldsymbol{\Gamma}\boldsymbol{Q}_1 + \boldsymbol{Q}_1\boldsymbol{\Gamma} > 0$ then, we conclude that $\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{Q}(\gamma,u) > 0$ for all $\gamma > 0$. On the other hand, $\boldsymbol{P}(\gamma,u)$ verifies

$$\boldsymbol{P}(\gamma,u)\Big(\boldsymbol{A}(u) - \boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}\Big)' + \Big(\boldsymbol{A}(u) - \boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}\Big)\boldsymbol{P}(\gamma,u) = -\boldsymbol{\Pi}(\gamma,u) \tag{28}$$

where $\boldsymbol{\Pi}(\gamma,u) = -\boldsymbol{Q}(\gamma,u) - \boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}\boldsymbol{P}(\gamma,u)$ then, we conclude that the time-varying matrix $\boldsymbol{A}(u) - \boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}$ is stable in the sense that the time-varying system

$$\dot{\eta} = \Big(\boldsymbol{A}(u) - \boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}\Big)\eta \tag{29}$$

is exponentially stable for all $\gamma > 0$, $\forall u$. Based on this result, and using the fact that the matrix $\boldsymbol{A}(u) - \boldsymbol{P}(\gamma,u)\boldsymbol{C}'\boldsymbol{C}$ verifies the Lyapunov equation (26) then, from the stability theorem of Lyapunov we conclude that $\frac{\mathrm{d}}{\mathrm{d}\gamma}\boldsymbol{P}(\gamma,u) > 0$.

iii) By considering the following partition of $\tilde{\boldsymbol{P}}^{-1}$

$$\tilde{\boldsymbol{P}}^{-1} = \begin{bmatrix} \tilde{\boldsymbol{P}}_{11}^{-1} & \tilde{\boldsymbol{p}}_{12} \\ \tilde{\boldsymbol{p}}_{12} & \tilde{p}_{22} \end{bmatrix} \tag{30}$$

where $\tilde{\boldsymbol{P}}_{11} \in \mathbb{R}^{n\times n}$, $\tilde{\boldsymbol{p}}_{12} \in \mathbb{R}^{n\times 1}$, $\tilde{p}_{22} \in \mathbb{R}_{>0}$ then,

$$\boldsymbol{P}^{-1}(\gamma,u) = \frac{1}{\gamma}\boldsymbol{D}(\gamma,u)\tilde{\boldsymbol{P}}^{-1}\boldsymbol{D}(\gamma,u) = \begin{bmatrix} \frac{1}{\gamma}D(\gamma)\tilde{\boldsymbol{P}}_{11}^{-1}D(\gamma) & \frac{\gamma^{n-1}}{u}D(\gamma)\tilde{\boldsymbol{p}}_{12} \\ \frac{\gamma^{n-1}}{u}D(\gamma)\tilde{\boldsymbol{p}}_{12} & \frac{\gamma^{2n-1}}{u^2}\tilde{p}_{22} \end{bmatrix} \tag{31}$$

This implies that

$$\left\| \begin{bmatrix} \mathscr{L}'_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \boldsymbol{P}^{-1}(\gamma,u) \begin{bmatrix} \mathscr{L}_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \frac{1}{\gamma}\mathscr{L}'_{n\times n}D(\gamma)\tilde{\boldsymbol{P}}^{-1}D(\gamma)\mathscr{L}_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \right\|_\infty = \left\| \frac{1}{\gamma}\mathscr{L}'_{n\times n}D(\gamma)\tilde{\boldsymbol{P}}^{-1}D(\gamma)\mathscr{L}_{n\times n} \right\|_\infty \leq \frac{c_1}{\gamma},\ \gamma > 1,\ c_1 > 0. \tag{32}$$

The second inequality of (20) is obvious due to the fact that all the entries of the matrix $\boldsymbol{D}(\gamma,u)\begin{bmatrix} \mathscr{L}_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}\boldsymbol{D}^{-1}(\gamma,u)$ are of the form $a_{i,j} + \frac{b_{i,j}}{\gamma}$ where $a_{i,j}$, $b_{i,j}$ are real constants. This ends the proof of the technical Lemma. The design of the nonlinear observer is summarized in the following statement.

