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Robust Tracking Control of an Underactuated Quadrotor Aerial-Robot Based on a Parametric Uncertain Model

DongBin Lee, Timothy C. Burg, Darren M. Dawson, Dule Shu
Department of Electrical and Computer Engineering
Clemson University
Clemson, SC 29634-0915, USA
dlee, tburg, dshu@clemson.edu, darren.dawson@ces.clemson.edu

Bin Xian
School of EE & Automation, Department of EEE
TianJin University, TianJin 300072, P.R. China
xbin@tju.edu.cn

Enver Tatlicioglu
Izmir Institute of Technology, Urla, Izmir 35430, Turkey
enver@envertatlicioglu.com

***Abstract*—In this paper, the tracking control of a underactuated quadrotor aerial vehicle is presented where position and yaw trajectory tracking is achieved using feedback control system. The control design is complicated by considering parametric uncertainty in the dynamic modeling of the quadrotor aerial-robot. Robust control schemes are then designed using a Lyapunov-based approach to compensate for the unknown parameters in each dynamic subsystem model. Lyapunov-type stability analysis suggests a global uniform ultimately bounded (GUUB) tracking result.**

***Index Terms*—uncertain, robust, Quadrotor UAV, robot, tracking, underactuated, Lyapunov stability**

I. Introduction

The quadrotor helicopter system is underactuated in the sense that the system is considered to have four control inputs to move and position the six degree-of-freedom system. Although the system has four independently controllable motor-rotor sets, a built-in strategy to group these motor-rotor sets to create vertical thrust, and roll, pitch, and yaw torques is presumed. An important property of the quadrotor system is that it can be modeled as coupled translational and angular rotational subdynamics where the output of the rotational dynamics are cascaded as the input to the translational dynamics. The coupled and cascaded dynamic model is used as the basis for the control design. The coupling between linear and angular dynamic subsystems occurs in the gyroscopic terms which are typically neglected in a hovering model.

Many researchers have proposed a variety control solutions for the underactuated quadrotor system. Of special significance, in [12], the control compensates for wind affects acting on the underactuated quadrotor. The work in [2] presented model-based control techniques applied to an underactuated quadrotor. Of particular note, the system dynamics include nonlinearities in the aerodynamic forces. The authors in [1] proposed a solution of the trajectory-tracking and path-following for underactuated autonomous vehicles such as underwater vehicle and a hovercraft in the presence of parametric modeling uncertainty.

A feasible control approach when the dynamic model includes gyroscopic effects is to use a backstepping approach. The work in [6] presented a trajectory tracking controller for an underactuated small helicopter using a backstepping procedure. In [7], a backstepping approach to control a specific model of a quadrotor, the X4 flyer, is presented. This work includes the dynamic complication of the aerodynamic and gyroscopic effects of the rotating blades. The work in [3] presented attitude stabilization of the quadrotor aircraft using the backstepping technique.

This paper emphasizes the control of an underactuated quadrotor to obtain position tracking along the x-, y-, and z-axes and the yaw angle in the presence of parametric uncertainty. In Section II, a brief outline of a quadrotor-helicopter model is presented along with a property and assumptions used in the control design. In Section III, the definitions of error signals are developed based on state feedback tracking control, a backstepping approach is introduced to the dynamic model of quadrotor, and a robust controller for compensating the unknown constant parameters is suggested. Stability analysis on the controller is considered in Section IV followed by a theorem. Concluding remarks are presented in Section V.

II. System Model

The translational and rotational kinematic equations of the quadrotor unmanned helicopter are given [11] by

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} R^T(\Theta) & O_{3\times 3} \\ O_{3\times 3} & T^{-1}(\Theta) \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{\Theta} \end{bmatrix} \in \mathbf{R}^6 \tag{1}$$

where $v(t)$, $\omega(t) \in \mathbf{R}^3$ denote the linear velocity and the angular velocity, respectively. The Euler-based rotation matrix $R(\Theta) = R_{x,\phi} \cdot R_{y,\theta} \cdot R_{z,\psi} \in SO(3)$ that translates a body-fixed frame referenced quantity into inertial coordinates is calculated from

$$R(\Theta) = \begin{bmatrix} c\psi c\theta & c\psi s\theta s\phi - s\psi c\phi & s\psi s\phi + c\psi c\phi s\theta \\ s\psi c\theta & c\psi c\phi + s\phi s\theta s\psi & s\theta s\psi c\phi - c\psi s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \tag{2}$$

where $\Theta(t) = [\phi, \theta, \psi]^T \in \mathbf{R}^3$ are the Euler angles, $c\cdot = \cos(\cdot)$ and $s\cdot = \sin(\cdot)$ are used. The body-fixed angular velocities are transformed by the matrix $T(\Theta) \in \mathbf{R}^{3\times 3}$, into the inertial frame and is given by

$$T(\Theta) = \begin{bmatrix} T_x(\Theta) \\ T_y(\Theta) \\ T_z(\Theta) \end{bmatrix} = \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix}. \quad (3)$$

