Robust Control for Robot Manipulators By Using Only Joint Position Measurements

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Abstract—In this work, we propose an output feedback sliding mode control (SMC) method for trajectory tracking of robotic manipulators. The design process has two steps. First, we design a stable SMC approach by assuming that all state variables are available for feedback. Then, an output feedback version of this SMC design is presented which incorporates a model-free linear observer to estimate unknown velocity signals. We then show that the tracking performance under output feedback design can asymptotically recover the performance achieved under state feedback based SMC design. A detailed stability analysis is given which shows semi-global uniform ultimately boundedness property of all the closed loop signals. The proposed method is implemented and evaluated on a 2-DOF robotic system to illustrate the effectiveness of the theoretical development.

KEY WORDS: Robotics, Output Feedback, Adaptive Control.

I. INTRODUCTION

Over the past decades, the sliding mode robot control technique has been extensively studied by many researchers, (see for example [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 23], to name a few). Most SMC designs reported in the literature, however, assume that velocity signals are available for feedback control design. In practice, it is very difficult to meet this assumption as advanced industrial robotic systems provide the user with only joint position measurements while the velocity sensors are removed from the system to reduce the cost and weight. In fact, the position measurements for manipulator joint can be obtained by means of encoders or resolvers. In contrast, the joint velocity measurements are obtained by means of tachometers or by differentiating the position measurements obtained with encoders or resolvers. The velocity signals are often contaminated by severe noise. The performance of the velocity-based SMC algorithms is limited as, in practice, the measurement noise amplifies with the increase of the values of the controller gains, see for example [8] and [12]. Specifically, due to the presence of the input and output disturbance, the SMC design generates excessive chattering and infinitely fast switching control causing poor tracking performance. As a matter of fact, such a large control effort based design may not be possible to realize in the real-time operation.

To deal with this practical problem, we propose to use observer to estimate the velocity signals in the SMC scheme. In contrast with the existing nonlinear observer for the certainty equivalence (CE) principle based adaptive output feedback design [13]-[17], the proposed estimator structure is very simple in the sense that it does not require system dynamics, uncertain parameters as well as the control inputs. The proof of stability of this method has two parts. In the first part, we derive the SMC scheme as a state feedback provided that all the state variables are available for feedback. Using the Lyapunov method, we show that the state feedback based SMC design achieves desired tracking property and guarantees that all the signals in the closed-loop system are bounded. This property is established by assuming that the parameters are unknown but belong to a known compact set which is relatively large.

In the second part of the proof, we replace the unknown velocity signals by the output of the model-free estimator. For the given set of initial conditions, we first define the estimated region of interest for the state feedback based SMC design. Then, we saturate the controller outside the estimated region of interest ensuring that the output feedback controller remains bounded over the estimated region. In the face of large scale parametric uncertainty, the bounded control allows the designer to increase the speed of the observer dynamics without sacrificing the tracking performance. The proposed method can be used to formulate an output feedback form of any state feedback based SMC or adaptive SMC design reported in the literature. The rest of the paper is organized as follows. We first formulate SMC and its stability property by assuming that position-velocity signals are available for feedback design. We then propose an output feedback form of this state feedback based SMC design via replacing velocity signals by the output of the linear observer. A detailed analysis of the proposed method is given to show that the performance under SMC design can be recovered by the OFBSMC design. Adaptive version of the OFBSMC design is also given in this section. This section evaluates the stability condition on a 2-DOF robotic system that demonstrates the theoretical development of this paper. Finally, section III concludes the paper.

II. OUTPUT FEEDBACK SLIDING MODE CONTROL

In this section, we first formulate the stability criterion for the SMC as a state feedback (position-velocity) control approach. Then, an output feedback version of this SMC is presented which incorporates a linear observer in order to remove the demand of the velocity signals from existing SMC design. Let us first consider the equation of motion for an n-link rigid robot [25] in error-state space form as

\[
\dot{e}_1 = \dot{e}_2 = \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau - \ddot{q}_d
\]  

