A novel Max-Plus algebra based wavelet transform and its applications in Image Processing

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Abstract—Max-plus algebra based wavelet transforms are considered currently to be the world’s fastest image processing methods. In the present paper a novel max-plus algebra based wavelet transform is proposed and studied both from the theoretical and practical points of view. The novel MP-wavelets considered here employ strictly neighbor pixels in the calculations and their most important properties are: very low computational complexity, flexible sampling window size and potentially, very easy hardware implementation.

Index Terms—max-plus algebra, MP-wavelets, mathematical morphology, image coding

I. INTRODUCTION

Nowadays, research efforts of several researchers in image processing are directed towards new image and video coding methods that satisfy the following properties:

• They should keep reasonable quality of images or video;
• They should have low computational complexity;
• They should be suitable for hardware implementation;
• They should be suitable for copyright protection;
• They should be adaptive to the application at hand.

The problem of designing cheap methods for the surveillance and storage of data collected from surveillance applications also motivates a search for new innovative methods.

A nonlinear wavelet-type transform was recently proposed in Heijmans’ works [4] [5], i.e., morphological wavelets, however, the morphological wavelets are defined based on the ordinal algebra on real numbers R together with the four arithmetic operations supplemented by max and min.

Max-plus (Tropical) algebra ([1], [9], [3], [11]) is an interesting algebraic structure (considered exotic by several Mathematicians [6]) over the set of integers \( \mathbb{Z} \) or reals \( \mathbb{R} \), endowed with maximum, minimum and standard addition as operations. It is interesting to remark that the idea to use max-plus algebra has recently appeared in many fields of mathematics. Tropical Algebra and Tropical Geometry ([10]) are fields of an increasing interest in the Mathematics Community. Also, Mathematical Morphology ([5]) is a novel, fruitful idea in the area of Computer Science.

Definition. Let us consider the (extended) set of integers \( \bar{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, \infty\} \), where \( \mathbb{Z} \) is the set of integers. Over the set \( \bar{\mathbb{Z}} \) one considers the operations \( \vee, \wedge \) and \( +, \min\{a, b\} \). The algebraic structure

\[ \mathbb{Z}_{\text{max}} = (\bar{\mathbb{Z}}, \vee, \wedge, +, -\infty, \infty, 0) \]

is called max-plus algebra, and the following properties hold true:

• \( \vee \) and \( \wedge \) are associative and commutative with neutral elements \(-\infty\) and \(\infty\) respectively;
• \( (\mathbb{Z}, +, 0) \) is the standard additive group of integers;
• + is distributive with respect to \(\vee\) and \(\wedge\);
• \(\vee\) is distributive with respect to \(\wedge\) and vice versa;
• \(-\infty\) is an absorbing element for \(\wedge\) while \(\infty\) is absorbing element for \(\vee\).

The convention that \(\wedge\) and \(\vee\) have a higher priority than + is used in the present paper.

Max-Plus algebra based wavelet transforms are defined using a multiresolution analysis approach. They are built using a nonlinear Analysis and a nonlinear Synthesis operator. The analysis operator is composed of an “approximation” \( \varphi^1_k : I_k \rightarrow I_{k+1} \) and a “details” \( \omega^1_k : I_k \rightarrow D_{k+1} \), while the synthesis operator is of the form \( \varphi^1_k : I_{k+1} \times D_{k+1} \rightarrow I_k \), where \( I_k \) and \( D_k \), \( k \geq 0 \) are nonempty. For a wavelet decomposition scheme the fulfillment of the pyramid condition (perfect reconstruction property) is essential. The pyramid condition reads \( \varphi^1_k \circ (\varphi^1_k, \omega^1_k) = 1_{I_k} \), where \( \circ \) denotes the usual composition and \( 1_{I_k} \) is the identity mapping over \( I_k \). The analysis process of MP-Wavelets using two types of signal decompositions: ‘approximation’ \( \varphi^1_k \) in our case to generate approximation signals in the space \( I_{k+1} \) and ‘detailed’ \( \omega^1_k \) to generate detailed signals in the space \( D_{k+1} \). The synthesis operator \( \varphi^1_k \) performs the reconstruction.

In Fig. 1 an example of a wavelet decomposition scheme is shown, using a sampling window size of 4x4 pixels. One approximation and three detail components are obtained.

![Fig. 1. A max-plus wavelet decomposition scheme with 4x4 sampling window size](image-url)
In the present paper the operators $\varphi_k^\perp, \omega_k^\perp$ and consequently $\psi_k^\perp$ are all nonlinear operators. Also the spaces $I_k$ and $D_k$, $k \geq 0$ are considered with a max-plus structure as considered above.