The dynamic model of the underactuated quadrotor is given in the body-fixed reference frame by

$$\begin{bmatrix} mI_3 & O_{3\times 3} \\ O_{3\times 3} & J \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \text{-}mS(\omega) & O_{3\times 3} \\ O_{3\times 3} & \text{-}S(\omega)J \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} - \begin{bmatrix} N_1(\theta_1, v, |v|) \\ N_2(\theta_2, v, |v|) \end{bmatrix} + \begin{bmatrix} G(R) \\ O_{3\times 1} \end{bmatrix} + \begin{bmatrix} B_1, O_{3\times 3} \\ O_{3\times 1}, B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (4)$$

where $N_1(\theta_1, v, |v|)$ are aerodynamic forces on the rigid-body and $N_2(\theta_2, v, |v|)$ denotes aerodynamic induced moments where θ_i is a parameter vector and $|v|$ is the norm of linear velocity vector $v(t)$, and $\text{-}S(\omega)Jw = S(J\omega)w$. (See [11] for details). $m \in \mathbf{R}^1$ is the mass of the quad-rotor, $J \in \mathbf{R}^{3\times 3}$ denotes a positive definite diagonal inertia matrix, and $g \in \mathbf{R}^1$ denotes the gravitational acceleration due to the gravity. These are all assumed to be unknown constants including the coefficients of the aerodynamic terms.

The dynamic system in (4) has the following property.

P1: The unknown system parameters are upper and lower bounded to satisfy the following inequalities

$$\underline{\theta}_{ij} \leq \theta_{ij} \leq \overline{\theta}_{ij} \quad (5)$$

where θ_i is the ith parameter of the jth vector θ_j. The aerodynamic forces and moments in (4) can be linearly parameterized in the form

$$\begin{aligned} N_1(\theta_1, v, |v|) &\equiv Y_1(v, |v|)\theta_1, \\ N_2(\theta_2, v, |v|) &\equiv Y_2(v, |v|)\theta_2 \end{aligned} \quad (6)$$

where $Y_1(v, |v|) \in \mathbf{R}^{3\times 3}$ and $Y_2(v, |v|) \in \mathbf{R}^{3\times 3}$ are known regression matrices, $\theta_1 \in \mathbf{R}^3$ and $\theta_2 \in \mathbf{R}^3$ are unknown constant parameter vectors.

The following assumption is made regarding the dynamic model.

A1: The pitch angle (θ) does not exceed $\pm\frac{\pi}{2}$ so that $T^{-1}(\Theta)$ is invertible.

III. Tracking Controller Design

The goal of the tracking controller is to force the aerial vehicle to track a desired trajectory. The error formulation for the tracking is developed and a backstepping approach for the coupled and cascaded system is utilized to facilitate the controller specification. The desired trajectories and up to their third derivatives are all bounded; i.e., $p_d(t)$, $\dot{p}_d(t)$, $\ddot{p}_d(t)$, and $\dddot{p}_d(t) \in \mathcal{L}_\infty$ and $\psi_d(t)$, $\dot{\psi}_d(t)$, and $\ddot{\psi}_d(t) \in \mathcal{L}_\infty$.

A. Error System Development

The position tracking error, denoted as $e_p(t)$, is defined in the body-fixed frame as the transformed difference between the inertial-frame based position, $p(t)$, and the inertial-frame based desired position, denoted as $p_d(t) \in \mathbf{R}^3$, in the manner

$$e_p = R^T(p - p_d) \in \mathbf{R}^3. \quad (7)$$

The position tracking error rate, $\dot{e}_p(t) \in \mathbf{R}^3$, is obtained by taking the time derivative of (7), and utilizing

$$\dot{e}_p = -S(\omega)e_p + v - R^T\dot{p}_d, \quad (8)$$

the definition of $e_p(t)$ in (7), $v(t) = R^\top\dot{p}$ from (1), and $\dot{R}^T = -S(\omega)R^T$. Note that the last two terms in (8) constitute the velocity error. For subsequent control development, adding and subtracting $\frac{1}{m}R^T\dot{p}_d(t)$ yields

$$\dot{e}_p = -S(\omega)e_p + \frac{1}{m}e_v + \frac{1}{m}R^T\dot{p}_d - R^T\dot{p}_d. \quad (9)$$

The virtual translational velocity tracking error, denoted by $e_v(t) \in \mathbf{R}^3$, in (9) is defined as

$$e_v = mv - R^T\dot{p}_d. \quad (10)$$

Note that this is not real velocity error but was manufactured for the control development. The final form of the position tracking error is obtained from (9) and (10) as follows

$$\dot{e}_p = -S(\omega)e_p + \frac{1}{m}e_v + (\frac{1}{m} - 1)R^T\dot{p}_d. \quad (11)$$