(1)
where $e_1 = q - q_d$, $e_2 = \dot{q} - \dot{q}_d$, $\phi_1(e, q_d, \dot{q}_d) = -M^{-1}(q)[C(q, \dot{q})\dot{q} + G(q)]$ and $\phi_2(e_1, q_d) = M^{-1}(q)$, $q \in \mathbb{R}^n$ are the joint position vectors, $\dot{q} \in \mathbb{R}^n$ are the joint acceleration vectors, $\tau \in \mathbb{R}^n$ is the input torque vector, $M(q) \in \mathbb{R}^{n \times n}$ is the symmetric positive definite inertia matrix, $C(q, \dot{q})\dot{q} \in \mathbb{R}^n$ is the coriolis and centrifugal loading vector and $G(q) \in \mathbb{R}^n$ is the gravitational loading vector. We first assume that the desired trajectory $q_d(t)$ and its first and second derivatives are bounded as $Q_d \in \mathbb{R}^{3n} = [q_d, \dot{q}_d, \ddot{q}_d]^T \in \Omega_d$ with compact set $\Omega_d$. Let us consider the reference state as $\dot{q}_r = \dot{q}_d - \lambda e_1$, where $\lambda = diag[\lambda_1, \lambda_2, ..., \lambda_n]$ with $\lambda_n > 0$. Then, we define the sliding surface $S = e_2 + \lambda e_1$. The control objective is to drive the joint position $q(t)$ to the desired position $q_d(t)$. To obtain the control objective, we consider the following control law for the system (1)

$$\tau(e, Q_d, \dot{\theta}) = \hat{M}(q)\dot{q}_r + \hat{C}(q, \dot{q})\dot{q}_r + \hat{G}(q) - KS - K_{sat}(S)$$

(2)

where $\dot{q}_r = \dot{q}_d - \lambda e_2$, $\hat{M}$, $\hat{C}$ and $\hat{G}$ are the estimates of $M(q)$, $C(q, \dot{q})$, $K = diag[K_1, K_2, ..., K_n]$, $\hat{K} = diag[\hat{K}_1, \hat{K}_2, ..., \hat{K}_n]$, where $K_i$ and $\hat{K}_i$ with $n = 1, 2, 3, ..., n$ are positive constants. Now using (1) and (2), we simplify the closed-loop dynamics as $\dot{S} = (C + K)S = \Delta \beta - K_{sat}(S)$ with $\Delta \beta = (\hat{M} - M)\dot{q}_r + (\hat{C} - C)\dot{q}_r + (\hat{G} - G) = \Delta M\dot{q}_r + \Delta C\dot{q}_r$, where $\Delta M = \hat{M} - M$, $\Delta C = \hat{C} - C$ and $\Delta G = \hat{G} - G$. We now explore the convergence condition of the closed-loop system in the Lyapunov-sense. To do that, we first define the following positive definite Lyapunov-like candidate function

$$V = \frac{1}{2}S^TMS$$

(3)

We take the derivative along the closed-loop trajectory, and then use the property 2 of [17] to obtain $\dot{V} = -S^T KS - \sum_{i=1}^n (S_i[K_i sat(|S_i|) - \Delta \beta_i])$. We now consider that model parameters are unknown but belong to a known compact set $\Omega$. We also consider that the initial state errors $e_1(0)$ and $e_2(0)$ belong to another known compact set $\Omega_{e0}$. Using $\sum_{i=1}^n (S_i[K_i sat(|S_i|) - \Delta \beta_i]) \geq 0$ with $K_i \geq \Delta \beta_i$, we can show that the derivative of the Lyapunov function $\dot{V}$ satisfies the inequality $\dot{V} \leq -S^T KS$. This implies that the sliding surface satisfies the sufficient condition as $\frac{1}{2}\dot{S}^T S \leq -\eta_0 |S|$, where $\eta_0$ is a positive constant [2] and [18]. Hence, the energy of $S$ will be decaying as long as $S \neq 0$. To generate the sliding control, one estimates the switching function $sgn(.)$ by using a smooth bounded function [18].

A. Output feedback sliding mode control (OFSMC) design

The presentation above is based on the strict assumption that all the state variables are available for feedback. To date, most existing SMC designs in the literature are based on this assumption. However, these algorithms cannot be applied directly on industrial manipulators. This is because advanced industrial manipulators (e.g., those from CRS Robotics Ltd, Applied AI Inc., etc.) do not have velocity sensors to reduce the weight and cost of the system. To obtain the velocity signal, the common practical approach is to differentiate the position measurements obtained from encoders or resolvers which are often contaminated by severe noise [8], [12] and [24]. As a consequence, the performance under state feedback based SMC approach is limited as, in practice, the measurement noise is amplified with the increase of the values of controller gains. In the face of the large parametric uncertainty, the demands of high control gains makes the design even more complex as high gains intensifies high-frequency control chattering activity. To deal with this practical problem, we propose an output feedback form of SMC (2) as $\tau(e, Q_d, \ddot{\theta}) = M(q)\dot{q}_r + \hat{C}(q, e_2 + \dot{q}_d)\dot{q}_r + \hat{G}(q) - K(e_2 + \lambda e_1) - K_{sat}(\frac{\dot{e}_1}{\epsilon})$, where $sat(.)$ is a bounded smooth saturation function, $\dot{q}_r = \dot{q}_d - \lambda e_1$, $\dot{q}_r = \dot{q}_d - \lambda e_2$, $\dot{S} = \dot{e}_2 + \lambda e_1$ and the unknown velocity signals $\dot{e}_2$ is replaced by the output of the following linear estimator