As key for the perfect reconstruction property, we will employ the following two identities:

$$a \land b_1 \land \ldots \land b_n + (a-b_1) \lor \ldots \lor (a-b_n) \lor 0 = a,$$  \hspace{1em} (2)

and the dual

$$a \lor b_1 \lor \ldots \lor b_n + (a-b_1) \land \ldots \land (a-b_n) \land 0 = a,$$  \hspace{1em} (3)

for any $a, b_1, b_2, \ldots, b_n \in \mathbb{Z}$.

It is easy to check the validity of these identities. Indeed, if e.g.

$$a \land b_1 \land \ldots \land b_n = b_k$$

then

$$(a-b_1) \lor \ldots \lor (a-b_n) = a-b_k > 0$$

and so

$$a \land b_1 \land \ldots \land b_n + (a-b_1) \lor \ldots \lor (a-b_n) \lor 0 = b_k + (a-b_k) = a.$$  

On the other hand if

$$a \land b_1 \land \ldots \land b_n = a$$

then

$$(a-b_1) \lor \ldots \lor (a-b_n) \leq 0$$

thus

$$a \land b_1 \land \ldots \land b_n + (a-b_1) \lor \ldots \lor (a-b_n) \lor 0 = a.$$  

II. TYPE I MP-WAVELETS

Type I and II MP wavelets were proposed and studied recently in [8], [7]. Let us recall here the definition of the Type I MP-wavelets.

The analysis operator is given by its approximation component $\varphi_k^\perp : I_k \rightarrow I_{k+1}$, and its detail components $\delta_{k,i}^\perp : I_k \rightarrow D_{k+1,i}$, $i = 1, \ldots, p-1$

$$\varphi_k^\perp(x)_n = y_{pn} = x_{pn} \land x_{pn+1} \land \ldots \land x_{pn+p-1}$$

$$\delta_{k,i}^\perp(x)_n = y_{pn+i} = x_{pn} - x_{pn+i}.$$  

In this case the synthesis is $\psi_k^\perp : I_{k+1} \times D_{k+1,1} \times \ldots \times D_{k+1,p-1} \rightarrow I_k$,

$$\psi_k^\perp(y)_{pn} = z_{pn} = y_{pn} + y_{pn+1} \lor \ldots \lor y_{pn+p-1} \lor 0$$

$$\psi_k^\perp(y)_{pn+i} = z_{pn+i} = z_{pn} - y_{pn+i}.$$  

It is easy to check that the pyramid condition holds by (2).

III. TYPE II MP-WAVELETS

The application of the type I, one-dimensional wavelet decomposition scheme successively on the horizontal and vertical direction gives a new type of MP-wavelets, different from type I MP-wavelets introduced above. The scheme is a biorthogonal-like wavelet decomposition scheme. The analysis is performed using a one dimensional type I scheme first on the horizontal, followed by the vertical direction. The synthesis is performed successively applying the 1-D synthesis scheme on the vertical direction followed by the 1-D scheme applied on the horizontal direction. As it was shown in [7] the perfect reconstruction property holds for this type of MP-wavelet too.

IV. TYPE III MP-WAVELETS

Let us first observe that Type I MP-wavelets have the detail component of the analysis operator defined based on a preferred center pixel $x_{pn}(\delta_{k,i}^\perp(x)_n = x_{pn} - x_{pn+i})$. The idea in the definition of MP-wavelets of type III is to use identity (2) in such a way that the neighbor pixels are involved in operations instead of a preferred pixel.

The analysis operator is given by its approximation component $\varphi_k^\perp : I_k \rightarrow I_{k+1}$, and the details $\omega_k^\perp$ having possibly several components $\omega_k^\perp = \left( \delta_{k,i}^\perp : I_k \rightarrow D_{k+1,i} \right)$, for $i = 1, \ldots, p-1$

$$\varphi_k^\perp(x)_n = y_{pn} = x_{pn} \land x_{pn+1} \land \ldots \land x_{pn+p-1}$$

$$\delta_{k,i}^\perp(x)_n = y_{pn+i} = x_{pn+i+1} - x_{pn+i}.$$  

The synthesis in this case is $\psi_k^\perp : D_{k+1,1} \times \ldots \times D_{k+1,p-1} \rightarrow I_k$,

$$\psi_k^\perp(z)_{pn} = z_{pn} = y_{pn} + 0 \lor y_{pn+1} \lor (y_{pn+1} + y_{pn+2})$$