After taking the time derivative of $e_v(t)$ in (10), substituting for $m\dot{v}(t)$ from (4), $-S(\omega)R^T$ from $\dot{R}^T$, and then applying the definition of $e_v(t)$ in (10), we get the velocity error rate as

$$\dot{e}_v = -S(\omega)e_v + G(R) - Y_1(v, |v|)\theta_1 - R^T\ddot{p}_d + B_1u_1 \quad (12)$$

where Property P1 was used to replace $N_1(\theta_1, v, |v|)$. The yaw angle tracking error, $e_\psi(t) \in \mathbf{R}^1$, is defined as

$$e_\psi = \psi - \psi_d. \quad (13)$$

The goal in the control development will be to ensure that $e_\psi(t)$ and $e_p(t)$ are driven to small values. The yaw angle rate error is derived by taking the time derivative of (13) as follows

$$\dot{e}_\psi = \dot{\psi} - \dot{\psi}_d = T_z(\Theta)\omega - \dot{\psi}_d \in \mathbf{R}^1 \quad (14)$$

where $T_z(\Theta) \in \mathbf{R}^{1\times 3}$ is the third row vector of $T(\Theta)$ from (3). Note that $T_z(\Theta)\omega(t) = \dot{\psi}(t)$ in $\dot{\Theta}(t)$ where $\dot{\psi}_d(t)$ is the desired yaw angular velocity in the body-fixed frame. In order to further develop the control design, the filtered position tracking error signal $r_p(t) \in \mathbf{R}^3$ is defined in the following manner [4]

$$r_p = e_v + \alpha e_p + \delta \quad (15)$$

where $\alpha \in \Re^1$ is a positive constant and $\delta = [0, 0, \delta_3]^T \in \Re^3$ is a constant design vector in which $\delta_3 \in \mathbf{R}^1$ is a constant. The filtered position tracking error can be combined with the yaw tracking error to create a composite tracking error $r(t) \in \mathbf{R}^4$ in the manner

$$r = [r_p^T,\ e_\psi]^T. \quad (16)$$

The filtered tracking error dynamics can be found by first differentiating (16) to yield

$$\dot{r}=[\dot{r}_p^T,\ \ \dot{e}_\psi]^T=[\dot{e}_v^T+\alpha\,\dot{e}_p^T,\ \ \dot{e}_\psi]^T\in\mathbf{R}^4. \tag{17}$$

The filtered position tracking error rate, $\dot{r}_p(t)$, is obtained by substituting (11) and (12), and the term $S(\omega)\delta$ has been added and subtracted to facilitate introduction of $\dot{r}_p(t)\in\mathbf{R}^3$ on the right-hand side as

$$\begin{aligned}\dot{r}_p &= \alpha v-S(\omega)r_p-\alpha R^T\dot{p}_d-R^T\,\ddot{p}_d-\frac{e_p}{m}+ \\ &\quad [G(R)-Y_1\theta_1+\frac{e_p}{m}]+[S(\omega)\delta+B_1u_1]\end{aligned} \tag{18}$$

where $\frac{e_p}{m}$ is subtracted and added for the subsequent stability analysis and $\alpha v-\alpha R^T\dot{p}_d=\frac{\alpha}{m}e_v+(\frac{\alpha}{m}-\alpha)R^T\dot{p}_d$ was used for separating the measurable and unknown terms. It is now a straightforward matter to substitute from (14) and (18) into (17) to yield the open-loop filtered tracking error dynamics in the following form

$$\begin{aligned}\dot{r}&=\begin{bmatrix}\alpha v-S(\omega)r_p-\alpha R^T\dot{p}_d-R^T\,\ddot{p}_d+W_1\Theta_1\\ -\dot{\psi}_d\end{bmatrix}\\ &+\begin{bmatrix}-\frac{1}{m}e_p\\ 0\end{bmatrix}+\begin{bmatrix}-S(\delta) & B_1\\ T_z(\Theta) & 0\end{bmatrix}\begin{bmatrix}\omega\\ u_1\end{bmatrix}\end{aligned} \tag{19}$$

where $S(\omega)\delta=-S(\delta)\omega$ was used [5]. Note that δ is a bounding constant which is utilized to incorporate the coupling between translational and rotational dynamics via the matrix $S(\delta)\omega$ in the ensuing backstepping approach, $\dot{r}(t)$ is derived from the position error rate $\dot{e}_p(t)$, yaw angle error rate $\dot{e}_\psi\,(t)$, and the translational dynamics $m\dot{v}(t)$ from (12) where $v(t)$ is coupled with the angular velocity $\omega(t)$.

P2: The combined term $W_1\Theta_1\in\mathbf{R}^3$ in (19) is defined as

$$W_1\Theta_1=G(R)-Y_1(v,|v|)\theta_1+\frac{e_p}{m}, \tag{20}$$

and satisfies a linear parameterization where $W_1\in\mathbf{R}^{3\times p}$ is a regression matrix, p is the number of uncertain parameters, and $\Theta_1\in\mathbf{R}^p$ is a constant parameter vector.

