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On Definability of Sets in Dominance-Based Approximation Space

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Abstract **–This paper introduces the representation of Dominance-based approximation space by binary neighborhood systems introduced by Lin. We use blocks indexed by pairs of decision values as elementary neighborhoods or granules for computing approximations of generalized multiple criteria decision tables. The concept of generalized decisions was introduced by Dembczynski et al. as a generalization of DRSA (Dominance-based Rough Set Approach) where criteria and decision attribute in a decision table may be assigned a range of values. We show that binary neighborhood systems provide a uniform representation of both singleton and generalized multi-criteria decision tables. The main task is to determine the family of binary relations to capture inconsistency caused by violating the dominance principle. In addition, some interesting relationships among lower, upper approximations and boundary sets are presented. The proposed approach is demonstrated by examples.**

I. INTRODUCTION

In rough set theory [11, 12, 13], information of objects is represented by an information system $IS = (U, A, V, f)$, where U is a finite set of objects, A is a finite set of attributes, $V = \cup_{q\in A} V_q$ and V_q is the domain of attribute q, and $f: U \times A \to V$ is a total information function such that $f(x,q) \in V_q$ for every $q \in A$ and $x \in U$. In many applications, data sets are represented by a special case of information systems called *decision tables*. In a decision table $(U, C \cup D = \{d\})$, there is a designated attribute d called *decision attribute*, and attributes in C are called *condition attributes*. Each attribute q in $C \cup \{d\}$ is associated with an equivalence relation R_q on the set of objects of U such that for each x and $y \in U$, $xR_q y$ means $f(x,q) = f(y,q)$. For each x and $y \in U$, x and y are *indiscernible* on attributes $P \subseteq C$ if and only if $xR_q y$ for all $q \in P$.

Dominance-based Rough Set Approach (DRSA) was introduced by Greco, Matarazzo and Slowinski [6, 7, 8, 9, 10]. The basic idea is that consistent decision making can be regarded a process that evaluation of objects and assignment of objects to decision classes obey the dominance principle, namely, the relationship of evaluation of objects and assignment of decision classes should be monotonic. In DRSA, attributes with totally ordered domains are called *criteria*. Each criterion q in C is associated with a outranking relation S_q on the set of objects of U such that for each x and $y \in U$, $xS_q y$ means $f(x,q) \geq f(y,q)$. The assignment of decision classes to objects $x, y \in U$ obeys the dominance principle based on a set $P \subseteq C$ of criteria when

$$xD_P y \text{ if and only if } f(x,q) \geq f(y,q) \text{ for all } q \in P. \quad (1)$$

Recently, Dembczynski et al. [4, 5] introduced the concept of generalized decisions as an extension of DRSA to deal with imprecise evaluation of objects and decision class assignment where imprecision is represented by a finite set of values, and the second-order of rough approximations is considered by extending the original dominance relation accordingly. The single-valued information function f is extended to a subset-valued function $\hat{f}$ consisting of a pair of information functions l and u: $\hat{f}(x,q) = \langle l(x,q), u(x,q)\rangle$ and $u(x,q) \geq l(x,q)$. The interval $\langle l(x,q), u(x,q)\rangle$ is a subset of V_q and is referred as an *interval evaluation* of object x on criterion q with $l(x, q)$ and $u(x, q)$ as the lower and upper boundary of the interval. It is called an *interval assignment* when q is the decision attribute.

In this work, we formulate the approximations of singleton and generalized multi-criteria decision tables based on the concepts of indexed blocks [1, 2, 3] and binary neighborhood systems [14, 15, 16]. A dominance-based approximation space is represented by a binary neighborhood system consisting of blocks indexed by decision pairs. In general, lower approximations are not definable in terms of elementary neighborhoods. We introduce a remedy based on the boundary set and upper approximations. We show that the representation of dominance-based approximation space by binary neighborhood systems is a generalization for computing approximations of any family of decision classes.

The rest of the paper is organized as follows. In Section II, we will present related concepts of binary neighborhood systems and indexed blocks. The definability of singleton and generalized decision tables and some relationships among approximations and boundary sets are presented in Section III. Conclusions are given in Section IV.

