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Relative phase control in a circularly coupled oscillator system and its application to timing regulation of a multicylinder engine

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***Abstract*— In this paper, dynamics that produce an oscillation pattern with homogeneous phase lags are designed in a circularly coupled oscillator system. Two key topics here are: a definition of the local oscillator interactions that converges the relative phases of oscillators to their references, and an adaptive modification of the references to make themselves achievable. Next, this design method is applied to the timing regulation of multicylinder engine. Simulation results show that the engine cycle pattern with homogeneous phase shift is achieved even if the number of cylinders in the engine varies due to the trouble or system extensions. The simulations also demonstrate that this oscillation pattern certainly reduces the fluctuation of rotational velocity of the engine shaft.**

I. INTRODUCTION

In physical and chemical systems such as a thermal liquid system or a chemical reaction system, we can find some spatiotemporal patterns [1], [2]. There, many homogeneous elements behave in an oscillatory manner through local interactions among them. A behavior of an oscillatory subsystem can be sometimes described using a coupled oscillator system, where their relative phases after the phase synchronization are regarded as spatiotemporal pattern. From the viewpoint of the system engineering, we here address a control and adaptation problem of the relative phases in a coupled oscillator system: how to design local oscillator interactions that converge the relative phases to their reference values, and how to adaptively change their references if they are not achievable. Finally, we pick up a circularly-coupled oscillator system and apply it to a controller that regulates the timing of multicylinder engine.

II. BASIC THEORY

A. Problem Setting

H. Yuasa and M. Ito proposed a method for controlling relative phases in a coupled oscillator system [3]. Their framework is based on parallel and distributed operations and is more general in the sense that linear relations between the one-dimensional state variables of two coupled subsystems can be regulated to their references. Of course, the linear relations include a difference in the state variables (the relative phase, in the case of a coupled oscillator system).

First, the following are assumed:

- The system under consideration consists of homogeneous subsystems.
- The state of the subsystem is represented by a scalar variable.
- The subsystems are connected locally, implying that there is no hub subsystem to which all subsystems connect to.
- Two connected subsystems can exchange state variables, each affecting the state variable of the other. This is called ”interaction”.
- A constraint (i.e., a variable calculated from the states of the two coupled subsystems) is defined for each connection. This variable is called a ”constraint variable”.
- Each constraint variable possesses a reference value that represents a purpose of the entire system.

Under these assumptions, the problem is described as follows:

- Define the dynamics of each subsystem with local interaction such that the constraint variables are regulated to their references.

Here, dynamics with local interaction means, mathematically, that the system never contains any state variables other than those of the connected subsystems.

B. A design method based on a gradient system

The state of each subsystem is denoted by $q_i \in R$ ($i = 1, \ldots, M$), where M is the number of the subsystems. $N(i)$ is a set of subsystems connected to subsystem i, and N_i is the number of the elements in $N(i)$. H. Yuasa and M. Ito [3] formulate a case where the constraint variable p_k is given by the linear relation:

$$p_k = L_{ki}q_i - L_{kj}q_j \tag{1}$$

This equation implies that the connection k ($k = 1, \cdots, K$) connects subsystem i and subsystem j. This relation can be represented using the matrix form

$$P = LQ \tag{2}$$

where $P = [p_1, \cdots, p_K]^T$, $Q = [q_1, \ldots, q_M]^T$, $L \in R^{K\times M}$ is a matrix whose (i, j) element is given as L_{ij}. Owing to (1), the number of non-zero elements is two in each row of the matrix L. Then, the problem is how to design subsystem dynamics

$$\dot{q}_i = f_i(Q_{N(i)}) \tag{3}$$

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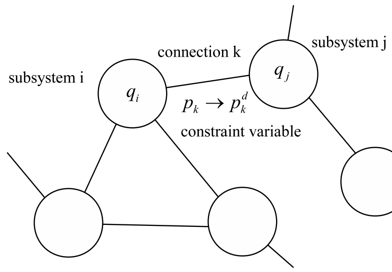

Fig. 1. System consisting of many homogeneous subsystem and constraint variable.

for P to converge to its reference $P^d = [p_1^d, \cdots, p_K^d]^T$. Here, $Q_{N(i)}$ is a vector in R^{N_i} whose element belongs to $N(i)$.