$$\dot{\hat{e}}_1 = \frac{H_1}{\epsilon} e_1 + \frac{H_2}{\epsilon^2} \dot{\hat{e}}_2$$

(4)

where $\hat{e}_1 = e_1 - \hat{e}_1$, $\hat{e}_2 = e_2 - \hat{e}_2$, and $H_1$ and $H_2$ are the positive constant matrices. In contrast with CE-based nonlinear adaptive output feedback design given in [13]-[17] and [20], the observer structure (4) is independent of the system dynamics, uncertain model parameters and control inputs. The observer dynamics (4) can be viewed as a simple chain of integrator plus correction terms injected by the output error term. To analyze the convergence rate for the OFBSMC design, we use singular perturbation method. To begin with that, we first define an observer error dynamics. Using OFBSMC, one obtains $\dot{e}_2 \approx \phi_2(e_1, q_d, \dot{q}_d)\tau(e, \dot{q}_d, \ddot{\theta})$. Then, the observer error becomes $\dot{\hat{e}}_1 = \frac{H_1}{\epsilon} e_1 + \frac{H_2}{\epsilon^2} \dot{\hat{e}}_2 = -\dot{q}_d + \phi_1(e_1, q_d, q_d) + \phi_2(e_1, q_d)\tau(e, \dot{q}_d, \ddot{\theta}) - \frac{H_2}{\epsilon^2} \dot{e}_1$. We now replace the observer errors by the scaled estimation error to form a singularly perturbed system as $\eta_1 = \frac{\epsilon}{\epsilon^2} m \Rightarrow \epsilon \eta_1 = \dot{\hat{e}}_1$ and $\eta_2 = e_2 - \hat{e}_2 = \hat{e}_2$ with a small positive parameter $\epsilon$. Using this scaled estimation error, one gets $\epsilon \eta_1 = Be_1 - \dot{\hat{e}}_1 + \phi_1(e_1, q_d, \dot{q}_d) + \phi_2(e_1, \dot{q}_d)\tau(e, \dot{q}_d, \ddot{\theta}) + A\eta_1$, where $A_0 = [-H_1 I]$ and $\zeta(\epsilon) = [\epsilon I_{n\times n} 0_{n\times n}]$. The positive constants $H_1$ and $H_2$ are chosen such that the matrix $A_0$ is Hurwitz. We now show that the performance achieved under
The transient period. To obtain such a bounded control, let

\begin{equation}
\dot{\epsilon} = B_\epsilon \left[ \phi_1(\epsilon, q_\epsilon, \dot{q}_\epsilon) + \phi_2(\epsilon_1, q_\epsilon) \tau((e - \zeta(\eta, \epsilon))Q_{\epsilon, \theta} - \dot{q}_\epsilon) \right] + A_\epsilon e
\end{equation}

\begin{equation}
\epsilon_\eta = B_{\epsilon, \eta} \epsilon_\eta = \phi_1(\epsilon, q_\epsilon, \dot{q}_\epsilon) + \phi_2(\epsilon_1, q_\epsilon) \tau((e - \zeta(\eta, \epsilon))Q_{\epsilon, \theta} - \dot{q}_\epsilon) + A_{\epsilon, \eta} \epsilon
\end{equation}

where \( A = \begin{bmatrix} 0 & I_{n \times n} \\ 0 & 0 \end{bmatrix} \). To begin with the recovery analysis, we first assume that the system output and its derivatives are available for feedback. Then, design a SMC as a state feedback control law (2) such that the design meets the desired tracking objectives. We then replace the unknown velocity state vectors in SMC by the output of the observer (4). If we increase the initial state estimates as well as the initial parameter estimates then the observer speed will also require to increase for robust reconstruction of unknown states. The problem with high speed observer is that it causes large control action during transient phase. To protect the plant from transient control effect, we saturate the control input within the estimated region of interest of the state feedback based SMC scheme. This saturation function will only be active during the transient period. To obtain such a bounded control, let us consider that \( \Omega_c = \{ e \mid e^T Q_{sm} e \leq c \} \), \( e > 0 \), be an estimate of the region of attraction of the state feedback based SMC design, \( \tau(e, Q_{\epsilon, \theta}) \), where \( Q_{sm} = \begin{bmatrix} \lambda^2 M & \lambda M \\ I & M \end{bmatrix} \).