II. RELATED CONCEPTS

A. Binary Neighborhood Systems

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The use of neighborhood systems to study approximations of database and knowledge-based systems was first introduced by Lin [14]. Let *V* and *U* be two sets. A binary neighborhood system is a mapping $B: V \rightarrow 2^U$. Each binary relation of $V \times U$ corresponds to one binary neighborhood system [15]. A refinement of rough set theory into granular computing was introduced in [16].

B. Indexed Blocks

The concept of indexed blocks was introduced in [1], and it has been used to study inconsistencies in a Multiple Criteria Decision Table (MCDT) caused by violating of dominance principle. *Indexed blocks* are sets of objects indexed by pairs of decision values. Blocks with indices (*i, i*) denote objects with consistent class assignment of decision value *i*, and blocks with other indices are inconsistent. Detailed construction of indexed blocks from a MCDT table can be found in [1]. In this work, we formulate the concept of indexed blocks using the concept of binary neighborhood systems.

Let $(U, C \cup D = \{d\})$ be a multi-criteria decision table with condition criteria *C* and decision attribute *d*. Decision class labeled by decision value *i* is denoted as D_i. For simplicity, in this following we also refer to D_i by its index *i*.

Let *B* be a mapping: $V_D \times V_D \rightarrow 2^U$ defined as follows.

$$B(i,i) = D_i - \bigcup_{j \neq i} B(i,j) \text{, and}$$

$$\begin{aligned} B(i,j) = \{ & x \in D_i : \exists y \in D_j, i \neq j, \textit{such that} \\ & \forall q \in C, f(x,q) \geq f(y,q) \textit{ and } i < j, or \\ & f(x,q) \leq f(y,q) \textit{ and } i > j\}. \end{aligned} \quad (2)$$

A block $B(i, j)$ indexed by the decision value pair (*i, j*) is called inconsistent indexed block, and it contains objects violating the dominance principle when evaluating all criteria in *C* with respect to the decision pair (i, j), $i \neq j$. It is clear that if $B(i, j)$ is not empty, then it has at least two objects. Consistent objects are those that do not belong to any inconsistent indexed blocks, therefore, they are members of consistent indexed blocks $B(i,i)$.

The mapping *B* or equivalently the family of indexed blocks $\{B(i, j)\}$ derived from the mapping is a binary neighborhood system. The indexed blocks are called *elementary neighborhoods* or *elementary granules*. Let $P \subseteq C$ and let B_P denote the restriction of the mapping *B* to *P*, then B_P (equivalently, $\{B_P(i, j)\}$) is also a binary neighborhood system.

Let *q* be a criterion in *C*, and let $I_q(i)$ denote the set of values of criterion *q* for objects in decision class D_i, which is the set of objects that are assigned a decision value *i*. More precisely, let $min_q(i)$ denote the minimum value of *q* among objects with decision value *i* and $max_q(i)$ denote the maximum value, then $I_q(i) = [min_q(i), max_q(i)]$ is the interval with $min_q(i)$ and $max_q(i)$ as its lower and upper boundary values. From $I_q(i)$, intervals related to pairs of decision values is defined by the mapping $I_q(i, j): D \times D \rightarrow \wp(V_q)$ as:

$$I_q(i, j) = I_q(i) \cap I_q(j) \text{ and } I_q(i,i) = I_q(i) - \bigcup_{j \neq i} I_q(i, j) \quad (3)$$

where $D = V_D$ and $\wp(V_q)$ denotes the power set of V_q.

Proposition 1. Let $x \in D_i$. If $\exists q \in C, f(x,q) \in I_q(i,i)$, then $x \in B(i,i)$.

Each block $B(i, j)$ of a binary neighborhood system $\{B(i, j)\}$ has one corresponding binary relation on $D_i \times D_j$. Let $x \in D_i$ and $y \in D_j$. and let $x, y \in B(i, j)$. We will treat $(x, y) \in B(i, j)$ the same as $x, y \in B(i, j)$.

