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Discrete Grey Model of Systematic Prediction

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***Abstract*—Based on the principle of grey systematic prediction, this paper constructed a discrete grey model of systematic prediction, which extends the application scope of the original grey model of systematic prediction and avoids optimizing the background value. This model is a linear equations set composed of N discrete GM (1, N) models. By using the row elementary transformation, the predicted values can be worked out. By means of computer, this process is very convenient to be operated. Finally, two examples are given to show that the model is feasible, and its simulated results are both good, thus it contributes to make a reasonable prediction.**

***Keywords*—Grey systematic prediction, Discrete GM (1, N), Linear equations, Row elementary transformation**

I. INTRODUCTION

Grey systematic prediction is a part of grey forecasting system. It is presented for there exist the mutual relations among the factors in a system, and believes that all the factors are not independent of each other, but regarded as a whole [1]. The basic model of grey systematic prediction is composed of some GM (1, N). Professor Deng proposed a kind of nested solution aiming at the special case of a certain structure in references [2~3], however, a great amount of actual examples rooting in life aren't such particular. For this reason, we presented a new solution for grey systematic prediction to generalize the basic model in reference [4]. Reference [5] proposed the MGM(1, N), in fact, in which the 1-AGO series of the original series are used to establish an ordinary differential equations set, by means of the knowledge of ordinary differential theory to seek for continuous solutions, and then discretize them to obtain the prediction values. Due to the discrete models have very good properties [6~9], and reference [10] established the multivariable discrete GM (n, h). On this basis, the discrete grey model of systematic prediction is proposed in this paper.

II. DISCRETE MODEL OF GREY SYSTEMATIC PREDICTION

Definition 1 Let $x_1, x_2, \cdots, x_N$ be the behavior variables of a given system S, and satisfy

$$
\begin{aligned}
x_1^{(0)}(k) + a_1 z_1^{(1)}(k) &= b_{12}x_2^{(1)}(k) + b_{13}x_3^{(1)}(k) + \cdots \\
&\quad + b_{1N}x_N^{(1)}(k) + b_{10} \\
x_2^{(0)}(k) + a_2 z_2^{(1)}(k) &= b_{21}x_1^{(1)}(k) + b_{23}x_3^{(1)}(k) + \cdots \\
&\quad + b_{2N}x_N^{(1)}(k) + b_{20} \\
&\ \ \vdots \\
x_N^{(0)}(k) + a_N z_N^{(1)}(k) &= b_{N1}x_1^{(1)}(k) + b_{N2}x_2^{(1)}(k) + \cdots \\
&\quad + b_{N.N-1}x_{N-1}^{(1)}(k) + b_{N0}
\end{aligned}
$$

then the above grey differential equations are called the grey model of systematic prediction, abbreviated as GMSP (1, N).

Definition 2 Let $x_1, x_2, \cdots, x_N$ be the behavior variables of a given system S, and satisfy

$$
\begin{aligned}
x_1^{(1)}(k) + \alpha_1 x_1^{(1)}(k-1) &= \beta_{12}x_2^{(1)}(k) + \beta_{13}x_3^{(1)}(k) + \cdots \\
&\quad + \beta_{1N}x_N^{(1)}(k) + \beta_{10} \\
x_2^{(1)}(k) + \alpha_2 x_2^{(1)}(k-1) &= \beta_{21}x_1^{(1)}(k) + \beta_{23}x_3^{(1)}(k) + \cdots \\
&\quad + \beta_{2N}x_N^{(1)}(k) + \beta_{20} \\
&\ \ \vdots \\
x_N^{(1)}(k) + \alpha_N x_N^{(1)}(k-1) &= \beta_{N1}x_1^{(1)}(k) + \beta_{N2}x_2^{(1)}(k) + \cdots \\
&\quad + \beta_{N,N-1}x_{N-1}^{(1)}(k) + \beta_{N0}
\end{aligned}
$$

then the above equations are called the discrete grey model of systematic prediction, abbreviated as DGMSP (1, N). Its parameter vectors

$$P_i = [\alpha_i, \beta_{i1}, \beta_{i2}, \cdots, \beta_{i(i-1)}, \beta_{i(i+1)}, \cdots, \beta_{iN}, \beta_{i0}],$$
$$i = 1, 2, \cdots, N.$$

Proposition 1 GMSP (1, N) is a kind of special case of DGMSP (1, N).