A method using a gradient system has been proposed for this problem.

Theorem 1 (H. Yuasa and M. Ito[3]): Let the dynamics of the entire system

$$\dot{Q} = F(Q) \tag{4}$$

Decompose F into two orthogonal components, $\bar{F}$ and $\tilde{F}$, using matrix L,

$$\bar{F} = [\bar{f}_1, \cdots, \bar{f}_M]^T = (I - L^\dagger L)F \in kerL \tag{5}$$

$$\tilde{F} = [\tilde{f}_1, \cdots, \tilde{f}_M]^T = L^\dagger LF \in (kerL)^\perp \tag{6}$$

where $L^\dagger$ is the pseudo-inverse matrix of L and $kerL$ is a null space of L. If $\tilde{F}$ is described using a function of the variable x_i defined below, such as

$$\tilde{f}_i = \tilde{f}_i(x_i) \tag{7}$$

then the dynamics of P become a gradient system whose potential function $V(P) = V_X(X)$ is given by

$$V_X(X) = \sum_{i=1}^{M} \int \tilde{f}_i(x_i) dx_i \tag{8}$$

Here, x_i is the i-th element of the vector X, calculated as

$$X = -L^T P = -L^T L Q \tag{9}$$

Because of the definition of L, x_i contains no state variables other than those of subsystems connected to subsystem i, implying that the conditions in (3) are satisfied. $\tilde{f}_i(x_i)$ expresses the effect of interaction, because this term contains only the states of connected subsystems.

C. Dynamics in orthogonal complementary kernel space

The above theorem results in the following subsystem dynamics:

$$\dot{q}_i = \bar{f}_i + \tilde{f}(x_i) \tag{10}$$

Now, what type of functional systems should be used to design $\tilde{f}(x_i)$ within the orthogonal complementary space of $kerL$?

The dynamics of the linear constraint variable P become

$$\dot{P} = L\tilde{F} \tag{11}$$

If P is controlled to its reference P^d, the time evolution of P should stop, i.e., $\dot{P} = 0$ should hold. One method is to define $\tilde{F}$ so that $\tilde{F} = 0$ when $P = P^d$. When P becomes P^d, X is

$$X^d = -L^T P^d \tag{12}$$

From the above equation, the next relation is obtained:

$$X - X^d = -L^T(P - P^d) \tag{13}$$

Thus, it is sufficient to define $\tilde{f}$ as

$$\tilde{f}_i = g(x_i - x_i^d) \tag{14}$$

where the function g satisfies the following three conditions [4]:

- $g(0) = 0$
- $\left.\frac{dg(x)}{dx}\right|_{x=0} > 0$
- $g(x) \neq 0 (x \neq 0)$

The first condition ensures $\tilde{f} = 0$ at $x_i = x_i^d$. The next one is needed to make $x_i = x_i^d$ the minimal point of the potential function of (8). The last one is required for $x_i = x_i^d$ to be the sole minimal point of this potential function. The odd function should be selected if the effect of the connected subsystem is the same in the bilateral direction.

D. Expression of interaction between subsystems

In Section II-A, the interaction was defined as an effect on the state of the other subsystem. Here, the expression of this interaction is discussed under the formulation in Section II-C.

$\tilde{f}_i$ in (10) is defined to achieve the reference P^d. Therefore, one reasonable idea is that the effect from the other subsystem is driven by the error, i.e., the difference in the constraint variable p_k from its reference p_k^d. Among such expressions, the simplest one is linearly represented as

$$I_k = -K_k(p_k - p_k^d) \tag{15}$$

Thus, I_k is regarded as a mathematical expression of the interaction that works at connection k. K_k is a parameter that adjusts the magnitude of the interaction. Using the i-th column of the matrix L, instead of K_k, (13) is rewritten as

$$x_i - x_{di} = -\sum_{k=1}^{K} (L^T)_{ik}(p_k - p_k^d) = \sum_{k \in C(i)} I_k \tag{16}$$

where $C(i)$ is a set of connections that couple the subsystem i. The second equality holds because of the following reason: L_{ij}^T is not zero if there is a connection between subsystems i and j. In other words, it becomes zero only between unconnected subsystems, implying that the summation in (16) actually sums the interaction only from subsystems connected to subsystem i.