Proposition 2. Let $x \in D_i$, $y \in D_j$, and $i \neq j$. Let q_a and q_b be two criteria. Then, $(x, y) \in B_{q_a}(i, j)$ and $(x, y) \in B_{q_b}(i, j)$ if and only if $(x, y) \in B_{\{q_a, q_b\}}(i, j)$.

Propositions 1 and 2 provide the basis for combining indexed blocks incrementally by criteria. Detailed construction can be found in [1].

C. Neighborhoods of Indexed Blocks

Based on the neighborhood system $\{B(i, j)\}$ of indexed blocks, we can define a binary neighborhood system of $\{B(i, j)\}$, which is a mapping $BNS: \{B(i, j)\} \rightarrow \wp(2^U)$ defined as, for all $i, j \in D$,

$$BNS(B(i, j)) = \{B(k,l): \{k,l\} \cap \{i, j\} \neq \emptyset, \forall k,l \in D\}. \quad (4)$$

Example 1.

Consider the following multi-criteria decision table, where *U* is the universe of objects, q_1 and q_2 are condition criteria and *d* is the decision with preference ordering 3 > 2 > 1.

Table 1. Example of a multi-criteria decision table.

U	q_1	q_2	*d*
x_1	1	2	2
x_2	1.5	1	1
x_3	2	2	1
x_4	1	1.5	1
x_5	2.5	3	2
x_6	3	2.5	3
x_7	2	2	3
x_8	3	3	3

The intervals of criterion q_1 for decision values are $I_{q_1}(1) = [1, 2]$, $I_{q_1}(2) = [1, 2.5]$, and $I_{q_1}(3) = [2, 3]$.

From definition (3), the intervals of decision pairs on q_1 are shown in Table 2.

Table 2. Decision pair intervals of q_1.

D×D	**1**	**2**	**3**
1	[ ]	[1, 2]	[2, 2]
2		[ ]	[2, 2.5]
3			[3, 3]

Based on definition (2), the indexed blocks $\{B_{q_1}(i,j)\}$ derived from the intervals in Table 2 are shown in Table 3. Note that it is a covering of *U*.

Table *3*. Index blocks derived from q_1.

D*×*D	**1**	**2**	**3**
1	Ø	{1, 2, 3, 4}	{3, 7}
2		Ø	{5, 7}
3			{6, 8}

Similarly, the decision pair intervals and indexed blocks based on q_2 are shown in Tables 4 and 5.

Table 4. Decision pair intervals of q_2.

D*×*D	***1***	***2***	***3***
1	[1, 1.5]	[2, 2]	[2, 2]
2		[ ]	[2, 3]
3			[ ]

Table *5*. Index blocks derived from q_2.

D*×*D	***1***	***2***	***3***
1	{2, 4}	{1, 3}	{3, 7}
2		*Ø*	{1, 5, 6, 7, 8}
3			*Ø*

Following the rules for combining two criteria q_1 and q_2, as proposed in [1], the indexed blocks after combining q_1 and q_2 in Table 1 are shown in the following table.

Table 6. Indexed blocks $\{B(i, j)\}$.

D*×*D	***1***	***2***	***3***
1	{2, 4}	{1, 3}	{3, 7}
2		*Ø*	{5, 7}
3			{6, 8}

From Table 6, we have the following neighborhoods of indexed blocks based on definition (4):

$BNS(B(1,1)) = \{B(1,1), B(1,2), B(1,3)\}$,
$BNS(B(1,2)) = \{B(1,1), B(1,2), B(1,3), B(2,3)\}$,
$BNS(B(1,3)) = \{B(1,1), B(1,2), B(1,3), B(2,3), B(3,3)\}$,
$BNS(B(2,3)) = \{B(1,2), B(1,3), B(2,3), B(3,3)\}$, and
$BNS(B(3,3)) = \{B(1,3), B(2,3), B(3,3)\}$.

III. Definability of Sets

A. Definability of Singleton MCDT

By a singleton MCDT we mean a multiple criteria decision table where criteria are assigned with singleton value. In the following, we will consider definability of decision classes in a singleton MCDT represented by a binary neighborhood system of indexed blocks.