B. Integrator Backstepping

Note that the angular velocity in the first equation of (4) is obtained by integrating the angular dynamics $\dot{\omega}(t)$ in the second equation of (4). Thus, this system can be seen as a cascaded and coupled system and here we suggest a backstepping approach, the reader is referred to [9] where the control of cascaded dynamic is addressed. As shown in the last term in the error dynamics in (19), $\omega(t)$ and $u_1(t)$ should be controlled simultaneously. Equation (19) can be described by using a general dynamic form as

$$\dot{r}=f_1(r)+g_1\mu \tag{21}$$

where the first bracketed term defines $f_1(r)\in\mathbf{R}^4$, the last matrix and vector define $g_1\in\mathbf{R}^{4\times4}$, and $\mu=\begin{bmatrix}\omega^T & u_1\end{bmatrix}^T\in\mathbf{R}^4$. Modifying $\mu(t)$ by a change in variables is achieved by adding and subtracting a new control signal $g_1\bar{u}_1(t)$ into (21) [8] and yields

$$\dot{r}=f_1(r)+g_1\bar{u}_1+g_1(\mu-\bar{u}_1). \tag{22}$$

Then, manipulating the last parenthetical term in (22) yields

$$\mu-\bar{u}_1=\begin{bmatrix}\omega-B_z\bar{u}_1\\ u_1-B_o\bar{u}_1\end{bmatrix}. \tag{23}$$

The actual translational control input $u_1(t)$ is designed by

$$u_1=\ B_o\bar{u}_1\ and B_o=\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}\in\mathbf{R}^{1\times4}. \tag{24}$$

Finally, we get

$$\mu-\bar{u}_1=\begin{bmatrix}\omega-B_z\bar{u}_1\\ 0\end{bmatrix}. \tag{25}$$

An auxiliary signal $z(t)\in\mathbf{R}^3$, in order to inject a control signal $\bar{u}_1(t)$ into the translational dynamics from the rotational (attitude) dynamics using $\omega(t)$, is defined as

$$z=\omega-B_z\bar{u}_1 \tag{26}$$

where $B_z=[I_3,O_{3\times1}]\in\mathbf{R}^{3\times4}$. Thus, the open-loop error signals are obtained

$$\begin{aligned}\dot{r}\ &=\ \begin{bmatrix}-S(\omega)r_p+\alpha v-\alpha R^T\dot{p}_d-R^T\,\ddot{p}_d+W_1\Theta_1\\ -\dot{\psi}_d\end{bmatrix}\\ &+\begin{bmatrix}-\frac{1}{m}e_p\\ 0\end{bmatrix}+B_b\begin{bmatrix}z\\ 0\end{bmatrix}+B_b\bar{u}_1\end{aligned} \tag{27}$$

where $B_b(\cdot)\in\mathbf{R}^{4\times4}$ is

$$B_b=\begin{bmatrix}-S(\delta) & B_1\\ T_z(\Theta) & 0\end{bmatrix}. \tag{28}$$

Taking the time derivative of $z(t)$ in (26) and multiplying by the inertia matrix, J, yields

$$J\dot{z}=J\dot{\omega}-JB_z\,\dot{\bar{u}}_1\,. \tag{29}$$

Substituting the second equation of (4) for $J\dot{\omega}(t)$ into (29), grouping terms, and invoking Property P1 for the linear parameterization of $N_2(\theta_2,v,|v|)$ produces

$$J\dot{z}=-S(\omega)J\omega-Y_2(v,|v|)\theta_2-JB_z\,\dot{\bar{u}}_1+B_2u_2 \tag{30}$$

where the control input $u_2(t)$ has finally appeared and it will be designed later to derive the closed-loop controller form and achieve system stability. Before designing the control input $u_2(t)$ in (30), the following property is stated.

P3: A linear parameterization $W_3\Theta_3$ is defined as

$$W_3\Theta_3=-S(\omega)Jw-Y_2(v,|v|)\theta_2-JB_z\,\dot{\bar{u}}_1$$

where $W_3(\dot{})\in\mathbf{R}^{3\times q}$ is a known regression matrix and $\theta_3\in\mathbf{R}^q$ is a constant parameters vector.

Using Property P3, (30) is rewritten as

$$J\dot{z}=W_3(\cdot)\Theta_3+\ B_2u_2. \tag{31}$$

IV. Lyapunov-based Robust Control

A. Stability Analysis

A Lyapunov analysis is used to guide the control design. The non-negative functions $V(t)$ is chosen as

$$V = \frac{1}{2}\left(e_p^T e_p + r^T r + z^T J z\right). \tag{32}$$

The function $V(t)$ has the following property

$$\frac{1}{2}\lambda_1 \left\|\eta\right\|^2 \leq V \leq \frac{1}{2}\lambda_2 \left\|\eta\right\|^2 \tag{33}$$

where $\lambda_1 = \min\{1,\ \lambda_{\min}\{J\}\}$ and $\lambda_2 = \max\{1, \lambda_{\max}\{J\}\}$. The time derivative of $V(t)$ yields

$$\dot{V} = e_p^T \dot{e}_p + r^T \dot{r} + z^T J \dot{z}. \tag{34}$$

After substituting (11) and (27) into (34) produces

$$\begin{aligned}\dot{V} &= e_p^T\left[\tfrac{1}{m}(r_p - \alpha e_p - \delta) - S(\omega)e_p + (\tfrac{1}{m} - 1)R^T \dot{p}_d\right] \\ &+ [r_p^T, e_\psi]\left(\begin{bmatrix} \alpha v - S(\omega) r_p - \alpha R^T \dot{p}_d - R^T\ \ddot{p}_d \\ -\dot{\psi}_d \end{bmatrix}\right. \\ &+ \left.\begin{bmatrix} W_1\Theta_1 - \frac{1}{m}e_p \\ 0 \end{bmatrix} + B_b \begin{bmatrix} z \\ 0 \end{bmatrix} + B_b \bar{u}_1\right) + z^T J \dot{z}\end{aligned} \tag{35}$$

