Proceedings of the 2009 IEEE International Conference on Systems, Man, and Cybernetics
San Antonio, TX, USA - October 2009

Study on the possibility degree of the optimal solution of grey dynamic programming model base on standard interval grey number

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***Abstract*—Considering the uncertainty of information, the paper puts forward an improved grey dynamic programming model. After that, treating profit value as positive interval grey number, the paper researches on dynamic programming model, and we could get the optimal strategy after solution by means of defining standard interval grey number. Furthermore, as we can not make the judgment of standard interval grey number, the author puts forward the possibility degree of the optimal solution, which represented as $\beta_{u_{ij}}(k)$ is only related with $f_{i+1,k}(\otimes)(S_{i+1,k})$, and it has the features of $0<\beta_{u_{ij}}(k)<1$ and $\sum_{k=1}^{t}\beta_{u_{ij}}(k)=1$. Finally, the author verifies the practicability of the model by case study.**

***Keywords*—Grey Systems, Dynamic programming model, Standard interval grey number, Optimal solution, Possibility degree**

I. INTRODUCTION

In reality, the process of decision-making can be divided into some interrelated stages, and each of them with a number of options. So the dynamic programming can be defined as making a reasonable choice by policy makers so as to obtaining the best results in the process of decision-making.

In resent years, the dynamic programming model is applied in the departments of engineering technology, enterprise management, industry, agriculture and military [1-7]. In the process of programming, the uncertain factors mainly come from two different aspects, one being outside programming system and the other lies inside factor under control. No matter in the programming model or the requirements of restrictions, if uncertain factors emerge, the normal programming methods cannot give effective solution. On the contrary, grey systems thinking and modeling methods of grey systems can be helpful for solving problems to some extent.

The concept of grey dynamic programming and the possibility degree of the optimal solution with an improved grey dynamic programming model is put forward in this paper. The optimal strategy solution can be got based on algorithms of interval grey numbers and grey algebraic system of interval grey number [8-13]. Finally, a calculating example is presented to verifies the practicability of the model.

II. GREY DYNAMIC PROGRAMMING MODEL

Definition 1 Suppose that there are n stages in a dynamic programming and S_k represents state in stage-k ,The value collection of the S_k known as the state assembly. If the value of the state variable is a grey number $\otimes$, then the dynamic programming is called grey dynamic programming.

For the convenience, we assume that:

1) In the whole process of dynamic programming, the system with k different stages states and the whole set is denoted as E={ $S_1(\otimes), S_2(\otimes)$,..., $S_k(\otimes)$ }.

2) Ihe system needs the external input in order to change the current state to reach the presumed object, the set of decision vectors is indicated as

$$U=\{u_1(s_1(\otimes)), u_1(s_2(\otimes)), \cdots, u_m(s_m(\otimes))\}.$$

3) The whole grey dynamic programming is completed during $[0,T]$. The interval $[0,T]$ is divided into n time intervals $[t_{k-1},t_k](k=1,2,\cdots,n)$, here

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$0=t_0<t_1<\cdots<t_n=\mathrm{T}$, and the grey decision variable of the k-th stage of time interval $(t_{k-1}, t_k]$ is denoted as $u(t_k) \in \mathrm{U}$.

4) Once the grey state and grey decision in a stage are known, the state of the following grey state is completely definitive. We employ the grey state transfer equation to signify this rule as denoted below:

$$\mathrm{S}_{k+1}(\otimes)=T_k^{\otimes}(s_k(\otimes), u_k(\otimes))$$

5) The grey index function is the quantitative function of grey strategy. The grey index function from stage-k to stage-n is expressed by $\mathrm{f}_k(\otimes)$, namely

$$f_k(\otimes)=f_k(s_k(\otimes), u_k(\otimes), s_{k+1}(\otimes), \ldots, s_{n+1}(\otimes))$$
$$k=1,2,\ldots,n$$

The index function constituting the dynamic programming should be divisible and conform to recurrence relation, which means that $\mathrm{f}_k(\otimes)$ could be represented as the functions of $S_k(\otimes)$, $u_k(s_k(\otimes))$ and $f_{k+1}(\otimes)$, that is:

$$f_k(s_k(\otimes), u_k(\otimes), s_{k+1}(\otimes), \ldots, s_{n+1}(\otimes))$$
$$=\psi_k[s_k(\otimes), u_k(\otimes), f_{k+1}(s_{k+1}(\otimes), \ldots, s_{n+1}(\otimes))]$$

III. THE POSSIBILITY DEGREE OF THE OPTIMAL SOLUTION OF GREY DYNAMIC PROGRAMMING MODEL BASE ON STANDARD INTERVAL GREY NUMBER

To solve a grey dynamic programming model, an interval grey number should be tranfered into a standard interval grey number.