$$\lor \ldots \lor (y_{pn+p-1} + y_{pn+p-1})$$

$$\psi_k^\perp(z)_{pn+i} = z_{pn+i} = z_{pn+i-1} - y_{pn+i}.$$  

It is easy to check that the pyramid condition holds by (2). Indeed, using the notations introduced above as,

$$\psi_k^\perp(\varphi_k^\perp(x), \delta_{k,i}^\perp(x))_n = y_{pn} + 0 \lor y_{pn+1} \lor (y_{pn+1} + y_{pn+2})$$

$$\lor \ldots \lor (y_{pn+p-1} + y_{pn+p-1})$$

$$= x_{pn} \land x_{pn+1} \land \ldots \land x_{pn+p-1}$$

$$+ 0 \lor (x_{pn} - x_{pn+1}) \lor (x_{pn} - x_{pn+2}) \lor (x_{pn} - x_{pn+p-1})$$

$$= x_{pn}$$

and by induction, if we suppose that $z_{pn+i-1} = x_{pn+i-1}$, then

$$\psi_k^\perp(\varphi_k^\perp(x), \delta_{k,i}^\perp(x))_{pn+i} = z_{pn+i} = z_{pn+i-1} - y_{pn+i}$$

$$= x_{pn+i-1} - (x_{pn+i-1} - x_{pn+i}) = x_{pn+i}.$$  

so, $\psi_k^\perp \circ (\varphi_k^\perp, \omega_k^\perp) = 1_{I_k}$.

Since $\delta_{k,i}^\perp(x)_n = x_{pn+i-1} - x_{pn+i}$ is a difference of colors of neighbor pixels one can expect that these MP-wavelets will work well with higher sampling window sizes as compared to MP-wavelets of types I.

We consider in the followings a biorthogonal-like transform for two-dimensional signals as follows. A Type-III MP-wavelet
analysis on the horizontal direction is followed by a Type III MP-wavelet analysis on the vertical direction. In order to reconstruct the original signal we employ a Type III MP-wavelet synthesis on the vertical direction followed by the synthesis on the horizontal direction.

V. EXPERIMENTAL RESULTS

An image compression/reconstruction experiment is performed, in order to study the properties of the new MP-wavelets and also to compare MP-wavelets of different types, with various sizes of the sampling windows. In this experiment, we use $3 \times 1$, $5 \times 1$, and $5 \times 5$ windows in Type-III MP-wavelets. The original image is shown in Fig. 2. It is easy to check also, that for the window size $3 \times 3$, the MP-wavelet of type I is the same as the MP-wavelet of type III. MP-wavelets of type II are not showing a significant difference with respect to type I MP-wavelets (see [7]) so we use the later ones for comparison in the present section.

The reconstructed images are shown in Figs. 3, 4, and 5, respectively.

The RMSE (Root Mean Square Error) comparison of $3 \times 1$, $5 \times 1$, and $5 \times 5$ windows in Type-III MP-wavelets is also presented in Fig. 6. As can be seen from the comparison, the performance of $5 \times 5$ window is better than that of $3 \times 1$, and $5 \times 1$.

Also, if we compare Type I and Type III MP-wavelets, we can conclude the followings. If the sampling window size is $3 \times 1$, $5 \times 1$, then there is not a significant difference in the performances of type I and III MP-wavelets. If the sampling window size is $3 \times 3$, then the two wavelets (type I and III)
coincide. If a sampling window size of 5×5 pixels is used, type III MP-wavelets outperform considerably type I MP-wavelets. Indeed, in Figure 8 reconstruction results are presented for image Lena using Type I and Type III MP-wavelets. Comparison between performances of 5x5 window based Type I and Type III MP-wavelets on different compression rates is analyzed in Fig. 7. The conclusion is that MP-wavelets of type III outperform their type I counterparts. This shows that higher sampling window size can be used with Type III MP-wavelets.

A. Conclusions

The present paper introduces and studies a new type of MP-wavelet based image coding method that keeps a good quality together with keeping the method very simple. It also works well with different window sizes. Future research is directed towards video coding using type III MP-wavelets.

Fig. 7. RMSE Comparison of type I and type III MP-wavelets with 5x5 sampling window size

Fig. 8. Reconstructed Images (5×5 channel), Left: Type I, Comp. Rate = 0.098, RMSE = 12.26, Right: Type III, Comp. Rate = 0.099, RMSE = 5.65

REFERENCES