Theorem 2: Consider the nonlinear system (2) subject to the non-symmetric dead-zone input (4). Assume that for a given bounded input $u(t) > d_2$ and some initial condition $x_0 \in \mathbb{R}^n$ the system states $(x_i)_{1\leq i\leq n}$ are globally bounded such that all the trajectories of system (2) are well-defined and contained in Ω defined as

$$\Omega^+ \triangleq \Big\{x \in \mathscr{M} \subset \mathbb{R}^n \big| -\underline{\omega}_i^+ \leq x_i(t) \leq +\bar{\omega}_i^+,\ 1 \leq i \leq n\Big\}, \tag{33}$$

Let $\mathrm{sat}^+(w)$, $\boldsymbol{A}(u)$, $\boldsymbol{Q}(\gamma,u)$, $\boldsymbol{C}$ be defined as in Eqs. (9), (17) and define the augmented nonlinearity vector

$\boldsymbol{f}(s_1, \cdots, s_n) = \begin{bmatrix} f_1(s_1) \\ f_2(s_1, s_2) \\ \vdots \\ f_n(s_1, s_2, \cdots, s_n) \\ 0 \end{bmatrix}$. Let $\hat{\xi}$ be the trajectory of the saturated-state observer

$$\begin{aligned}
\dot{\hat{\xi}} &= \boldsymbol{A}(u^{\Delta})\,\hat{\xi} + \boldsymbol{f}\big(\mathrm{sat}_1^+(\hat{\xi}_1), \mathrm{sat}_2^+(\hat{\xi}_2), \cdots, \mathrm{sat}_n^+(\hat{\xi}_n)\big) \\
&\qquad + \boldsymbol{P}(\gamma, u)(y - \boldsymbol{C}\hat{\xi}), \\
&\boldsymbol{P}(\gamma, u^{\Delta})\boldsymbol{A}'(u^{\Delta}) + \boldsymbol{A}(u^{\Delta})\boldsymbol{P}(\gamma, u^{\Delta}) \\
&\qquad - \boldsymbol{P}(\gamma, u^{\Delta})\boldsymbol{C}'\boldsymbol{C}\boldsymbol{P}(\gamma, u^{\Delta}) + \boldsymbol{Q}(\gamma, u^{\Delta}) = 0, \\
\dot{\gamma} &= (y - \boldsymbol{C}\hat{\xi})^2, \ \gamma(0) > 1,
\end{aligned} \tag{34}$$

then, for any initial condition $\hat{\xi}(0)$, we have

$$\lim_{t\to\infty}(\hat{\xi}_i - x_i) = 0, \ 1 \le i \le n, \ \lim_{t\to\infty}\hat{\xi}_{n+1} = \theta_2. \tag{35}$$

Proof: Let us note $\xi \triangleq \begin{bmatrix} x \\ \theta_2 \end{bmatrix}$, and let $e \triangleq \hat{\xi} - \xi$ be the observation error between system (34) and (15). For $u > d_2$, this error verifies the following state-space dynamics

$$\begin{aligned}
\dot{e} &= \Big(\boldsymbol{A}(u^{\Delta}) - \boldsymbol{P}(\gamma, u)C'C\Big)\,e \\
&\quad + \boldsymbol{f}\big(\mathrm{sat}_1^+(\hat{\xi}_1), \mathrm{sat}_2^+(\hat{\xi}_2), \cdots, \mathrm{sat}_n^+(\hat{\xi}_n)\big) \\
&\qquad - \boldsymbol{f}\big(\mathrm{sat}_1^+(\xi_1), \mathrm{sat}_2^+(\xi_2), \cdots, \mathrm{sat}_n^+(\xi_n)\big).
\end{aligned} \tag{36}$$