Then, the skew-symmetric terms in the first and second row can be removed. After rearranging, the equation (35) becomes

$$\begin{aligned}\dot{V} \;=\; & -\frac{\alpha}{m}e_p^T e_p + e_p^T\left[(\frac{1}{m} - 1)R^T \dot{p}_d - \frac{1}{m}\delta\right] \\ & + [r_p^T, e_\psi]\left(\begin{bmatrix} \alpha v - \alpha R^T \dot{p}_d - R^T\ \ddot{p}_d \\ -\dot{\psi}_d \end{bmatrix}\right. \\ & + \left.\begin{bmatrix} W_1\Theta_1 \\ 0 \end{bmatrix} + \bar{B}_b^T z + B_b \bar{u}_1\right) + z^T J \dot{z}\end{aligned} \tag{36}$$

where $\bar{B}_b^T \in \mathbf{R}^{4\times 3}$ is obtained from the left three columns in (28) and the transposition of $\bar{B}_b(\cdot) \in \mathbf{R}^{3\times 4}$ in the parenthesis in (36) is formed as

$$\bar{B}_b^T = \begin{bmatrix} -S(\delta) \\ T_z(\Theta) \end{bmatrix},\ \bar{B}_b = [S(\delta),\ T_z^T(\Theta)] \tag{37}$$

where $-S(\delta)^T = S(\delta)$ is used. Thus, the last bracketed term in (27) can be rewritten as

$$B_b \begin{bmatrix} z \\ 0 \end{bmatrix} = \bar{B}_b^T z \in \mathbf{R}^3 \tag{38}$$

B. Translational Input Design

The control input $\bar{u}_1(t)$ in (27) can be designed based on (35) in the Lyapunov stability analysis as

$$\begin{aligned}\bar{u}_1 = B_b^{-1}(-k_r r &+ \begin{bmatrix} -\alpha v + \alpha R^T \dot{p}_d + R^T\ \ddot{p}_d \\ \dot{\psi}_d \end{bmatrix} \\ - \begin{bmatrix} W_1\hat{\Theta}_1 \\ 0 \end{bmatrix} &- \begin{bmatrix} \frac{\rho_1^2(\|\zeta_1\|_s)}{\varepsilon_1} r_p \\ 0 \end{bmatrix}) = B_b^{-1} U\end{aligned} \tag{39}$$

where $k_r = \mathrm{diag}(k_{r1},\ k_{r1},\ k_{r1},\ k_{r2}) \in \mathbf{R}^{4\times 4}$ is a positive constant gain matrix, the term $-k_r r(t) \in \mathbf{R}^4$ is a feedback term designed to promote the convergence of $r(t)$ to zero, the measurable signals in the first bracket of the second row of (35) can be directly canceled, and other estimated terms in $W_1\hat{\Theta}_1$ can be obtained as

$$W_1\hat{\Theta}_1 = G(\hat{mg}) - Y_1(v, |v|)\hat{\theta}_1 + \frac{1}{\hat{m}}e_p \tag{40}$$

where $\hat{\Theta}_1 \in \mathbf{R}^p$ is the unknown constant parameter vector. The last term, $\rho_1^2(\|\zeta_1\|_s)\frac{r_p}{\varepsilon_1}$, where $\zeta_1 = \begin{bmatrix} p^T & p_d^T & v^T \end{bmatrix}^T$, is a robust term to compensate for parameter mismatch term in (46), the bounding function $\rho_1(\|\zeta_1\|)$ is defined as

$$\rho_1(\|\zeta_1\|_m) \leq \rho_1(\|\zeta_1\|) \leq \rho_1(\|\zeta_1\|_s) \tag{41}$$

where the functions $\|\zeta_1\|_m$ and $\|\zeta_1\|_s$ are defined in the following manner

$$\begin{aligned}\|\zeta_1\|_s &= \sqrt{\zeta_1^T\zeta_1 + \sigma} \\ \|\zeta_1\|_m &= \sqrt{\zeta_1^T\zeta_1 + \sigma} - \sqrt{\sigma} = \|\zeta_1\|_s - \sqrt{\sigma}\end{aligned} \tag{42}$$

in which $\sigma \in \mathbf{R}^1$ represents a small positive constant and $\|\zeta_1\|_m \leq \|\zeta_1\| \leq \|\zeta_1\|_s$. Finally, the closed-loop filtered tracking error dynamics for $\dot{r}(t)$ is formed from the translational input design by substituting (39) into (27) to yield

$$\dot{r} = \begin{bmatrix} W_1\tilde{\Theta}_1 - S(\omega)r_p - \frac{e_p}{m} - \frac{\rho_1^2(\|\zeta_1\|_s)}{\varepsilon_1}r_p \\ 0 \end{bmatrix} - k_r r + \bar{B}_b^T z \tag{43}$$