Let $S = (U, C \cup D = \{d\})$ be a MCDT. Let $K \subseteq V_D$ and $\neg K = V_D - K$. Let $D_K = \bigcup_{i \in K} D_i$ and $D_{\neg K} = \bigcup_{i \in \neg K} D_i$. Then, S is a singleton MCDT table if and only if for $i \in V_D$, $D_{\{i\}} \cap D_{\neg\{i\}} = \varnothing$ and $D_{\{i\}} \cup D_{\neg\{i\}} = U$.

Let $B = \{B(i, j)\}$ be the binary neighborhood system derived from a MCDT table, and let $K \subseteq V_D$.

The upper approximation of D_K by *B* is defined as

$$\begin{aligned}\overline{B}D_K &= \bigcup\{B(i,j) : B(i,j) \cap D_K \neq \varnothing\} \\ &= \bigcup\{B(i,j) : \{i,j\} \cap K \neq \varnothing\}\end{aligned} \tag{5}$$

Similarly, the upper approximation of $D_{\neg K}$ by *B* is defined as

$$\begin{aligned}\overline{B}D_{\neg K} &= \bigcup\{B(i,j) : B(i,j) \cap D_{\neg K} \neq \varnothing\} \\ &= \bigcup\{B(i,j) : \{i,j\} \cap (D - K) \neq \varnothing\}\end{aligned} \tag{6}$$

The boundary set of D_K by *B* is defined as

$$BN_B(D_K) = \overline{B}D_K \cap \overline{B}D_{\neg K} = BN_B(D_{\neg K}) \tag{7}$$

One way to define the lower approximation of D_K by *B* is

$$\underline{B}D_K = \bigcup\{B(i,j) : B(i,j) \subseteq D_K\} \tag{8}$$

Similarly, the lower approximation of $D_{\neg K}$ by *B* is

$$\underline{B}D_{\neg K} = \bigcup\{B(i,j) : B(i,j) \subseteq D_{\neg K}\} \tag{9}$$

However, in general, the above definitions do not hold when the underlying approximation space $B = \{B(i, j)\}$ is a covering of *U*, because an indexed block $B(i, j)$ may be partially overlapped with other indexed blocks of its neighborhood as defined in (4). It is clear that (8) and (9) will hold if and only if there is no overlapping between $B(i, j)$ and all other blocks of its neighborhood, which is the same as that the underlying approximation space forms a partition on *U*.

Alternatively, we propose the following definitions for the lower approximations of decision classes.

The lower approximation of D_K by *B* is defined as

$$\begin{aligned}\underline{B}D_K &= \bigcup\{B(i,j) : B(i,j) \subseteq D_K\} - BN_B(D_K) \\ &= D_K - BN_B(D_K)\end{aligned} \tag{10}$$

, and the lower approximation of $D_{\neg K}$ by *B* is defined as

$$\begin{aligned}\underline{B}D_{\neg K} &= \bigcup\{B(i,j) : B(i,j) \subseteq D_{\neg K}\} - BN_B(D_{\neg K}) \\ &= D_{\neg K} - BN_B(D_{\neg K}).\end{aligned} \tag{11}$$

From definitions (5), (6), (10), and (11), we say that a decision class D_K is *B-definable* if and only if $\underline{B}D_K = \overline{B}D_K$ if and only if $\underline{B}D_{\neg K} = \overline{B}D_{\neg K}$. A singleton MCDT is *B-definable of degree K* if and only if there exists D_K that is *B*-definable.

The boundary set $BN_B(D_K) = BN_B(D_{\neg K})$ can be further divided into two symmetric parts: the lower boundary and upper boundary of D_K ($D_{\neg K}$). The lower boundaries of D_K and $D_{\neg K}$ are defined as

$$\underline{BN}_B(D_K) = D_K - \underline{B}D_K \text{ and} \tag{12}$$

$$\underline{BN}_B(D_{\neg K}) = D_{\neg K} - \underline{B}D_{\neg K}. \tag{13}$$

The upper boundaries of D_K and $D_{\neg K}$ are defined as

$$\overline{BN}_B(D_K) = \overline{B}D_K - D_K \text{ and} \quad (14)$$

$$\overline{BN}_B(D_{\neg K}) = \overline{B}D_{\neg K} - D_{\neg K}\,. \quad (15)$$

Their symmetric relationships are characterized in the following proposition.