Using the differential Mean-Value Theorem which states that for two given vectors $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^n$ the difference $\varphi(x_1) - \varphi(x_2)$ can be rewritten as

$$\varphi(x_1) - \varphi(x_2) = \int_0^1 \frac{\partial \varphi(s)}{\partial s}\bigg|_{s = x_1 - \lambda(x_1 - x_2)} (x_1 - x_2)\mathrm{d}\lambda, \tag{37}$$

for any smooth vector $\varphi(s) \in \mathbb{R}^n$. Let $s_\lambda \triangleq \mathrm{sat}^+(\hat{\xi}) - \lambda(\mathrm{sat}^+(\hat{\xi}) - \mathrm{sat}^+(\xi))$ then, from Eqs. (36) and (37), we have

$$\begin{aligned}
\dot{e} &= \Big(\boldsymbol{A}(u^{\Delta}) - \boldsymbol{P}(\gamma, u)C'C\Big)\,e \\
&\quad + \int_0^1 \frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s = s_\lambda} (\mathrm{sat}^+(\hat{\xi}) - \mathrm{sat}^+(\xi))\,\mathrm{d}\lambda.
\end{aligned} \tag{38}$$

Let us associate the following time-varying Lyapunov function $V(e) \triangleq e'\boldsymbol{P}^{-1}(\gamma, u)\,e$ to the dynamics (38) with $u > d_2$. This gives

$$\begin{aligned}
\dot{V}(e) &= \dot{e}'\boldsymbol{P}^{-1}(\gamma, u)\,e + e'\boldsymbol{P}^{-1}(\gamma, u)\dot{e} + e'\frac{\mathrm{d}}{\mathrm{d}\,t}\boldsymbol{P}^{-1}(\gamma, u)\,e \\
&= \dot{e}'\boldsymbol{P}^{-1}(\gamma, u)\,e + e'\boldsymbol{P}^{-1}(\gamma, u)\dot{e} + \dot{\gamma}\,e'\frac{\mathrm{d}}{\mathrm{d}\,\gamma}\boldsymbol{P}^{-1}(\gamma, u)\,e
\end{aligned} \tag{39}$$

Using the result of Lemma 1, we have $e'\frac{\mathrm{d}}{\mathrm{d}\,\gamma}\boldsymbol{P}^{-1}(\gamma, u)\,e < 0$ for $e \neq 0$; therefore,

$$\begin{aligned}
\dot{V}(e) &\le \dot{e}'\boldsymbol{P}^{-1}(\gamma, u)\,e + e'\boldsymbol{P}^{-1}(\gamma, u)\,\dot{e} \\
&= e'\Big(\boldsymbol{A}(u^{\Delta}) - \boldsymbol{P}(\gamma, u)C'C\Big)'\boldsymbol{P}^{-1}(\gamma, u)\,e \\
&\quad + e'\boldsymbol{P}^{-1}(\gamma, u)\Big(\boldsymbol{A}(u^{\Delta}) - \boldsymbol{P}(\gamma, u)C'C\Big)\,e \\
&\quad + 2e'\boldsymbol{P}^{-1}(\gamma, u)\int_0^1 \frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s = s_\lambda} \\
&\qquad \times (\mathrm{sat}^+(\hat{\xi}) - \mathrm{sat}^+(\xi))\,\mathrm{d}\lambda.
\end{aligned} \tag{40}$$

The matrix $\boldsymbol{P}^{-1}(\gamma, u)$ verifies the following ARE

$$\begin{aligned}
&\boldsymbol{A}'(u^{\Delta})\boldsymbol{P}^{-1}(\gamma, u^{\Delta}) + \boldsymbol{P}^{-1}(\gamma, u^{\Delta})\boldsymbol{A}(u^{\Delta}) \\
&\qquad = \boldsymbol{C}'\boldsymbol{C} - \boldsymbol{P}^{-1}(\gamma, u^{\Delta})\boldsymbol{Q}(\gamma, u)\boldsymbol{P}^{-1}(\gamma, u^{\Delta}).
\end{aligned} \tag{41}$$