A. The expression and calculation of interval grey number

Definition 2 An interval grey number in form (1)

$$\otimes_i = a_i + c_i \cdot \gamma_i, \quad i=1,2,\cdots \tag{1}$$

is called a standard interval grey number.

Where a_i is called the write part of $\otimes_i$, $c_i \cdot \gamma_i$ is called the grey part of $\otimes_i$, where c_i is grey coefficient and γ_i is unit grey number.

Theorem 1 Any interval grey number $\otimes_i = [a_i, b_i], a_i \le b_i, i=1,2,\cdots$ can be expressed as a standard interval grey number.

Proof

$$\begin{aligned}\otimes_i &= [a_i, b_i] = [a_i, b_i] + a_i - a_i = a_i + [0, b_i - a_i] \\ &= a_i + (b_i - a_i) \cdot [0,1] \\ &= a_i + (b_i - a_i) \cdot \gamma_i \\ &= a_i + c_i \cdot \gamma_i\end{aligned}$$

(2)

$$c_i = (b_i - a_i), \gamma_i \in [0,1], i=1,2,\cdots$$

The equation (2) proved that any interval grey number can be expressed by its standard form through standard transformation.

Suppose that $\otimes_1, \otimes_2, \cdots, \otimes_n$ all be the standard interval grey number, $F(\Theta) = \Theta\{\otimes_1, \otimes_2, \cdots, \otimes_n\}$ is the outcome of some algebraic operation with $\otimes_1, \otimes_2, \cdots, \otimes_n$, then the operation process can be conducted as follows

Step1 Let $\gamma_i, \gamma_i \in [0,1], i=1,2,\cdots$ all be a certain constants;

Step2 Evaluate the minimum value

$$\min\{F(\Theta)\} = \min \Theta\{\otimes_1, \otimes_2, \cdots, \otimes_n\} \Big|_{\gamma_i = c_i, c_i \in [0,1], i=1,2,\cdots,n}$$

and the maximum value

$$\max\{F(\Theta)\} = \max \Theta\{\otimes_1, \otimes_2, \cdots, \otimes_n\} \Big|_{\gamma_i = c_i, c_i \in [0,1], i=1,2,\cdots,n}$$

Step3

$$F(\Theta) = \Theta\{\otimes_1, \otimes_2, \cdots, \otimes_n\} \in [\min\{F(\Theta)\}, \max\{F(\Theta)\}]$$

Definition 3 Grey number $\otimes_1 \in [\mathrm{a,b}](\mathrm{a} \le \mathrm{b})$ is called comparable with grey number $\otimes_2 \in [\mathrm{c,d}](\mathrm{c} \le \mathrm{d})$, if the following two conditions are satisfied.

(1) when $b \le c$, then $[a,b] \le [c,d]$, that is $[a,b](a \le b)$ is less than or equal to $[c,d](c \le d)$; this can be expressed as figure (1) and figure (2);

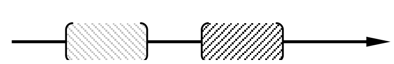

Fig 1 [a,b]<[c,d]

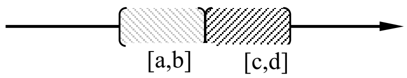

Fig 2 [a,b]≤[c,d]

(2) when $a = c, b = d$,

If $\gamma_1 = \gamma_2$, then $[a,b] = [c,d]$, that is, grey number $[a,b](a \le b)$ is equal to grey number $[c,d](c \le d)$; this can be expressed as figure (3);

If $\gamma_1 < \gamma_2$, then $\otimes_1 > \otimes_2$ (that is $[a,b] > [c,d]$), and grey number $[a,b](a \le b)$ is larger than grey number $[c,d](c \le d)$; this can be expressed as figure (4);

If $\gamma_1 > \gamma_2$, then $\otimes_1 < \otimes_2$ (that is $[a,b] < [c,d]$), and grey number $[a,b](a \le b)$ is smaller than grey number $[c,d](c \le d)$; this can be expressed as figure (5).