where the parameter mismatch term $\tilde{\Theta}_1(\cdot) \in \mathbf{R}^p$ is introduced as follows

$$\tilde{\Theta}_1 = \Theta_1 - \hat{\Theta}_1 \tag{44}$$

where

$$\tilde{\Theta}_1 = \begin{bmatrix} \tilde{mg} & \tilde{\theta}_1^T & \tilde{m}_2 \end{bmatrix}^T,$$

in which $\tilde{mg} = mg - \hat{mg}$ and $\tilde{m}_2 = \frac{1}{m} - \frac{1}{\hat{m}}$, and the parameter mismatch term is upper bounded by

$$\left\|W_1\tilde{\Theta}_1\right\| \leq \rho_1(\|\zeta_1\|). \tag{45}$$

Thus, substituting (39) into (36) yields

$$\begin{aligned}\dot{V} &= -\tfrac{\alpha}{m}e_p^T e_p + e_p^T\left[(\tfrac{1}{m} - 1)R^T \dot{p}_d - \tfrac{\delta}{m}\right] + [r_p^T, e_\psi] \\ &\left(\begin{bmatrix} W_1\tilde{\Theta}_1 \\ 0 \end{bmatrix} - k_r r - \begin{bmatrix} \frac{\rho_1^2(\|\zeta_1\|_s)}{\varepsilon_1} r_p \\ 0 \end{bmatrix} + \bar{B}_b^T z\right) + z^T J \dot{z}.\end{aligned} \tag{46}$$

C. Torque Input Design

The control input $u_2(t) \in \mathbf{R}^3$ is now formulated from (31) in the following form

$$u_2 = B_2^{-1}(-k_z z - W_3(p, R, v, \omega)\hat{\Theta}_3 - \frac{\rho_2^2(\|\zeta_2\|_s)}{\varepsilon_2}z - \bar{B}_b^T r) \tag{47}$$

where the control input $u_2(t)$ would be designed to stabilize the $z(t)$-dynamics using the feedback term $z(t)$ and finally $\omega(t)$ rotational dynamics and in order to compensate $W_3(\cdot)\Theta_3$, in a similar way to $W_1\hat{\Theta}_1$ in (40), $W_3(\cdot)\hat{\Theta}_3$ is given by

$$W_3\hat{\Theta}_3 = -S(\omega)\hat{J}\omega - Y_2(v, |v|)\hat{\theta}_2 - JB_z\ \dot{\bar{u}}_1 \in \mathbf{R}^3 \tag{48}$$

where $W_3 \in \mathbf{R}^{3\times q}$ and $\hat{\Theta}_3(\cdot) \in \mathbf{R}^q$ is a regression vector updated by on-line parameter estimation, the robust term

$\rho_2^2(\|\zeta_2\|_s)\frac{r_p}{\varepsilon_2}$, where $\zeta_2 = \begin{bmatrix} p^T & v^T & \omega^T \end{bmatrix}^T$, the bounding function $\rho_2(\|\zeta_2\|)$ are defined in a similar manner as in (39), and the last term is added for canceling the opposite term. It is clear that $Y_2(v,|v|)\hat{\theta}_2$ terms was already parameterized and $S(\omega)\hat{J}\omega(t)$ can be easily linearly parameterized. The parameterizable model for $\dot{\bar{u}}_1(t)$ needs to be developed. The time derivative of $\bar{u}_1(t)$ can be calculated using the estimated definition in (39) as follows

$$\dot{\bar{u}}_1 = \frac{d}{dt}(B_b^{-1})U + (B_b^{-1})\frac{d}{dt}U \tag{49}$$

where $U(t)$ is the parenthetical term on the right equation as mentioned in (39) and the time derivative of $U(t)$ is calculated as

$$\frac{d}{dt}U = \begin{bmatrix} -\alpha\dot{v} - \dot{W}_1(v)\hat{\Theta}_1 - \frac{d}{dt}\left(\frac{\rho_1^2(\|\zeta_1\|_s)}{\varepsilon_1}r_p\right) \\ 0 \end{bmatrix} + \begin{bmatrix} -\alpha S(\omega)R^T\dot{p}_d + (\alpha - S(\omega))R^T\ddot{p}_d + R^T\dddot{p}_d \\ -\ddot{\psi}_d \end{bmatrix} - k_r\dot{r} \tag{50}$$

where $\dot{v}(t)$ will be substituted from (4) including unknown parameters as

$$\dot{v} = -S(\omega)v + \frac{1}{m}(W_1\hat{\Theta}_1 + B_1u_1), \tag{51}$$

and the time derivative of $W_1(\cdot)$ in (50) is defined as

$$\dot{W}_1\hat{\Theta}_1 = \dot{G}(\hat{m},\hat{g}) - \dot{Y}_1(v)\hat{\theta}_1 + \frac{1}{\hat{m}}\dot{e}_p, \tag{52}$$

and $\dot{e}_p(t)$ is same as (8) and the time derivative of the robust term related to $\rho_1(\|\zeta_1\|_s)$ is developed in [10]. $\dot{r}(t)$ in (43) uses $r(t)$ but $r_p(t)$ can be linearly divided up into known and unknown terms each as