This gives

$$\begin{aligned}
\dot{V}(e) &\le e'\bigg(-\boldsymbol{C}'\boldsymbol{C} - \boldsymbol{P}^{-1}(\gamma, u^{\Delta})\boldsymbol{Q}(\gamma, u)\boldsymbol{P}^{-1}(\gamma, u^{\Delta})\bigg)e \\
&\quad + 2e'\boldsymbol{P}^{-1}(\gamma, u)\int_0^1 \frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s = s_\lambda} \\
&\qquad \times (\mathrm{sat}^+(\hat{\xi}) - \mathrm{sat}^+(\xi))\,\mathrm{d}\lambda.
\end{aligned} \tag{42}$$

Using the following inequality $2a'b \le a'Xa + b'X^{-1}b$ where $a \in \mathbb{R}^q$, $b \in \mathbb{R}^q$, $\forall X = X' > 0 \in \mathbb{R}^{q\times q}$ then, by tacking $X = \boldsymbol{P}(\gamma, u), a = \boldsymbol{P}^{-1}(\gamma, u)\,e$, $b = \int_0^1 \frac{\partial \boldsymbol{f}(s)}{\partial s}\Big|_{s = s_\lambda} (\mathrm{sat}^+(\hat{\xi}) - \mathrm{sat}^+(\xi))\,\mathrm{d}\lambda$ then,

$$\begin{aligned}
&2e'\boldsymbol{P}^{-1}(\gamma, u)\int_0^1 \frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s = s_\lambda} (\mathrm{sat}^+(\hat{\xi}) - \mathrm{sat}^+(\xi))\,\mathrm{d}\lambda \\
&\le e'\boldsymbol{P}^{-1}(\gamma, u)\,e \\
&+ \bigg(\int_0^1 \frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s = s_\lambda} (\mathrm{sat}^+(\hat{\xi}) - \mathrm{sat}^+(\xi))\,\mathrm{d}\lambda\bigg)' \\
&\times \boldsymbol{P}^{-1}(\gamma, u^{\Delta})\bigg(\int_0^1 \frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s = s_\lambda} (\mathrm{sat}^+(\hat{\xi}) - \mathrm{sat}^+(\xi))\,\mathrm{d}\lambda\bigg).
\end{aligned} \tag{43}$$

Since

$$\left(\int_0^1 \frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda} (\text{sat}^+(\hat{\xi}) - \text{sat}^+(\xi))\, \mathrm{d}\lambda\right)' \times \boldsymbol{P}^{-1}(\gamma, u)\left(\int_0^1 \frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda} (\text{sat}^+(\hat{\xi}) - \text{sat}^+(\xi))\, \mathrm{d}\lambda\right) \le \int_0^1 \left(\frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda} (\text{sat}^+(\hat{\xi}) - \text{sat}^+(\xi))\right)' \times \boldsymbol{P}^{-1}(\gamma, u^\Delta)\left(\frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda} (\text{sat}^+(\hat{\xi}) - \text{sat}^+(\xi))\right) \mathrm{d}\lambda. \quad (44)$$

Consequently,

$$\dot{V}(e) \le V(e) - e'\left(\boldsymbol{P}^{-1}(\gamma, u^\Delta)\boldsymbol{Q}(\gamma, u)\boldsymbol{P}^{-1}(\gamma, u^\Delta)\right)e + \int_0^1 \left(\frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda} (\text{sat}^+(\hat{\xi}) - \text{sat}^+(\xi))\right)' \times \boldsymbol{P}^{-1}(\gamma, u^\Delta)\left(\frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda} (\text{sat}^+(\hat{\xi}) - \text{sat}^+(\xi))\right) \mathrm{d}\lambda. \quad (45)$$