Proposition 3. From definitions (5), (6), (7), (10), and (11), we have the following:

$\underline{BN}_B(D_K) = D_K - \underline{B}D_K = \overline{B}D_{\neg K} - D_{\neg K} = \overline{BN}_B(D_{\neg K})$, and

$\underline{BN}_B(D_{\neg K}) = D_{\neg K} - \underline{B}D_{\neg K} = \overline{B}D_K - D_K = \overline{BN}_B(D_K)$.

Proposition 4.

$BN_B(D_K) = \underline{BN}_B(D_K) \cup \underline{BN}_B(D_{\neg K}) = BN_B(D_{\neg K})$, and

$BN_B(D_K) = \overline{BN}_B(D_K) \cup \overline{BN}_B(D_{\neg K}) = BN_B(D_{\neg K})$.

From Proposition 3, we have the following proposition.

Proposition 5. $\underline{B}D_K = D_K - (\overline{B}D_{\neg K} - D_{\neg K}) = D_K - \overline{B}D_{\neg K}$.

Similarly, $\underline{B}D_{\neg K} = D_{\neg K} - (\overline{B}D_K - D_K) = D_{\neg K} - \overline{B}D_K$.

Example 2.

Consider the singleton MCDT shown in Table 1 and its indexed blocks in Table 6. The decision classes are $D_1 = \{x_2, x_3, x_4\}$, $D_2 = \{x_1, x_5\}$, and $D_3 = \{x_6, x_7, x_8\}$. Let $K = \{1,2\}$, then $\neg K = \{3\}$. We have $D_K = \{x_1, x_2, x_3, x_4, x_5\}$ and $D_{\neg K} = \{x_6, x_7, x_8\}$.

$$\overline{B}D_K = \overline{B}D_{\{1,2\}} = B(1,1) \cup B(1,2) \cup B(1,3) \cup B(2,3)$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_7\}, \textit{and}$$
$$\overline{B}D_{\neg K} = \overline{B}D_{\{3\}} = B(1,3) \cup B(2,3) \cup B(3,3)$$
$$= (x_3, x_5, x_6, x_7, x_8\}.$$

Thus, $BN_B(D_K) = \overline{B}D_K \cap \overline{B}D_{\neg K} = BN_B(D_{\neg K}) = \{x_3, x_5, x_7\}$.

From definitions (8) and (9), we have the following lower approximations:

$$\underline{B}D_K = \bigcup\{B(i,j) : B(i,j) \subseteq D_K\}$$
$$= B(1,1) \cup B(1,2) = \{x_1, x_2, x_3, x_4\}, \textit{and}$$
$$\underline{B}D_{\neg K} = \bigcup\{B(i,j) : B(i,j) \subseteq D_{\neg K}\}$$
$$= B(3,3) = \{x_6, x_8\}.$$

Since x_3 is in the boundary set $BN_B(D_K)$, it should not belong to $\underline{B}D_K$. The lower approximation of $D_{\neg K}$ happens to be correct in this case.

Now, from definitions (10) and (11), we have

$$\underline{B}D_K = \bigcup\{B(i,j) : B(i,j) \subseteq D_K\} - BN_B(D_K)$$
$$= D_K - BN_B(D_K) = \{x_1, x_2, x_4\}, \textit{and}$$
$$\underline{B}D_{\neg K} = \bigcup\{B(i,j) : B(i,j) \subseteq D_{\neg K}\} - BN_B(D_{\neg K})$$
$$= D_{\neg K} - BN_B(D_{\neg K}) = \{x_6, x_8\}.$$

From Proposition 3, we have

$$\underline{BN}_B(D_K) = D_K - \underline{B}D_K$$
$$= \{x_1, x_2, x_3, x_4, x_5\} - \{x_1, x_2, x_4\} = \{x_3, x_5\},$$
$$\overline{BN}_B(D_{\neg K}) = \overline{B}D_{\neg K} - D_{\neg K}$$
$$= \{x_3, x_5, x_6, x_7, x_8\} - \{x_6, x_7, x_8\} = \{x_3, x_5\}, \textit{and}$$
$$\underline{BN}_B(D_K) = \overline{BN}_B(D_{\neg K}).$$