Since \( \tau(e, Q_{\epsilon, \theta}) \) is a continuous function over \( e, Q_{\epsilon, \theta} \), then there always exists a maximum control \( \tau_{i, \text{max}} = \max|\tau_i(e, Q_{\epsilon, \theta})| \), \( 1 \leq i \leq n \), such that the input can be saturated over the compact set \( \Omega_c \) that satisfies \( \tau_i(e, Q_{\epsilon, \theta}) = \tau_{i, \text{max}} \text{sat}(\tau_i(e, Q_{\epsilon, \theta})) \), \( \forall 0 \leq e \in \Omega_c \), \( \theta \in \Theta \), \( Q_{\epsilon, \theta} \in \Omega_{Q_{\epsilon, \theta}} \) and \( \tau_{i, \text{max}} \) is taken over for all \( e \in \Omega_c \), with \( c_r > c \) and \( \text{sat}(.) \) is a smooth bounded saturation function. Then, \( \forall \theta(t) \in \Theta, \epsilon(t) \in \Omega_c, \text{sat}(.) \) is one has \( |\tau_i(e, Q_{\epsilon, \theta})| \leq \tau_{i, \text{max}} \) \( \forall t \geq 0 \) and \( \tau_i(e, Q_{\epsilon, \theta}) \) is globally bounded [19]. We now replace the state vectors \( e \) in the control law by state estimator \( \hat{e} \). Then, the bounded OFBSMC can also be achieved via saturating the outside of the region of interest \( \Omega_c \) as \( \tau_{i}(\hat{e}, Q_{\epsilon, \hat{\theta}}) = \tau_{i, \text{max}} \text{sat}(\tau_{i}(\hat{e}, Q_{\epsilon, \hat{\theta}})) \), \( \forall \hat{e} \in \Omega_{c, \hat{\theta}}, \forall \hat{\theta} \in \Theta, \forall e \in \Omega_c, \forall \theta \in \Theta, \forall \hat{\theta} \in \Theta \). Note that the bounded output feedback control will be required when unknown parameters and initial conditions belong to large compact set. We now summarize our main results for the OFBSMC design.

**Theorem 1:** Consider the closed-loop system (5) and (6). Then, for any given compact set of \( e(0) \in \Omega_{c, \epsilon} \), \( \hat{e}(0) \in \Omega_{c, \theta} \) and \( \hat{\theta}(0) \in \Omega_{\epsilon, \hat{\theta}} \) there exists a small \( c_\epsilon^* \) such that for all \( 0 < \epsilon < c_\epsilon^* \), all the state variables of the closed-loop systems are bounded by a bound that can be made arbitrary small by using small value of \( c_\epsilon^* \).

**Proof:** The proof of the above Theorem 1 consists of two parts. In the first part, we show that there exists a short transient period \( T_1(e) \in [0, T_2] \) during which the fast variable \( \eta \) approaches a function of the order \( O(\epsilon) \), while the slow variables \( (e, \hat{\theta}) \) remain in a subset of the domain of attraction. In the second part, we establish the boundedness of the signal \( e(t) \) for all \( t \in [T_1(e), T_2] \), where \( T_1(e) \in [0, \frac{\epsilon}{\epsilon^*}] \) and \( T_2 \geq T_2 \) is the first time \( (e(t), \hat{\theta}(t)) \) exists from the set \( \Omega_c \). This part shows that the state variables \( (e(t), \hat{\theta}(t)) \) remain bounded for \( t \geq 0 \).