E. Unachievable reference and its automatic adjustment within kernel

If the system contains a loop connection, such as in a circularly coupled system, the achievability of the reference is critical. For example, assume an oscillator system with n circularly coupled oscillators. A reference that causes each oscillator to oscillate with a $1/n$-period lag is certainly achievable, but a reference with $1/(n-1)$-period lag is not.

What happens in the stationary state if unachievable reference is set? The dynamics of P are given by (11), thus, $\dot{P} = 0$ gives its stationary state. If $\tilde{F}$ is defined using an odd function, as discussed in Section II-C, $X - X^d = 0$ holds because $\tilde{F} = 0$. Ideally, $X - X^d = 0$ should be satisfied under the condition that P converges to P^d, i.e., $P - P^d = 0$. According to (16), however, it is sufficient for $X - X^d = 0$ that the summation of $p_k - p_k^d$ becomes zero. In other words, if $P - P^d (\neq 0)$ stays at the kernel of L^T, $X - X^d$ becomes zero.

In the latter case, the interaction in (15) does not disappear – it takes non-zero value. This implies that non-zero interactions work permanently. This situation is not desirable from two points of view. Interactions are supposed to work to achieve the references. However, in this situation, the references are never achieved, even if the interactions are always working. This is inconsistent from a system control viewpoint. Secondarily, the stationary state is maintained by the mutual cancellation of non-zero interactions. Such non-zero interactions are disadvantageous from a cost viewpoint and occur when unachievable references are set. Thus, modification of the unachievable reference must be introduced. Reducing non-zero interactions is a natural criterion of the modification. Accordingly, using a cost function given as the squared sum of the interactions,

$$V_I = \frac{1}{2} \sum_{k=1}^{K} I_k^2 \tag{17}$$

the adjustment rule is defined to decrease as

$$\frac{dp_i^d}{dt} = -\tau_p \frac{\partial V_I}{\partial p_i^d} \tag{18}$$

This is equivalent to memorizing a currently emerging P as P^d. Here, P^d obtained from the above rule as a reference appropriate to the current environmental conditions in the sense that it can be maintained with zero interactions.

III. Application to Regulating the Timing of a Multicylinder Engine

A. Circularly coupled oscillator system

1) Formulation: The previous section proposed a distributed control method for controlling constraint variables between subsystems. Here, it is applied to the relative phase control of a circularly coupled oscillator system. In this system, N oscillators are connected in a circle, where each oscillator is numbered, in order from 1 to N. The connections in this coupled oscillator system are local – oscillator i is connected only to neighboring oscillators, oscillators $i-1$ and $i+1$. The phase of each oscillator is q_i $(i = 1, \cdots, N)$.

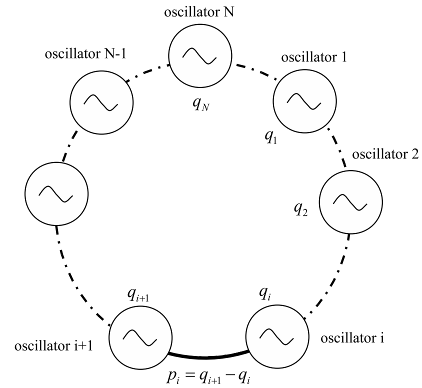

Fig. 2. A circularly-coupled oscillator system.