Similarly, we have

$$\underline{BN}_B(D_{\neg K}) = D_{\neg K} - \underline{B}D_{\neg K} = \overline{B}D_K - D_K = \overline{BN}_B(D_K) = \{x_7\}.$$

B. Definability of Generalized MCDT

Generalized MCDT tables (or called generalized decision tables) with interval evaluations and assignments were studied by Dembczynski et al. [4, 5]. A generalized MCDT table can be decomposed into a family of singleton MCDT tables based on the combination of all interval evaluations and assignments in the table. One way to represent a generalized decision table is by a pair of singleton decision tables, called *lower and upper singleton tables* as proposed in [17]. The lower table is obtained from a generalized decision table by replacing all interval values with their lower boundary values, and they are replaced by upper boundary values in the upper table. The pair of lower and upper tables represents the boundary tables of a generalized decision table. We say that a generalized decision table is *B*-definable if both its lower and upper singleton decision tables are *B*-definable. In general, the lower and upper tables may not be definable. However, we can try to make them definable by reassignment of objects as indicated by [5], and an algorithm for objects reassignment is introduced in [17].

In the following, we propose a representation of generalized MCDT tables by binary neighborhood systems as formulated in previous section.

Let $S = (U, C \cup D = \{d\})$ be a generalized MCDT table where criteria of objects in U may have intervals or sets of values, i.e., $f(x,q) \subseteq V_q$ instead of $f(x,q) \in V_q$. Let $x_q = f(x,q) \subseteq V_q$, x_q is represented by a pair of lower and upper boundary values as <*min*(x_q), *max*(x_q)>.

Let σ denote a mapping: $\wp(X) \to X$. It is a selection function for any nonempty set X. The decision classes of S are defined as $D_i = \{x \in U : \sigma(f(x,d)) = i\}$.

Let $P \subseteq C$ and BR_P^{ij} be a binary relation on $D_i \times D_j$, $i \neq j$, defined as

$$BR_P^{ij}(x) = \{y \in D_j : \forall q \in P,$$
$$min(x_q) \geq max(y_q) \wedge max(x_d) < min(y_d), or \quad (16)$$
$$max(x_q) \leq min(y_q) \wedge min(x_d) > max(y_d)\}$$

where *min*(V) returns the minimum value of V, and *max*(V) returns the maximum. We have $BR_P^{ij} = BR_P^{ji}$.

Let $BR^{ij} = BR_C^{ij}$, and let $B(i,j) \subseteq U$ be the block induced from $BR^{ij} \subseteq D_i \times D_j$. Objects in block $B(i,j)$ are those that violate the dominance principle with respect to decision value pair (i,j). The family of binary relations BR^{ij} captures the inconsistency that may appear in a generalized MCDT. For simplicity, we consider only the minimum and maximum values of interval evaluations and assignments in this work. This can be generalized in future work.

Based on BR^{ij}, the representation of a generalized MCDT by a binary neighborhood system with the mapping B: $V_D \times V_D \rightarrow 2^U$, which is defined as:

$B(i, j) = BR^{ij}$ for $i \neq j$, and

$$B(i,i) = D_i - \bigcup_{j \neq i} BR^{ij} = D_i - \bigcup_{j \neq i} B(i, j) \quad (17)$$

The mapping B or equivalently, the family $\{B(i, j) : i, j \in V_D\}$ of indexed blocks represents a generalized MCDT.

C. Examples

For simplicity, let us consider the following table used in Dembczynski et al. [4] as an example. In Table 7, criteria and decision of some objects are assigned a range of integer values delimited by the lower and upper boundary values.

Table 7. A generalized multi-criteria decision table.