**Part 1:** We first show that there exists a finite time \( T_2 \), independent of \( \epsilon \), such that for all \( t \in [0, T_2] \) slow variable remains bounded in the set \( \Omega_c \). To show that, we define the positive definite Lyapunov like function candidate \( V = \frac{1}{2} S^T M S \). We also consider that all initial conditions are bounded. We choose \( e(0) \in \Omega_{c, \epsilon} \subset \Omega_{c, \theta} \), that includes \( e(0) \in \Omega_{c, \epsilon} \) and \( e_\eta(0) \in \Omega_{c, \eta} \), where \( \Omega_{c, \eta} \) is the domain of attraction, \( \Omega_{c, \epsilon} \) is the compact set chosen to carry any bounded initial condition. Then, for the given initial set of \( e_1 \) and \( e_2 \), we have \( c_4 = \max_{e_1 \in \Omega_{c, \epsilon}, e_2 \in \Omega_{c, \eta}} \frac{1}{2} S^T M S \), where \( c_4 > 0 \). Then define the compact set as \( e(t) \in \Omega_{c, \epsilon} \) with \( c > c_4 \). Now, our aim is to prove that the energy function \( V \) remains bounded by a constant \( c \). To verify that we first take the derivative of the Lyapunov function (3). Then we use the property 2 of [17] and the bounded output-feedback law on the set \( \Omega_{c, \epsilon} \) as \( \tau_{i}(\hat{e}, Q_{\epsilon, \hat{\theta}}) = \left[ \tau_{i}(e, Q_{\epsilon, \hat{\theta}}) - \tau_{i}(\zeta(\eta, e), Q_{\epsilon, \hat{\theta}}) \right] \) with \( \tau(e, Q_{\epsilon, \hat{\theta}}) = \tau_{i}(e, Q_{\epsilon, \hat{\theta}}) \) to simplify the derivative of \( V \) as

\begin{equation}
\dot{V} = -S^T KS - \sum_{i=1}^{n} \left( S_i \left[ K_{i, \eta} \left( \frac{|S_i|}{\phi_i} - \Delta i \right) \right] - S_i^T \tau_{i}(e, Q_{\epsilon, \hat{\theta}}) \right)
\end{equation}
\( C \lambda \eta \eta + (G - G) \| \leq k_{sm} \) for \( k_{sm} > 0 \). Then, we can write \( \hat{V} \) as \( \hat{V} \leq -\sum_{i=1}^{n} \left( S_i K_{sat} \frac{\| S_i \|}{\| \phi_i \|} - \Delta \eta_i - k_{sm} \right) - S^T K S - S^T \tau_s (\zeta(e, \eta_i, Q_d, \hat{\theta})) \). Now, for \( \| K_i \| \geq |\Delta \beta_i| \), we have \( \| S_i K_{sat} \| - \Delta \beta_i \geq 0 \). As \( \eta \) enters into slow subsystem via bounded function then \( \hat{V} \) can be simplified further as \( \hat{V} \leq -\eta_0 V + \alpha_0 \forall \theta \in \Omega, \forall \theta \in \Omega, \) where \( \alpha_0 = \gamma_a k_{sm} + \gamma_a \alpha_1 \), \( \gamma_a = \frac{\lambda_{min}(K)}{4}, \gamma_a \) and \( \alpha_1 \) are the bound for \( \| S \| \) and \( \| \eta \| \) over the set \( \Omega \), respectively. Therefore, the result for \( \hat{V} \) can be derived as \( V(t) \leq V(0) e^{-\eta_0 \epsilon + \alpha_0 (1 - e^{-\eta_0 \epsilon})} \). As \( V(0) \leq c_3 < c \), then we conclude that there always exists a finite time \( T_2 \), independent of \( \epsilon \), such that for all \( t \in [0, T_2] \), \( V(t) \leq c \). We now prove that over the time interval \( [0, T_2] \) the fast variable \( \eta \) converges to a very small value. To show that, let us consider the following Lyapunov function candidate for the fast observer-error model (6)

\[
W(\eta) = \eta^T P \eta
\]

where \( P = P^T > 0 \) is the solution of the Lyapunov equation \( PA_\omega + A_\omega^T P = -I \). Applying \( PA_\omega + A_\omega^T P = -I \), one can simplify the derivative of (8) along the trajectory (6) as \( \dot{W}(\eta) \leq -\frac{1}{2} \| \eta \|^2 + 2\gamma \epsilon |P| [-\eta_0 + \phi_1(e, q_d, \eta_d) + \phi_2(e, q_d) \tau_s ((e - \zeta(\epsilon, \eta), Q_d, \hat{\theta})] \). Now, for any given \( e(0) \in \Omega_{cov}, \hat{e}(0) \in \Omega_{cov}, \theta(0) \in \Omega \) and \( \theta(0) \in \Omega \), we have the following bound

\[
\| [-\eta_0 + \phi_1(e, q_d, \eta_d) + \phi_2(e, q_d) \tau_s ((e - \zeta(\epsilon, \eta), Q_d, \hat{\theta})] \| \leq k_1
\]

for \( k_1 > 0 \). We omit the remaining proof of the part 1 for brevity.