The connection k represents a connection that connects oscillator k to oscillator $k+1$. To this connection k, a constraint variable p_k is defined. In this problem, the constraint variable is the difference of the states of the two connected oscillators, i.e., the relative phase. This is obviously a linear relation, given as

$$p_i = q_{i+1} - q_i \tag{19}$$

This matrix expression corresponding to (2) is given using the matrix L

$$L = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & 0 & -1 & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} \tag{20}$$

The kernel of L, i.e., $\bar{f}$ becomes

$$\bar{f} = [1, \cdots, 1]^T \tag{21}$$

Regarding the orthogonal complementary space of this kernel, a sine function is selected as $g(x)$ in (14), taking the periodicity of the oscillator phase into account:

$$g(x) = \tau \sin(x) \tag{22}$$

Then, the oscillator dynamics are defined as follows:

$$\dot{q}_i = \omega + \tau \sin(q_{i-1} - 2q_i + q_{i+1} - p_{i-1}^d + p_i^d) \tag{23}$$

Here, ω is a constant corresponding to the natural angular frequency, $\tau > 0$ is a parameter adjusting the magnitude of the interactions, and $q_{N+1} = q_1$, $q_0 = q_N$, $p_{N+1} = p_1$, $p_0 = p_N$, $p_{N+1}^d = p_1^d$, $p_0^d = p_N^d$.

If the potential function is defined using (8) as

$$V = \sum_{i=1}^{N} \tau \cos((p_{i-1} - p_{i-1}^d) - (p_i - p_i^d)) \tag{24}$$

the dynamics of p_i are certainly described as the gradient system of this potential function:

$$\begin{aligned}\dot{p}_i &= q_{i+1} - q_i \\ &= \tau \sin((p_i - p_i^d) - (p_{i+1} - p_{i+1}^d)) \\ &\quad - \tau \sin((p_{i-1} - p_{i-1}^d) - (p_i - p_i^d)) \\ &= -\frac{\partial V}{\partial p_i} \end{aligned} \tag{25}$$

2) Stationary state: At the stationary state, $X - X^d = 0$ is satisfied. Using (13) and (20), the solutions of $X - X^d = 0$ are given as

$$p_1 = \cdots = p_K = const. \tag{26}$$

This means that an oscillation pattern finally emerges in which all oscillators oscillate with a constant relative phase to the connected one. Of course, this oscillation pattern includes a completely phase-coherent oscillation, where all the oscillators have the same phase, i.e., the zero relative phase.

3) Adjustment of the reference relative phase: In the previous section, an oscillation pattern with the same relative phases was achieved regardless of the reference P^d. Based on the analysis in Section II-E, if the constraint variable p_k is not equal to the reference p_k^d in the stationary state, this reference is unachievable and inappropriate under the current conditions. In such a case, the reference is adjusted according to (18) to be appropriate and consistent with current conditions.

4) Application to the timing regulation of a multicylinder engine: Timing regulation of a multicylinder engine is one possible example of the application of the circularly coupled oscillator system formulated in this section. In a four-stroke cycle engine, four strokes are repeated: induction, compression, explosion, and exhaust. Because of its periodicity, this engine stroke is characterized by the phase of the engine cycle. Appropriate phase shifts among the cylinders provide stable output with minimal fluctuation. Inappropriate phase timings, however, causes vibration or degradations of engine efficiency. In short, 'homogeneous phase shifts' among the cylinders are important for a multicylinder engine. This requirement well matches the behavior of the circularly coupled oscillator system considered in Section III-A.1.

The production of homogeneous phase shifts has no relationship with the reference of the constraint variable in the circularly coupled oscillator system. This may enhance the fault tolerance of this engine. For example, assume that one of the cylinders fails. In this situation, the relative phases of the engine cycle are not equal among the remaining cylinders. However, drive with the same interval is feasible with the remaining cylinders if the cylinder movement phase is adjusted properly. To this adjustment the above property must be applied. If possible, the relative oscillator phases should automatically change when the number of oscillators increases or decreases because of problems or system modification.

In addition, if the reference is also adjusted based on the current oscillation, the coupled oscillator system can always memorize the appropriate oscillation pattern for the current condition.