U	*q₁*	*q₂*	*d*
x_1	⟨46, 50⟩	⟨48, 52⟩	4
x_2	⟨44, 48⟩	⟨48, 50⟩	4
x_3	⟨49, 52⟩	44	3
x_4	26	⟨28, 35⟩	3
x_5	30	⟨26, 32⟩	2
x_6	24	10	⟨2, 3⟩
x_7	⟨6, 8⟩	⟨14, 20⟩	⟨1, 2⟩
x_8	⟨9, 10⟩	⟨16, 20⟩	⟨1, 2⟩
x_9	8	11	1
x_{10}	⟨15, 27⟩	11	1

From Table 7, we have the following decision classes:

$D_1 = \{x_7, x_8, x_9, x_{10}\}$,

$D_2 = \{x_5, x_6, x_7, x_8\}$,

$D_3 = \{x_3, x_4, x_6\}$, and

$D_4 = \{x_1, x_2\}$.

We have $BR^{23}_{q_1}(x_5) = \{x_4\}$, because $f(x_5, q_1) = 30$ > $f(x_4, q_1) = 26$ and $f(x_5, d) = 2$ < $f(x_4, d) = 3$. However, $x_4 \notin BR^{23}_{q_2}(x_5)$, since $max(f(x_4, q_2)) \geq max(f(x_5, q_2))$. Therefore, $(x_5, x_4) \notin BR^{23}$.

Similarly, consider objects x_{10} and x_6. Although (x_{10}, x_6) is in $BR^{12}_{q_2}$ based on definition (16). The pair (x_{10}, x_6) is not in $BR^{12}_{q_1}$, because $min(f(x_{10}, q_1)) = 15 < min(f(x_6, q_1)) = 24$ and $min(f(x_{10}, d)) = 1$ < $min(f(x_6, d)) = 2$. Therefore, combining q_1 and q_2, $(x_{10}, x_6) \notin BR^{12}$.

The blocks induced from $BR^{ij}_{q_1}$ and $BR^{ij}_{q_2}$ are shown in Tables 8 and 9, and the blocks induced from $BR^{ij}_{\{q_1, q_2\}} = BR^{ij}$ are given in Table 10, which shows that the generalized MCDT is B-definable by the binary neighborhood system $\{B(i, j)\}$.

Table 8. The indexed blocks of $B_{q_1}(i, j)$.

D×D	**1**	**2**	**3**	**4**
1	$\{x_7, x_8, x_9, x_{10}\}$	Ø	Ø	Ø
2		$\{x_6, x_7, x_8\}$	$\{x_5, x_4\}$	Ø
3			$\{x_6\}$	$\{x_3, x_2\}$
4				$\{x_1\}$

Table 9. The indexed blocks of $B_{q_2}(i, j)$.

D×D	**1**	**2**	**3**	**4**
1	Ø	$\{x_6, x_7, x_8, x_9, x_{10}\}$	$\{x_6, x_7, x_8, x_9, x_{10}\}$	Ø
2		Ø	$\{x_6, x_7, x_8, x_9, x_{10}\}$	Ø
3			$\{x_3, x_4\}$	Ø
4				$\{x_1, x_2\}$

Table 10. The indexed blocks of $B_{\{q_1, q_2\}}(i, j)$.

D×D	**1**	**2**	**3**	**4**
1	$\{x_7, x_8, x_9, x_{10}\}$	Ø	Ø	Ø
2		$\{x_5, x_6, x_7, x_8\}$	Ø	Ø
3			$\{x_3, x_4, x_6\}$	Ø
4				$\{x_1, x_2\}$

IV. CONCLUSIONS

We have introduced the representation of both singleton and generalized multiple-criteria decision tables by binary neighborhood systems consisting of blocks indexed by decision value pairs. The main task is in determining a family of binary relations to capture inconsistency caused by violating the dominance principle. We have considered only minimum and maximum boundary values in determining inconsistency. Further generalization can be done in the future. In general, lower approximations are not definable by taking the union of elementary neighborhoods, we introduced a remedy in terms of boundary sets to compute lower approximations. We have shown that binary neighborhood systems provide a uniform framework for representing a general dominance-based approximation space. Properties and algorithms for determining definable generalized decision tables need more studies. Another future work is to develop efficient rule learning algorithms for generalized decision tables based on the proposed representation.