**Part 2:** Let us now study the slow subsystems over the time interval \([T_1(\epsilon), T_3(\epsilon)]\). That is to study the property of the slow variable \((e, \hat{\theta})\) when \( \| \eta \| \) converges closed to the origin. For the time interval \([T_1(\epsilon), T_3(\epsilon)]\), we choose \( e(0) \in \Omega_{cov} \) and \( \theta(0) \in \Omega \). Then we can write the tracking error model as

\[
\dot{e}_2 = \phi_1(e, q_d, \hat{q}_d) + \phi_2(e, q_d) \tau_s ((e, Q_d, \hat{\theta}) - \hat{q}_d) + \phi_2(e, q_d) \tau_s (e, Q_d, \hat{\theta})
\]

This can be viewed as a perturbed closed-loop model under state feedback over the time interval \([T_1(\epsilon), T_3(\epsilon)]\). From part 1, we already know that the perturbation term \( \| \eta \| \) decays to a level where \( \| \eta \| \) is of order \( O(\epsilon) \) and \( W(\eta(t)) \leq \epsilon^2 \beta \). Due to space limitation, we remove the remaining proof of the part 2.

**B. Adaptive sliding mode control (ASMC)**

The level of uncertainty in classical SMC and OFBSMC design can be reduced by adding an adaptation term. For this purpose, we first propose to introduce an estimation algorithm to develop an adaptive SMC algorithm as a state feedback

\[
\dot{\tau}(e, Q_d, \hat{\theta}) = Y(e, q_r, \hat{q}_r)^T \hat{\theta} - K S - K sat \left( \frac{S}{\phi} \right)
\]

with \( \dot{\hat{\theta}} = -\Gamma Y^T (e, q_r, \hat{q}_r) S \) where \( (e, q_r, \hat{q}_r) = M(q) \hat{q}_d + \hat{C}(q, \hat{q}_r) \hat{q}_d + \hat{G}(q) \), \( \hat{\theta} \) is the estimate of the robot dynamics and its operating environments such as link masses and payloads, \( \Gamma = diag[\Gamma_1, \Gamma_2, \ldots, \Gamma_n] \) with constant diagonal elements \( \Gamma_n > 0 \) and \( \hat{Y}(e, \hat{q}_r, \hat{q}_r) \) is the regressor matrix. The parameter estimates \( \hat{\theta} \) can be adjusted with the smooth parameter projection scheme [22] as \( \hat{\theta} = [Proj(\hat{\theta}, \Psi_i)] \), where \( \theta(0) = \{ \theta \mid a_i \leq \theta_i \leq b_i, 1 \leq i \leq p \} \), where \( \Psi_i \) is the i-th element of the column vector \(-Y^T (e, \hat{q}_r, \hat{q}_r) S - K sat \left( \frac{S}{\phi} \right) \). The proposed adaptive control law is designed by using the control Lyapunov function as \( V = \frac{1}{2} S^T M S + \frac{1}{2} \hat{\theta}^T \hat{\theta} \), where \( \hat{\theta} = (\theta - \hat{\theta}) \). Using the property 2 of [17], the time derivative \( V \) along the closed-loop error trajectories can be simplified as \( \dot{V}(e, \hat{\theta}) \leq -\lambda_{max}(K) || S \|^2 - K || S \| \leq 0 \forall e(0) \in \Omega_{cov}, \forall \theta(0) \in \Omega \) and \( \theta(t) \in \Omega \).

**C. Adaptive output feedback sliding mode control (AOFSMC)**

The above design is implementable only when all the process states are measurable. To relax this strict assumption, we now replace the unknown state vectors in (11) by the output of the estimator (4) as \( \tau(e, Q_d, \hat{\theta}) = Y(e, \hat{q}_r, \hat{q}_r)^T \hat{\theta} = \hat{K} S - K sat \left( \frac{S}{\phi} \right) \) with \( \hat{\theta} = (\theta - \hat{\theta}) \). Using the parameter estimates, we may use parameter projection scheme, where the state estimates replaced by the output of the observer (4). Then the closed-loop model under AOFBSMC approach has the following form

\[
\epsilon = B \left[ \phi_1(e, q_d, \hat{q}_d) + \phi_2(e, q_d) \tau_s ((e - \zeta(\epsilon, \eta), Q_d, \hat{\theta}) - \hat{q}_d) \\
+ A e \right] + \hat{\theta} = \frac{\tau(e, q_d, \hat{q}_d) + \phi_2(e, q_d)(e - \zeta(\epsilon, \eta), Q_d, \hat{\theta}) + A e}{\eta_i}
\]

with \( \hat{\theta} = Proj(\hat{\theta}, \Psi(e - \zeta(\epsilon, \eta), Q_d, \hat{\theta})) \). Then, we consider that for any given \( e(0) \in \Omega_{cov}, \hat{e}(0) \in \Omega_{cov}, \theta(0) \in \Omega \) and
in practice. Note also that these inequalities do not enforce
for brevity.