B. Multicylinder engine and controller

1) Controller specifications: A four-stroke cycle engine is considered. The number of cylinders is set to N. The dynamics of this engine are described by the motion equation and the state equation for an ideal gas [5].

The specifications of the controller that regulates the timing of the engine cycle among the cylinders are set as follows:

- It produces an oscillation pattern with a $1/n$-period phase lag, if the number of cylinders in operation is n.
- It adjusts the relative phase automatically when the number of cylinders increases or decreases.
- It learns or memorizes the appropriate references for the current conditions.

Here, the controller is assumed to detect the rotational velocity of the engine shaft A.

2) Controller design: The circularly coupled oscillator system in Section III-A is used. N oscillators are prepared for the controller. This number is the same as that of the engine cylinder. An oscillator is assigned to each cylinder, and the phase of the engine cycle is assumed to be controlled by the phase of the corresponding oscillator.

The rotational velocity of the engine shaft A is sent from the engine to the controller as a feedback signal. Each oscillator in the controller is driven based on this signal, i.e., using the parameter ω in (23). This ω is determined from the rotational velocity of the shaft $\omega = \omega(A)$ to synchronize the shaft and stroke cycles.

IV. Simulations

A. Conditions

To examine the effect of homogeneous phase shifts, consider the case where the number of cylinders decreases from initial four cylinders as follows:

- Cylinder four breaks down at 3.0 [s].
- For a while, the engine works without any phase shift.
- Oscillator control for relative phase regulation is operated at 6.0 [s].

Here, it is assumed that the oscillator connection is kept in the circular form, regardless of the increment/decrement of the cylinders. The parameters are set as follows: $\tau = 10.0$ and $\tau_p = 0.5$. Because the number of cylinders is known at the initial state, each oscillator phase is set such that they are shifted $\pi/2$ from each other and the references of the relative phases satisfy this condition. Simulations are executed using the fourth order Runge-Kutta method with a step size of 0.1 [ms].

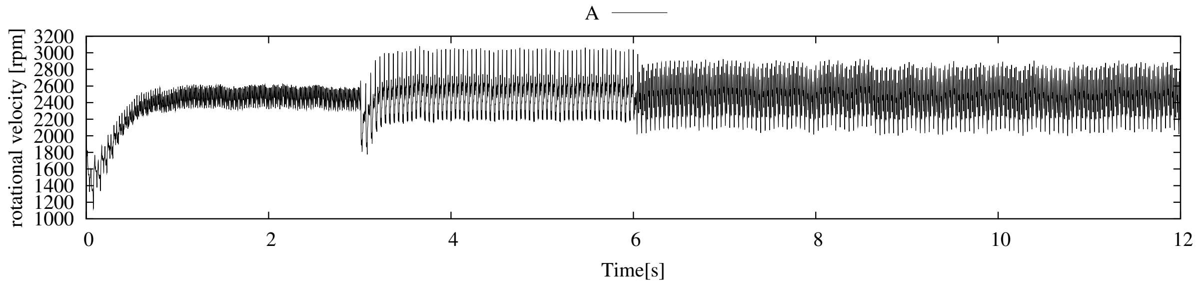

Fig. 3. Simulation of the rotation velocity of the engine shaft.