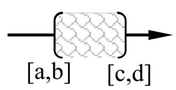

Fig 3 [a,b]=[c,d]

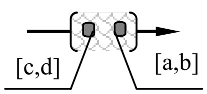

Fig 4 $\gamma_1 < \gamma_2$, [a,b]>[c,d]

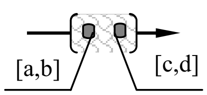

Fig 5 $\gamma_1 > \gamma_2$, [a,b]<[c,d]

The simplest forms of grey numbers which easy be compared each other are given in definition 3. When γ_1, γ_2 is unknown, two interval grey numbers could be compared each other only as the condition (1) is satisfied.

B. The solution of grey dynamic programming model

Assume that grey dynamic programming could be divided into n stages and m states, the j th state in the i th stage is denoted as $s_{i,j}(\otimes)$, the corresponding decision variables is denoted as $u_{i,j}(s_{i,j}(\otimes))$. The index funtions of($s_{i,j}(\otimes)$, $u_{i,j}(s_{i,j}(\otimes))$) is denoted as

$$f_{i,j}(\otimes)(S_{i,j}) = \underset{\{u_i,\cdots u_n\}}{opt}\left\{d_{i,j}(i,k)(S_{i,j},u_{i,j}) + f_{i+1,k}(\otimes)(S_{i+1,k})\right\} \tag{3}$$

where $d_{i,j}(i,k)(s_{i,j},u_{i,j}), k = 1,2,\cdots,t.$, and $t < j$ is the kth index.

The grey dynamic programming satisfies the optimality principle and the optimality theorem[14].

Definition 4 If all of decision makers could make judgment by comparison in the process of solving the grey dynamic programming model, the optimal value will be obtained, which means the possibility degree of optimal value is 1.

Definition 5 In the process of solving the grey dynamic programming model,

$$\beta_{u_{ij}}(k) = \frac{d_{i,j}(i,k)(S_{i,j},u_{i,j}) + f_{i+1,k}(\otimes)(S_{i+1,k})}{\sum_{k=1}^{t}(d_{i,j}(i,k)(S_{i,j},u_{i,j}) + f_{i+1,k}(\otimes)(S_{i+1,k}))} \tag{4}$$

is called the possibility degree of the optimal strategy in the i th stage.

IV. STUDY ON THE PROPERTIES OF THE OPTIMAL SOLUTION'S POSSIBILITY DEGREE OF GREY DYNAMIC PROGRAMMING MODEL

Let $d_{i,j}(i,k)(S_{i,j},u_{i,j}) = a_{i,j}(k) + b_{i,j}(k)\gamma_{i,j}(k)$, $k = 1,2,\cdots,t.$ and we treat $\gamma_{ij}(k)$ as standard unit grey number. Meanwhile, $f_{i+1,k}(\otimes)(S_{i+1,k}) = \sum A_{i+1,k} + \sum B_{i+1,k}\gamma_{i+1,k}$, among the equation, the $A_{i+1,k}$, $B_{i+1,k}$ and $\gamma_{i+1,k}$ are respective represents the white part, grey part and standard unit grey number.

Then the equation will be transformed into:

$$\beta_{u_{ij}}(k) = \frac{a_{i,j}(k) + b_{i,j}(k)\gamma_{i,j}(k) + \sum A_{i+1,k} + \sum B_{i+1,k}\gamma_{i+1,k}}{\sum_{k=1}^{t}(a_{i,j}(k) + b_{i,j}(k)\gamma_{i,j}(k) + \sum A_{i+1,k} + \sum B_{i+1,k}\gamma_{i+1,k})} \tag{5}$$

Property 1: $0 < \beta_{u_{ij}}(k) < 1$

Proof:

$$\beta_{u_{ij}}(k) = \frac{a_{i,j}(k) + b_{i,j}(k)\gamma_{i,j}(k) + \sum A_{i+1,k} + \sum B_{i+1,k}\gamma_{i+1,k}}{\sum_{k=1}^{t}(a_{i,j}(k) + b_{i,j}(k)\gamma_{i,j}(k) + \sum A_{i+1,k} + \sum B_{i+1,k}\gamma_{i+1,k})}$$

Because

$$a_{i,j}(k) + b_{i,j}(k)\gamma_{i,j}(k) + \sum A_{i+1,k} + \sum B_{i+1,k}\gamma_{i+1,k} > 0,$$

$$\sum_{k=1}^{t}(a_{i,j}(k) + b_{i,j}(k)\gamma_{i,j}(k) + \sum A_{i+1,k} + \sum B_{i+1,k}\gamma_{i+1,k}) > 0,$$

then,

$$\beta_{u_{ij}}(k) = \frac{a_{i,j}(k) + b_{i,j}(k)\gamma_{i,j}(k) + \sum A_{i+1,k} + \sum B_{i+1,k}\gamma_{i+1,k}}{\sum_{k=1}^{t}(a_{i,j}(k) + b_{i,j}(k)\gamma_{i,j}(k) + \sum A_{i+1,k} + \sum B_{i+1,k}\gamma_{i+1,k})} > 0$$