REFERENCES

[1] Chan, C.-C., Tzeng, G.-H., "Dominance-based rough sets using indexed blocks as granules," *RSKT 2008*, Wang et al. (eds.), *LNAI vol. 5009*, pp. 244 – 251 (2008).

[2] Chan, C.-C., "Approximate dominance-based rough sets using equivalence granules," *Proceedings of IEEE WCCI 2008*, June 1-6, Hong Kong, pp. 2433-2438 (2008).

[3] Chan, C.-C., Tzeng, G.-H., "Computing approximations of dominance-based rough sets by bit-vector encodings," *RSCTC 2008*, October 23-25, Akron, OH, USA, *LNAI vol. 5306*, pp. 131-141 (2008).

[4] Dembczynski, K., G. Salvatore, and R. Slowinski, "Second-order rough approximations in multi-criteria classification with imprecise evaluations and assignments," *Proceedings of RSFDGrC 2005, LNAI 3641*, D. Slezak et al. (Eds.), pp. 54-63 (2005), Springer-Verlag, Berlin Heidelberg.

[5] Dembczynski, K., G. Salvatore, W. Kotlowski, and R. Slowinski, "Optimized generalized decision in dominance-based rough set

approach," *Proceedings of RSKT 2007, LNAI 4481*, J.T.Yao et al. (Eds.), pp. 118-125 (2007), Springer.

[6] Greco, S., Matarazzo, B., Slowinski, R., "Rough approximation of a preference relation by dominance relations," *ICS Research Report* 16/96, Warsaw University of Technology, Warsaw (1996). *European Journal of Operational Research* 117(1), 63–83 (1999).

[7] Greco, S. Matarazzo, B., Slowinski, R., "A new rough set approach to evaluation of bankruptcy risk," in Zopounidis, C. (Ed.), *Operational Tools in the Management of Financial Risks*. Kluwer Academic Publishers, Dordrecht, pp. 121–136 (1998).

[8] Greco, S., Matarazzo, B., Slowinski, R., "The use of rough sets and fuzzy sets in MCDM," in Gal, T., Stewart, T., Hanne, T. (Eds.), *Advances in Multiple Criteria Decisions Making*. Kluwer Academic Publishers, Dordrecht, pp. 14.1–14.59 (Chapter 14) (1999).

[9] Greco, S., Matarazzo, B., Slowinski, R., "Rough sets theory for multicriteria decision analysis," *European Journal of Operational Research* 129(1), 1 – 47 (2001).

[10] Greco, S., Matarazzo, B., Slowinski, R., "Rough sets methodology for sorting problems in presence of multiple attributes and criteria," European Journal of Operational Research 138(2), 247 – 259 (2002).

[11] Pawlak, Z., "Rough sets: basic notion," *International Journal of Computer and Information Science* 11(15), 344-56 (1982).

[12] Pawlak, Z., "Rough sets and decision tables," *Lecture Notes in Computer Science* 208, 186-196, Berlin, Heidelberg, Springer-Verlag (1985).

[13] Pawlak, Z., Grzymala-Busse, J., Slowinski, R., Ziarko, W., "Rough sets," *Communication of ACM*, 38(11), 89-95 (1995).

[14] Lin, T.Y., "Neighborhood systems and approximation in database and knowledge base systems," *Proceedings of the Fourth International Symposium on Methodologies of Intelligent Systems*, Poster Session, October 12-15, pp. 75-86 (1989).

[15] Lin, T.Y., "Granular computing on binary relations I: data mining and neighborhood systems," In: L. Polkowski and A. Skowron (eds), *Rough Sets in Knowledge Discovery*, Physica-Verlag, Heidelberg, 107-120 (1998).

[16] Lin, T.Y., "A road map from rough set theory to granular computing," RSKT 2006, Wang et al. (eds.), *LNAI vol. 4062*, pp. 33 – 41 (2006).

[17] Chan, C.-C. and Tzeng, G.-H., "Representation of second-order dominance-based approximation space by neighborhood systems," (Submitted to IEEE Granular Computing conference, 2009).