Proof. Let

$$x_i^{(0)}(k) + a_i z_i^{(1)}(k) = \sum_{j=1, j\neq i}^{N} b_{ij}x_j^{(1)}(k) + b_{i0},$$

where

$$z_i^{(1)}(k) = \lambda x_i^{(1)}(k) + (1-\lambda)x_i^{(1)}(k-1) \ (0 < \lambda < 1),$$
$$x_i^{(0)}(k) = x_i^{(1)}(k) - x_i^{(1)}(k-1).$$

Substitute respectively, then

$$\left[x_i^{(1)}(k)-x_i^{(1)}(k-1)\right]+a_i\left[\lambda x_i^{(1)}(k)+(1-\lambda)x_i^{(1)}(k-1)\right]$$
$$=\sum_{j=1,\,j\neq i}^{N} b_{ij}x_j^{(1)}(k)+b_{i0}$$
$$\Leftrightarrow\ \left(1+a_i\lambda\right)x_i^{(1)}(k)+\left(a_i(1-\lambda)-1\right)x_i^{(1)}(k-1)$$
$$=\sum_{j=1,\,j\neq i}^{N} b_{ij}x_j^{(1)}(k)+b_{i0}$$
$$\Leftrightarrow\ x_i^{(1)}(k)+\left(\frac{a_i(1-\lambda)-1}{1+a_i\lambda}\right)x_i^{(1)}(k-1)$$
$$=\sum_{j=1,\,j\neq i}^{N}\left(\frac{b_{ij}}{1+a_i\lambda}\right)x_j^{(1)}(k)+\frac{b_{i0}}{1+a_i\lambda}.$$

Only let $\alpha_i=\dfrac{a_i(1-\lambda)-1}{1+a_i\lambda}$, $\beta_{ij}=\dfrac{b_{ij}}{1+a_i\lambda}$ $(j=1,2,\cdots,N;j\neq i)$, $\beta_{i0}=\dfrac{b_{i0}}{1+a_i\lambda}$, and when $\lambda=\dfrac{1}{2}$, that is proved.

Proposition 2 For identifying the parameters in DGMSP (1, N), generally, we can use the following two ways [2]:

A. Least square formula

$$P_i=(Q_i^TQ_i)^{-1}Q_i^Ty_i\ \text{ (When } n-1\geq N+1)$$

B. Least norm formula

$$P_i=Q_i^T(Q_iQ_i^T)^{-1}y_i\ \text{ (When } n-1< N+1)$$

Where

$$Q_i=\begin{bmatrix}-x_i^{(1)}(1) & x_1^{(1)}(2) & \cdots & x_{i-1}^{(1)}(2) & x_{i+1}^{(1)}(2) & \cdots & x_N^{(1)}(2) & 1\\ -x_i^{(1)}(2) & x_1^{(1)}(3) & \cdots & x_{i-1}^{(1)}(3) & x_{i+1}^{(1)}(3) & \cdots & x_N^{(1)}(3) & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -x_i^{(1)}(n-1) & x_1^{(1)}(n) & \cdots & x_{i-1}^{(1)}(n) & x_{i+1}^{(1)}(n) & \cdots & x_N^{(1)}(n) & 1\end{bmatrix},\quad y_i=\begin{bmatrix}x_i^{(1)}(2)\\ x_i^{(1)}(3)\\ \vdots\\ x_i^{(1)}(n)\end{bmatrix}.$$

From the Proposition 2, we can obtain all parameters, denoted by $P=[P_1,P_2,\cdots,P_N]^T$. In order that they are uniform in the form, the parameter matrix P is written as

$$P=\begin{bmatrix}\alpha_1 & \beta_{12} & \cdots & \beta_{1,N-1} & \beta_{1N} & \beta_{10}\\ \beta_{21} & \alpha_2 & \cdots & \beta_{2,N-1} & \beta_{2N} & \beta_{20}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \beta_{N1} & \beta_{N2} & \cdots & \beta_{N,N-1} & \alpha_N & \beta_{N0}\end{bmatrix}.$$

III. THE MODEL SOLUTION

Let

$$x_i^{(1)}(k)+\alpha_ix_i^{(1)}(k-1)=\sum_{j=1,\,j\neq i}^{N}\beta_{ij}x_j^{(1)}(k)+\beta_{i0}\qquad(1)$$

be the *i*-th equation of DGMSP (1, N), it follows that

$$\sum_{j=1,\,j\neq i}^{N}\beta_{ij}x_j^{(1)}(k)-x_i^{(1)}(k)=\alpha_ix_i^{(1)}(k-1)-\beta_{i0}$$

Thus, the discrete model of systematic prediction is transformed into the following equivalent linear equations:

$$-x_1^{(1)}(k)+\beta_{12}x_2^{(1)}(k)+\beta_{13}x_3^{(1)}(k)+\cdots+\beta_{1N}x_N^{(1)}(k)=\alpha_1x_1^{(1)}(k-1)-\beta_{10}$$
$$\beta_{21}x_1^{(1)}(k)-x_2^{(1)}(k)+\beta_{23}x_3^{(1)}(k)+\cdots+\beta_{2N}x_N^{(1)}(k)=\alpha_2x_2^{(1)}(k-1)-\beta_{20}$$
$$\vdots$$
$$\beta_{N1}x_1^{(1)}(k)+\beta_{N2}x_2^{(1)}(k)+\cdots+\beta_{N,N-1}x_{N-1}^{(1)}(k)-x_N^{(1)}(k)=\alpha_Nx_N^{(1)}(k-1)-\beta_{N0}$$