After that

$$\beta_{u_{ij}}(k)=\frac{a_{i,j}(k)+b_{i,j}(k)\gamma_{i,j}(k)+\sum A_{i+1,k}+\sum B_{i+1,k}\gamma_{i+1,k}}{\sum_{k=1}^{t}(a_{i,j}(k)+b_{i,j}(k)\gamma_{i,j}(k)+\sum A_{i+1,k}+\sum B_{i+1,k}\gamma_{i+1,k})}$$

$$=\frac{a_{i,j}(k)+b_{i,j}(k)\gamma_{i,j}(k)+\sum A_{i+1,k}+\sum B_{i+1,k}\gamma_{i+1,k}}{(a_{i,j}(k)+b_{i,j}(k)\gamma_{i,j}(k)+\sum A_{i+1,k}+\sum B_{i+1,k}\gamma_{i+1,k})+\sum_{p=1,p\neq k}^{t}(a_{i,j}(p)+b_{i,j}(p)\gamma_{i,j}(p)+\sum A_{i+1,p}+\sum B_{i+1,p}\gamma_{i+1,p})}$$

$$\sum_{p=1,p\neq k}^{t}(a_{i,j}(p)+b_{i,j}(p)\gamma_{i,j}(p)+\sum A_{i+1,p}+\sum B_{i+1,p}\gamma_{i+1,p})>0$$

So $\beta_{u_{ij}}(k)=\dfrac{a_{i,j}(k)+b_{i,j}(k)\gamma_{i,j}(k)+\sum A_{i+1,k}+\sum B_{i+1,k}\gamma_{i+1,k}}{\sum_{k=1}^{t}(a_{i,j}(k)+b_{i,j}(k)\gamma_{i,j}(k)+\sum A_{i+1,k}+\sum B_{i+1,k}\gamma_{i+1,k})}<1$

Property 2: $\sum_{k=1}^{t}\beta_{u_{ij}}(k)=1$

Proof:

$$\sum_{k=1}^{t}\beta_{u_{ij}}(k)=\sum_{k=1}^{t}[\frac{a_{i,j}(k)+b_{i,j}(k)\gamma_{i,j}(k)+\sum A_{i+1,k}+\sum B_{i+1,k}\gamma_{i+1,k}}{\sum_{k=1}^{t}(a_{i,j}(k)+b_{i,j}(k)\gamma_{i,j}(k)+\sum A_{i+1,k}+\sum B_{i+1,k}\gamma_{i+1,k})}]$$

$$=\frac{\sum_{k=1}^{t}a_{i,j}(k)+b_{i,j}(k)\gamma_{i,j}(k)+\sum A_{i+1,k}+\sum B_{i+1,k}\gamma_{i+1,k}}{\sum_{k=1}^{t}(a_{i,j}(k)+b_{i,j}(k)\gamma_{i,j}(k)+\sum A_{i+1,k}+\sum B_{i+1,k}\gamma_{i+1,k})}=1$$

(Proof ends)

Property 3: $\beta_{u_{ij}}(k)$ is only related with $f_{i+1,k}(\otimes)(S_{i+1,k})$, but not $d_{i,j}(i,k)(S_{i,j},u_{i,j})$.

Proof: From the definition (4) we could know that as $f_{i,j}(\otimes)(S_{i,j})=\underset{\{u_i,\cdots u_n\}}{opt}\{d_{i,j}(i,k)(S_{i,j},u_{i,j})+f_{i+1,k}(\otimes)(S_{i+1,k})\}$, $k=1,2\cdots t$ can not be judged, the optimal solution of grey dynamic programming will face risks, which means that $d_{i,j}(i,p)(S_{i,j},u_{i,j})+f_{i+1,p}(\otimes)(S_{i+1,p})$ could not be compared with $d_{i,j}(i,q)(S_{i,j},u_{i,j})+f_{i+1,q}(\otimes)(S_{i+1,q})$ (among them $p,q\in k, p\neq q$).

However, $d_{i,j}(i,p)(S_{i,j},u_{i,j})=d_{i,j}(i,q)(S_{i,j},u_{i,j})$ happens as in the same state $S_{i,j}(\otimes)$, so the risks mentioned above originate from the non judgment of $f_{i+1,p}(\otimes)(S_{i+1,p})$ and $f_{i+1,q}(\otimes)(S_{i+1,q})$, namely they only related with $f_{i+1,k}(\otimes)(S_{i+1,k})$, but not $d_{i,j}(i,k)(S_{i,j},u_{i,j})$.