$$r = \begin{bmatrix} \alpha e_p + \delta - R^T\dot{p}_d \\ e\psi \end{bmatrix} + \begin{bmatrix} mv \\ 0 \end{bmatrix}, \tag{53}$$

from the definition of (15) by substituting (10)

$$r_p = \alpha e_p + \delta + mv - R^T\dot{p}_d, \tag{54}$$

the last term in (50) yields via (43)

$$\begin{aligned} \dot{r} &= -k_r\left(\begin{bmatrix} mv \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha e_p + \delta - R^T\dot{p}_d \\ e\psi \end{bmatrix}\right) + \bar{B}_b^T z \\ &\quad + \begin{bmatrix} W_1\tilde{\Theta}_1 - S(\omega)r_p - \frac{e_p}{m} - \frac{\rho_1^2(\|\zeta_1\|_s)}{\varepsilon_1}r_p \\ 0 \end{bmatrix}. \end{aligned} \tag{55}$$

Finally, (49) can now be implemented using (43), (51), (52), (55), (8), and the time derivative of the robust term, $\rho_1(\|\zeta_1\|_s)$, to produce parameterization as follows

$$\dot{\bar{u}}_1 = \Phi_1 + \Phi_2\hat{\Theta}_2 \tag{56}$$

where $\Phi_1(\cdot) \in \mathbf{R}^3, \Phi_2 \in \mathbf{R}^{3\times l}$ are a signal vector and matrix without having unknown parameters where $\hat{\Theta}_2 \in \mathbf{R}^l$, is a unknown parameter vector. After substituting (47) into (31) and then rearranging the equation, we have the final form for the closed-loop system as shown below

$$J\dot{z} = -k_z z + W_3\tilde{\Theta}_3 - \bar{B}_b^T r - \frac{\rho_2^2(\|\zeta_2\|_s)}{\varepsilon_2}z \tag{57}$$

where the regression estimation error, $\tilde{\Theta}_3(\cdot)$, is defined as $\tilde{\Theta}_3 = \Theta_3 - \hat{\Theta}_3$.and

$$\left\|W_3\tilde{\Theta}_3\right\| \le \rho_2(\|\zeta_2\|). \tag{58}$$

The first bracketed term in (46) can be upper bounded by δ_1 in the following manner $\left\|\frac{(1-m)R^T\dot{p}_d - \delta}{m}\right\| \le \delta_1$ where $\delta_1 \in \mathbf{R}^1$ is a constant. Using the Young's inequality, the first bracketed term can be further upper bounded as

$$\|e_p\|\,\delta_1 \le \frac{1}{2}\left(\varepsilon_3\|e_p\|^2 + \frac{1}{\varepsilon_3}\delta_1^2\right) \tag{59}$$

where $\varepsilon_3 \in \mathbf{R}^1$ is a positive constant. After substituting (39), (57) by (47), and (59) into (46), the following terms, $r_p^T\bar{B}_b z$ and $z^T\bar{B}_b^T r_p$, are then canceled each other and then $\dot{V}(t)$ produces the following upper boundedness

$$\begin{aligned} \dot{V} \le\ & -\left(\frac{\alpha}{m} - \frac{\varepsilon_3}{2}\right)\|e_p\|^2 - \lambda_{\min}\{k_r\}\|r\|^2 - \lambda_{\min}\{k_z\}\cdot \\ & \|z\|^2 + \|r_p^T\|\left\|W_1\tilde{\Theta}_1\right\| - \frac{\rho_1^2(\|\zeta_1\|)}{\varepsilon_1}\|r_p\|^2 \\ & + \|z^T\|\left\|W_3\tilde{\Theta}_3\right\| - \frac{\rho_2^2(\|\zeta_2\|)}{\varepsilon_2}\|z\|^2 + \frac{\delta_1^2}{2\varepsilon_3}. \end{aligned} \tag{60}$$

Using (45) and (58), the following upper bound yields

$$\begin{aligned} & \|r_p^T\|\left\|W_1\tilde{\Theta}_1\right\| - \frac{\rho_1^2(\|\zeta_1\|)}{\varepsilon_1}\|r_p\|^2 \\ & \le -\frac{1}{\varepsilon_1}\left(\rho_1(\|\zeta_1\|)\|r_p\| - \frac{\varepsilon_1}{2}\right)^2 + \frac{\varepsilon_1}{4} \le \frac{\varepsilon_1}{4}, \end{aligned} \tag{61}$$

and also the following is obtained in a similar manner as $\|z^T\|\left\|W_3\tilde{\Theta}_3\right\| - \frac{\rho_2^2(\|\zeta_2\|)}{\varepsilon_2}\|z\|^2 \le \frac{\varepsilon_2}{4}$ and then, specifying upper bounds by combining the above constant terms yields $\varepsilon_0 > \frac{\delta_2^2}{2\varepsilon_3} + \frac{\varepsilon_1}{4} + \frac{\varepsilon_2}{4}$. Then, upper bound can be written as

$$\dot{V} \le -\gamma\|\eta\|^2 + \varepsilon_0 \tag{62}$$

where a positive constant γ is given by

$$\gamma = \min\{\left(\frac{\alpha}{m} - \frac{1}{2\lambda_1}\right),\ \lambda_{\min}\{k_r\},\ \lambda_{\min}\{k_z\}\}. \tag{63}$$