Proof: The proof of stability of Theorem 2 can be shown along
the line of Theorem 1. So we removed the stability analysis for brevity.

Remark 1: It is worth noting that the bounded inequality (9)
and (14) does not impose any restriction either in theory or
in practice. Note also that these inequalities do not enforce
growth condition on systems nonlinearities. The constant $k_1$
and $k_{2o}$ (that used in observer error analysis) are used to
establish the semi-global stability property of theorems 1 and
2. For the given set of initial condition of interest, the designer
can calculate the value of $k_1$ and $k_{2o}$ over the domain of
attraction $\Omega$. The value of $k_1$ and $k_{2o}$ can be calculated
as follows. For the given $e(0) \in \Omega$ and $\theta(0) \in \Omega$ and $\bar{\theta}(0) \in \Omega$
we calculate the terms $\phi_1(e, q_d, \dot{q}_d)$ and $\phi_2(e, q_d).
Then, for given $q_d$ and initial conditions of interest, we define
the saturation level $\tau_{max}$ (maximum bound on $\tau(e, \phi, \dot{\theta})$). For a
small value of $\epsilon$ and initial error estimates $e(0) \in \Omega_c$, we then
calculate the bound on output feedback $\tau^* = \max(\epsilon - \zeta(\epsilon \eta), Q_d, \dot{\theta})$.

Design and Implementation process: In this part, we
analyze the property of the Lyapunov stability argument estab-
lished in Theorem 2 on a 2-DOF serial link manipulator, which
has the following dynamical model [17]:

$$m_{11} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

where $m_{11} = (\theta_1 + 2\theta_2 + 2\theta_2 \cos q_2)$, $m_{12} = (\theta_2 + 2\theta_2 \sin q_2)$, $m_{21} = (\theta_2 + 2\theta_2 \cos q_2)$, $m_{22} = 2\theta_2$, $c_{11} = -2q_2\theta_2 \sin q_2$, $c_{12} = -q_2\theta_2 \sin q_2$, $c_{21} = q_2\theta_2 \sin q_2$, $c_{22} = 0$, where $\theta_1 = m_1 l^2$, $\theta_2 = m_2 l^2$ and $l$ is the link
lengths and $m_1$ and $m_2$ are the masses of link 1 and link
2, respectively. Due to the space limit, we have removed the
evaluation of Theorem 1. We assume that the parameters $\theta_1$
and $\theta_2$ of the given robot dynamics are unknown but belong to
comparatively large compact sets as $\theta \in R^2 \in \Omega = \{-10 < \theta_1 < 10, -10 < \theta_2 < 10\}$. Suppose that the initial states
$[e(0)] \leq 2, \bar{e}(0) \leq 2$ and the initial estimated parameters as
$\theta(0) = 0$. In our evaluation, we will examine the tracking
convergence property of ASMC and AFOSMC approach
with respect to estimation errors as $\theta = 10$. To generate
smooth control input, we use bounded function sat(.) instead of
sign(.) function. The value of $\lambda$ and $\bar{\lambda}$ are chosen such that
they ensures an acceptable transient and steady state tracking
error objectives [26]. For our evaluation, we consider $\bar{\lambda}$ as
$\bar{\lambda}_1 = 125$ and $\bar{\lambda}_2 = 125$. The bandwidth of the controller is
chosen as $\lambda_1 = \lambda_2 = 2$. The value of $\Gamma$, $K$ and $\frac{1}{\phi}$ are selected
in such a way that guarantees fast parameter learning. As the
results are semi-global, then the designer can select the control
gains $\Gamma$, $\bar{\lambda}$ and provide that $\theta \in \Omega$. For our evaluation,
we choose the value of discontinuous and learning gains $\Gamma$, $K$ and $\frac{1}{\phi}$ as $\Gamma = 10 I_{2 \times 2}$, $K = 15 I_{2 \times 2}$ and $\phi = 0.7$. Note that,
for the given set, the designer can increase the value of $\phi$ and
decrease the value of $\bar{\lambda}$. With these control design parameters,
let us apply ASMC design (11) on the given model. The
tested results are given in Figure 1 with the chosen parameter
designator errors as $\theta = 10$ (black-solid). It is worth noting
that the value of $\Gamma$, $K$ and $\frac{1}{\phi}$ will increase with the increase
of the parametric uncertainties, that is, $\Omega$ and $\Omega_{co}$. The main
practical problem is that high gains are mainly responsible for
amplifying the output and input disturbances in state feedback
based SMC approach. In practice, high control gain may
excite hidden unmodeled dynamics resulting poor tracking
performance. This is because high-gain intensifies the control
chattering activity as it amplifies the disturbances associated
with the output (position-velocity measurements), input and
disturbance such as backlash, friction, external disturbance,
entering into the closed system [8], [12] and [25]. To illustrate
this argument, we now examine the tracking performance of
ASMC algorithms under non ideal situation where we add a
band limited white disturbance noise, $\omega(t)$, into the output
$q(t)$ (position measurement) input $\tau(t)$ to the system. For our
evaluation, the level $\omega(t)$ for the output $q(t)$ and input $\tau(t)$
are considered as given in Figure 2. Then, we apply ASMC
design (11) on the given system with the estimation errors
as $\theta = 10$. The tested results under $\theta = 10$ are depicted
in Figure 3. We notice from our implementation that the
tracking errors increase with the increase of the parametric
uncertainty. More specifically, the control effort becomes very
large that may not be possible to realize as available control
inputs are restricted in real-world operation. This large control
action is mainly because of the derivative action of the noisy
position signals that used in position-velocity based ASMC
law (11). Now our aim is to show that the large control effort
under position-velocity based ASMC design can be reduced
by using AOFBSMC. We keep the same controller design
parameters as used for the evaluation of the position-velocity