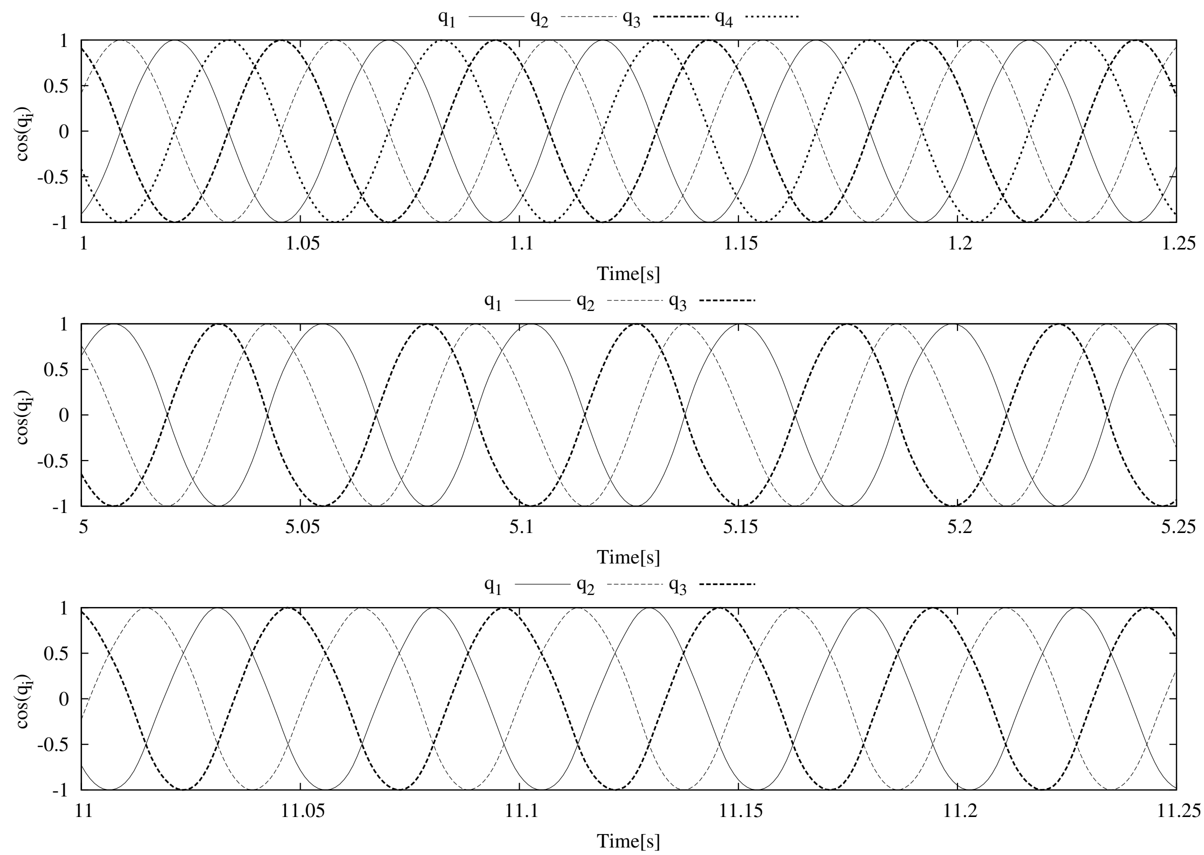

Fig. 4. Simulation of oscillator phases.

B. Results

The simulation results are illustrated in Fig. 3 - 5. The graphs in Fig. 3, Fig. 4 and Fig. 5 are time-based plots of the rotational velocity of the engine shaft A, the phase of each oscillator, and the relative phases and their references, respectively. The shaft rotational velocity at each condition is summarized in Table I. Period N represents the period from 1.0 [s] to 2.9 [s], where the engine operates in a stationary state under normal conditions. Period B does from 3.0 [s] to 6.0 [s], where the broken engine works without any control. Period C is from 17.0 [s] to 20.0 [s], when the relative phase control is sufficiently in effect.

C. Discussions

The shaft rotational velocity settled into the stationary state after a transient period of about 0.5 [s], as shown in Fig. 3. The breakdown of cylinder four at 3.0 [s] noticeably enlarges its fluctuation. However, this fluctuation is somewhat suppressed thanks to the relative phase regulation of the oscillator control. Table I shows that the controller decreases the fluctuations.