Thus, $\dot{V}$ can be upper bounded by utilizing (33) as

$$\dot{V} \le -\frac{2\gamma}{\lambda_2}V + \varepsilon_0, \tag{64}$$

using the upper boundedness of V by $\|\eta\|^2$ in (33) as $\|\eta\|^2 \ge \frac{2}{\lambda_2}V$. Solve the inequality (64) to obtain

$$V(t) \le V(0)\exp\left(-\frac{2\gamma}{\lambda_2}t\right) + \frac{\lambda_2\varepsilon_0}{2\gamma}\left(1 - \exp\left(-\frac{2\gamma}{\lambda_2}t\right)\right)$$

where $V(0) \le \frac{1}{2}\lambda_2\|\eta(0)\|^2$. Therefore, $V(t) \le \frac{\lambda_2}{2}\|\eta(0)\|^2 + \frac{\lambda_2\varepsilon_0}{2\gamma}$ where $V(t)$ is exponentially driven into a ball with a radius of ε_0 from the term $\frac{\lambda_2\varepsilon_0}{2\gamma}$. The tracking result yields

$$\begin{aligned} & \|\eta(t)\| \le \sqrt{\frac{2}{\lambda_2}V(t)} \\ & = \sqrt{\frac{\lambda_2}{\lambda_1}\|\eta(0)\|^2\exp\left(-\frac{2\gamma}{\lambda_2}t\right) + \frac{\lambda_2\varepsilon_0}{2\gamma}\left(1 - \exp\left(-\frac{2\gamma}{\lambda_2}t\right)\right)}, \end{aligned} \tag{65}$$

provided the following control gain condition holds $\alpha > \frac{\varepsilon_3}{2} m$.

Theorem 1: The control laws given in (39) and (47), respectively ensure that the tracking error is globally uniformly ultimately bounded as

$$\|\eta(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1} \|\eta(0)\|^2 e^{\left(-\frac{2\gamma}{\lambda_2} t\right)} + \frac{\lambda_2 \varepsilon_0}{2\gamma} \left(1 - e^{\left(-\frac{2\gamma}{\lambda_2} t\right)}\right)} \tag{66}$$

where $\eta(t) \in \mathbf{R}^{10}$ is defined as $\eta = [e_p^T, r^T, z^T]^T$, provided the following control gain condition holds

$$\alpha > \frac{\varepsilon_3 m}{2}. \tag{67}$$

Remark: Based on Theorem 1, it can be shown that all signals remain bounded in the closed-loop system (see [10]).

V. Conclusion

The goal of designing a state feedback controller for tracking control of a quad-rotor UAV system considering parametric uncertainties has been achieved. The parameter uncertainty is compensated by the robust terms designed by a Lyapunov-based control approach. A backstepping approach is introduced to the highly nonlinear coupled and cascaded system. The controller is then designed for position tracking in the three linear dimensions and the yaw angle A global uniformly ultimately bounded (GUUB) tracking result is achieved using Lyapunov-type stability analysis.

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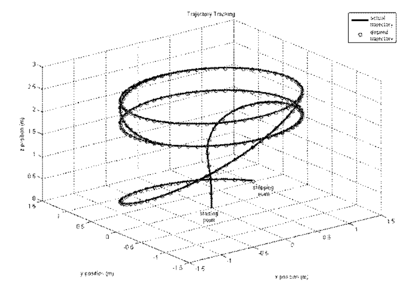

Fig. 1. Trajectory Tracking in 3D Space

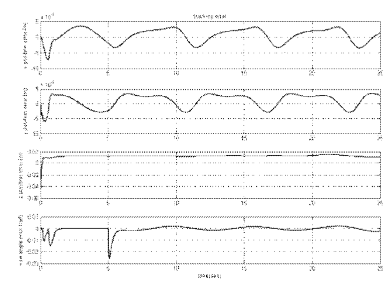

Fig. 2. Tracking Errors

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Appendices

A. Figures and Simulation

The simulation was run by a control software Qmotor on QNX real time operating system. Figure 1 shows the tracking result, where the line shows the actual trajectory of UAV and circles marks out the desired trajectory. Figure 2 shows the tracking errors in x, y, z direction and in yaw angle. The tracking control in the presence of uncertainty was simulated using reference values of a small quad-rotor UAV as $m = 1.2$ $[kg]$, $g = 9.81[ms^2]$ and J =diag(0.40, 0.40, 0.60) $[kg \cdot m^2]$. The constant control parameters for controller were empirically chosen to be k_r =diag(5, 5, 5, 5), k_z =diag(5, 5, 5), $\alpha = 75$, $\delta = [0, 0, -1]^T$, $\varepsilon_1 = \varepsilon_2 = 1$, $\hat{\Theta}_1 = 1.2\Theta_1$, $\hat{\Theta}_3 = 1.2\Theta_3$, $\sigma = 1$. More simulation results and notes can be seen in [10].