V. A CALCULATING EXAMPLE

Supposing an enterprise facing pricing problems, the unit price of the product should be selected from ¥5 to ¥8. After that, the price only could be changed early in each year and the range could not be over ¥1. Furthermore, the enterprise supposes that the product will be ready sale for five years before being washed out. Finally, according to the prediction of sales situation, we work out the anticipated profit Tab. as follows:

TABLE I. ANTICIPATED PROFIT OF A NEW PRODUCT IN DIFFERENT CONDITIONS

Unit price	1st year	2nd year	3rd year	4th year	5th year
¥5	10-11	12-14	15-16	19-20	25-26
¥6	13-16	13-15	16-18	20-21	24-25
¥7	14-16	15-17	14-15	18-19	18-19
¥8	15-16	14-16	15-16	14-15	14-15

As $k=5$, $f_5(5)=25+\gamma_{11}$, $f_5(6)=24+\gamma_{21}$, $f_5(7)=18+\gamma_{31}$, $f_5(8)=14+\gamma_{41}$

As $k=4$,

$$f_4(5)=\max\begin{Bmatrix}19+\gamma_{12}+25+\gamma_{11}\\19+\gamma_{12}+24+\gamma_{21}\end{Bmatrix}=44+\gamma_{12}+\gamma_{11};$$

$$f_4(6)=\max\begin{Bmatrix}20+\gamma_{22}+25+\gamma_{11}\\20+\gamma_{22}+24+\gamma_{21}\\20+\gamma_{22}+18+\gamma_{31}\end{Bmatrix}=45+\gamma_{22}+\gamma_{11};$$

$$f_4(7)=\max\begin{Bmatrix}18+\gamma_{32}+24+\gamma_{21}\\18+\gamma_{32}+18+\gamma_{31}\\18+\gamma_{32}+14+\gamma_{41}\end{Bmatrix}=42+\gamma_{32}+\gamma_{21};$$

$$f_4(8)=\max\begin{Bmatrix}14+\gamma_{42}+18+\gamma_{31}\\14+\gamma_{42}+14+\gamma_{41}\end{Bmatrix}=32+\gamma_{42}+\gamma_{31}.$$

In the same way, we can get the optimal index function as $k=3, k=2$, and $k=1$ and then

$$\max\{f_1(5), f_1(6), f_1(7), f_1(8)\}$$
$$=\max\begin{Bmatrix}89+3\gamma_{25}+2\gamma_{34}+2\gamma_{23}+\gamma_{22}+\gamma_{11}\\90+2\gamma_{35}+2\gamma_{34}+2\gamma_{23}+\gamma_{22}+\gamma_{11}\\91+\gamma_{45}+2\gamma_{34}+2\gamma_{23}+\gamma_{22}+\gamma_{11}\end{Bmatrix}$$

According to equation (4), we could get the possibility degree of each optimal index function, namely the possibility degree of each optimal path, which is as shown in Tab. (2).

TABLE II. THE POSSIBILITY DEGREE OF EACH OPTIMAL INDEX FUNCTION AND PATH

The optimal index function	The optimal path	Possibility degree
$f_1(6)$	$f_1(6)\to f_2(7)\to f_3(6)\to f_4(6)\to f_5(5)$	0.331425
$f_1(7)$	$f_1(7)\to f_2(7)\to f_3(6)\to f_4(6)\to f_5(5)$	0.333199
$f_1(8)$	$f_1(8)\to f_2(7)\to f_3(6)\to f_4(6)\to f_5(5)$	0.335005

VI. CONCLUSION

The concept of grey dynamic programming is put forward in this paper. Treating profit value as positive interval grey number and the dynamic programming model is studied.And the optimal strategy solution is got by means of defining standard interval grey number. The model is useful to realize optimal control in the process of uncertain manufacture such as production plan, storage, source distribution, investment and sorting problems.

ACKNOWLEDGMENT

This work was partially supported by National Natural Science Foundation of China (No.70701017), the Foundation for Doctoral Programs and the Foundation for Humanities and Social Sciences of the Chinese National Ministry of Education(No.08JA630039), the Key Program of Soft Science Foundation (No.2008GXS5D115), the authors would like to acknowledge the partial support of the Science Fund for distinguished Professor of NUAA and Jiangsu Province (No.1009-316011).

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