Let $F(t) = \frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda}$. Since

$$\int_0^1 \left(\frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda} (\text{sat}^+(\hat{\xi}) - \text{sat}^+(\xi))\right)' \times \boldsymbol{P}^{-1}(\gamma, u^\Delta)\left(\frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda} (\text{sat}^+(\hat{\xi}) - \text{sat}^+(\xi))\right) \mathrm{d}\lambda \le \int_0^1 (\text{sat}^+(\hat{\xi}) - \text{sat}^+(\xi))' \left(\frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda}\right)' \boldsymbol{P}^{-1}(\gamma, u^\Delta) \times \left(\frac{\partial \boldsymbol{f}(s)}{\partial s}\bigg|_{s=s_\lambda}\right) (\text{sat}^+(\hat{\xi}) - \text{sat}^+(\xi))\, \mathrm{d}\lambda \le \left\|\boldsymbol{D}^{-1}(\gamma, u)F'(t)\boldsymbol{P}^{-1}(\gamma, u)F(t)\boldsymbol{D}^{-1}(\gamma, u)\right\| \left\|\boldsymbol{G}(\gamma, u)\, e\right\|^2 \quad (46)$$

then, by putting $z \triangleq \boldsymbol{D}(\gamma, u)\, e$ we can always find a constant $\alpha > 0$ such that $\boldsymbol{Q}_1 > \alpha\, \tilde{\boldsymbol{P}}$ which implies that

$$\begin{aligned} &-e'\left(\boldsymbol{P}^{-1}(\gamma, u^\Delta)\boldsymbol{Q}(\gamma, u^\Delta)\boldsymbol{P}^{-1}(\gamma, u^\Delta)\right)e \\ &\le -\alpha\, e'\left(\boldsymbol{D}(\gamma, u^\Delta)\tilde{\boldsymbol{P}}^{-1}\boldsymbol{D}(\gamma, u^\Delta)\right)e \quad (47) \\ &= -\alpha\,\gamma\, V(e) = -\alpha_1\, z'z, \;\; \alpha_1 = \alpha\lambda_{\min}\left(\tilde{\boldsymbol{P}}^{-1}\right). \end{aligned}$$

Taking into account $V(e) \le \frac{\lambda_{\max}(\tilde{\boldsymbol{P}}^{-1})}{\gamma}$ and using the result of Lemma 1, we can find two constants c_2 and c_3 such that $\left\|\boldsymbol{D}(\gamma, u)F(t)\boldsymbol{D}^{-1}(\gamma, u)\right\| \le c_2 + \frac{c_3}{\gamma}$. This implies from (45) and (46) that

$$\dot{V}(e) \le -\left(\alpha_1 - \frac{\bar{\lambda}}{\gamma} - \frac{\bar{\lambda}}{\gamma}\left(c_1 + \frac{c_2}{\gamma}\right)^2\right) z'z \quad (48)$$

where $\bar{\lambda} = \lambda_{\max}(\tilde{\boldsymbol{P}}^{-1})$. Since $\gamma(t)$ is increasing then, there exists a time $t^\star > t_0$ such that $\dot{V}(e) \le -c^\star z'z,\;\; c^\star > 0,\; t \ge t^\star$. Since for $t \ge t^\star$; $\dot{V}(e) \le -c^\star\,(\boldsymbol{C}\hat{\xi}_1 - y)^2$ then, it is easy to prove that $\gamma(t)$ converges to a some constant when time tends to infinity. Furthermore, during the period $t_0 \le t \le t^\star$ the states of the observer do not escape to infinity due to the fact the observer nonlinearities are bounded for all $t \ge 0$. This ends the proof of the Theorem.

III. Conclusion

In this note, we gave a systematic method to estimate the parameters of dead-zone input systems exhibiting bounded-state behaviors. The proposed procedure is able to reconstruct constant and time-varying parameters without restrictive conditions. Based on this proposed design it is then possible to reconstruct dead-zone inverses to cancel the effects of unknown dead-zone nonlinearities in feedback control systems.

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