As shown in the top graph of Fig. 4, before the failure each oscillator oscillates with the same phase lag. However,

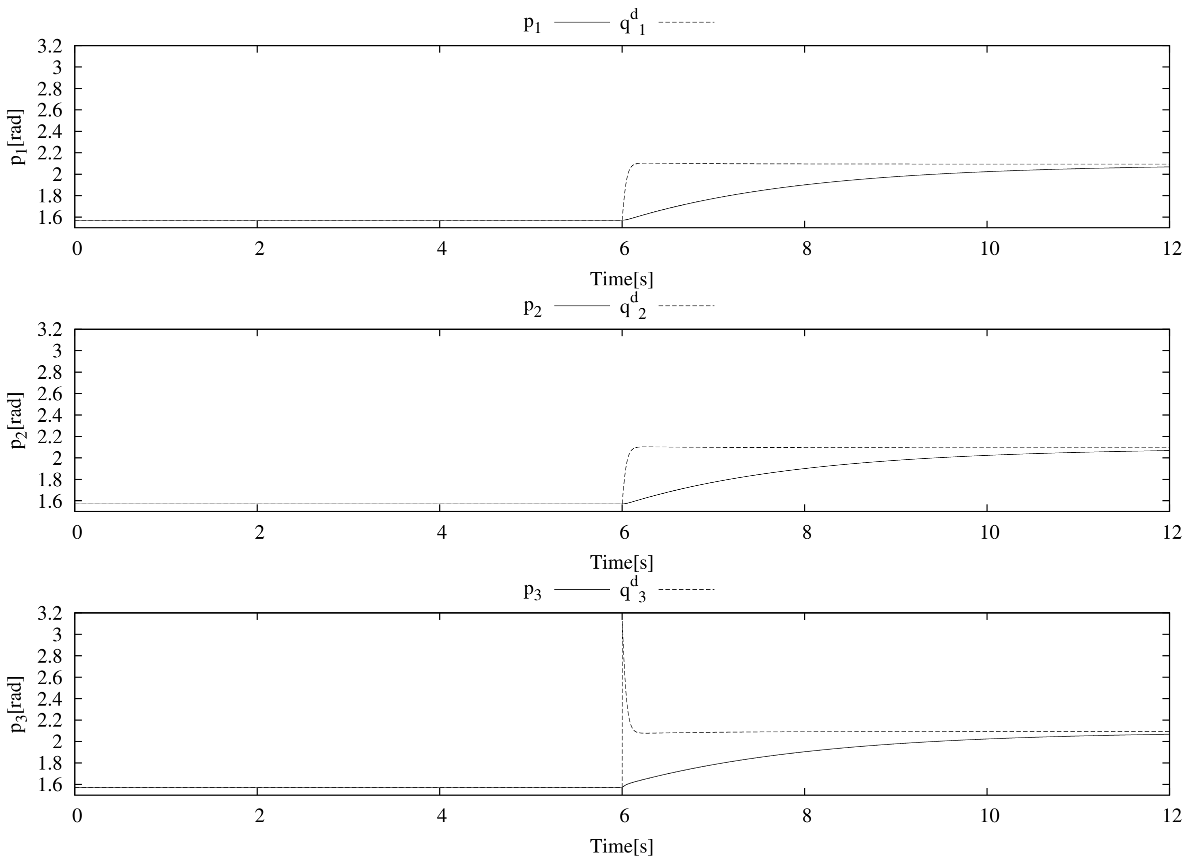

Fig. 5. Simulation of relative phases and their references.

TABLE I

COMPARISON OF SHAFT ROTATION VELOCITY

	Avg.	SD	Min.	Max.	Fluctuation
Period N	2460.07	78.44	2628.91	2298.93	329.98
Period B	2488.63	219.50	3078.15	1776.17	1301.98
Period C	2459.45	212.60	2917.66	1986.66	931.00

unequal relative phases are observed in the middle graph of Fig. 4, because cylinder four has failed. The homogeneous phase shift is recovered in the bottom graph of Fig. 4, thanks to the oscillator control. As shown in Fig. 5, the relative phase control allows the relative phase and its desired values to converge to $3\pi/2$, which is an achievable phase lag with three oscillator systems.

V. CONCLUDING REMARKS

This paper demonstrated, by simulation, a possibility that the controller using oscillators can adjust the phase lag of the engine cycle to shift it to a constant lag, which prevents the engine shaft speed from fluctuating. To realize an actual mechanical system, some assumptions must be dealt with. In particular, a mechanism for changing the phase of the engine cycle with an oscillator signal is required.

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