\[ \dot{\theta}(0) \in \Omega, \text{ the following inequality holds} \]

\[ ||\dot{\theta}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e, q_d)\tau^*(e - \zeta(\epsilon \eta), Q_d, \dot{\theta})|| \leq k_{2o} \]

for $k_{2o} > 0$. We now state the main results for the AOFBSMC
design in the following Theorem 2.

**Theorem 2**: Let us consider the closed-loop system (12)-(13).
Suppose that $\dot{\theta}(0) \in \Omega$ and $(e(0), \bar{e}(0)) \in \Omega_{co}$ is bounded.
Then, there exists $\epsilon_1$ such that for all $0 < \epsilon < \epsilon_1$, all the
signals in the closed-loop system will be bounded by a bound
that can be made arbitrarily small for small value of the
observer design constant $\epsilon$.

**Proof**: The proof of stability of Theorem 2 can be shown along
the line of Theorem 1. So we removed the stability analysis for brevity.
based ASMC algorithm. Then, we choose observer design constants of $H_1$ and $H_2$ as $H_1 = I_{2 \times 2}$, $H_2 = I_{2 \times 2}$ and $\epsilon = 0.05$ that guarantees fast recovery of unknown velocity signal. With these set up, now apply the AOFBSMC on the given system. The results are given in Figure 4. We can see from our results that the performance under AOFBSMC design is similar to the performance under ASMC design (11). We now add a band limited white noise, $w(t)$, into the output measurement $q(t)$ and input $\tau(t)$ of the system as depicted in figure 3. Then, using the following observer-controller design parameters as $K_1 = 125$, $K_2 = 125$, $\Gamma = 10I_{2 \times 2}$, $K_1 = 15$, $K_2 = 15$, $\phi_1 = 0.7$, $\phi_2 = 0.7$, $\epsilon = 0.05$, $H_1 = 5I_{2 \times 2}$ and $H_2 = 5I_{2 \times 2}$, we apply AOFBSMC on the given robot system. The tested results are presented in figure 5 with $\dot{\theta} = 10$. By comparing figures 3 (ASMC) and 5 (AOFBSMC), we can observe that the tracking performance under AOFBSMC is almost closed to zero while relatively large tracking errors can be seen under ASMC design. Notice also from these results that the required control effort under AOFBSMC (see figure 5, for $\dot{\theta} = 10$) is much smaller than the control effort demanded under ASMC (see figure 3, for $\dot{\theta} = 10$) design.

III. CONCLUSION

In this paper, an output feedback sliding mode robot control method has been proposed to deal with the problem associated with the velocity based SMC design. The Lyapunov method has been utilized to establish the semi-global stability condition of all the closed-loop signals. This property has been shown using parameter projection and control saturation technique. The tested results validate the Lyapunov stability arguments of the proposed method. The evaluation on a 2-DOF robotic system has been shown to demonstrate the theoretical development for the real-time application.

REFERENCES