Let

$$C=\begin{bmatrix}-1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1N}\\ \beta_{21} & -1 & \beta_{23} & \cdots & \beta_{2N}\\ \beta_{31} & \beta_{32} & -1 & \cdots & \beta_{3N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \beta_{N1} & \beta_{N2} & \beta_{N3} & \cdots & -1\end{bmatrix},$$

$$D=\begin{bmatrix}\alpha_1x_1^{(1)}(k-1)-\beta_{10}\\ \alpha_2x_2^{(1)}(k-1)-\beta_{20}\\ \vdots\\ \alpha_Nx_N^{(1)}(k-1)-\beta_{N0}\end{bmatrix},\ x^{(1)}(k)=\begin{bmatrix}x_1^{(1)}(k)\\ x_2^{(1)}(k)\\ \vdots\\ x_N^{(1)}(k)\end{bmatrix},$$

then,

$$Cx^{(1)}(k)=D\ .\qquad(2)$$

Due to grey systematic prediction is aimed at the k-th moment, thus the values of the k-1 moment can be regarded as a constant. So it is easy to solve Eq.(2). By the row elementary transformation, transform the matrix C into an upper triangular (or echelon form) matrix. We can use the function "rref" in MATLAB to fulfill. Let $G=\left[C\ D\right]$, then

$$\text{rref}(G) \to \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \cdots & \rho_{1N} & \sigma_1 \\ 0 & \rho_{22} & \rho_{23} & \cdots & \rho_{2N} & \sigma_2 \\ 0 & 0 & \rho_{33} & \cdots & \rho_{3N} & \sigma_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \rho_{NN} & \sigma_N \end{bmatrix}.$$

If $\text{rank}(C) = N$, we have

$$\text{rref}(G) \to \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \gamma_1 \\ 0 & 1 & 0 & \cdots & 0 & \gamma_2 \\ 0 & 0 & 1 & \cdots & 0 & \gamma_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \gamma_N \end{bmatrix}.$$

Therefore, we easily obtain

$$\hat{x}_1^{(1)}(k) = \gamma_1, \hat{x}_2^{(1)}(k) = \gamma_2, \cdots, \hat{x}_N^{(1)}(k) = \gamma_N.$$

According to

$$x_i^{(0)}(k) = x_i^{(1)}(k) - x_i^{(1)}(k-1),$$

we obtain

$$\hat{x}_i^{(0)}(k) = \hat{x}_i^{(1)}(k) - \hat{x}_i^{(1)}(k-1), \ i = 1, 2, \cdots, N.$$

When $2 \le k \le n$, $\hat{x}_i^{(0)}(k)$ is the simulated value of $x_i^{(0)}(k)$;

When $k > n$, $\hat{x}_i^{(0)}(k)$ is the forecasting value of $x_i^{(0)}(k)$.

IV. Model Checking

Example 1 In order to check the effect of the new model, we take a regional economic data cited in Reference [12] as an example to explain it. The original data are as follows:

$$\overline{x}_1^{(0)} = (111, 143, 252, 332, 441)$$
$$\overline{x}_2^{(0)} = (13148, 14107, 15806, 22182, 28784)$$
$$\overline{x}_3^{(0)} = (31013, 33656, 37390, 51531, 65231)$$
$$\overline{x}_4^{(0)} = (388100, 396360, 399336, 403895.67, 420018)$$

In order to minimize the condition number, let

$$x_i^{(0)}(k) = \overline{x}_i^{(0)}(k) / \overline{x}_i^{(0)}(1); k = 1,2,3,4,5, i = 1,2,3,4.$$

We have

$$x_1^{(0)} = (1, 1.2882, 2.2702, 2.9909, 3.9729)$$
$$x_2^{(0)} = (1, 1.0729, 1.2021, 1.6871, 2.1892)$$
$$x_3^{(0)} = (1, 1.0852, 1.2056, 1.6615, 2.1033)$$
$$x_4^{(0)} = (1, 1.02128, 1.02895, 1.0407, 1.0822)$$

From the References [10, 11], we know that the multiply transformation to original data will not change the relative error of the multi-variable discrete grey model DGM (n, h). Therefore, initialization will not affect the effect of simulation and predication of DGMSP (1, N). Their 1-AGO series are respectively as follows:

$$x_1^{(1)} = (1.0000, 2.2882, 4.5584, 7.5493, 11.5222)$$
$$x_2^{(1)} = (1.0000, 2.0729, 3.2750, 4.9621, 7.1513)$$
$$x_3^{(1)} = (1.0000, 2.0852, 3.2908, 4.9523, 7.0556)$$
$$x_4^{(1)} = (1.0000, 2.02128, 3.05023, 4.09093, 5.17313)$$

By means of the least norm formula, we obtain the parameter matrix

$$P = \begin{bmatrix} 0.983308 & 2.443024 & 1.118644 & -0.727439 & -2.654874 \\ 0.327416 & 0.162035 & 0.594259 & -0.081434 & 0.411193 \\ 0.308052 & 0.558899 & 0.326721 & 0.183647 & 0.177290 \\ 0.099534 & -0.069677 & -0.073516 & -0.955054 & 1.136202 \end{bmatrix}.$$

Further, we have

$$C = \begin{bmatrix} -1 & 2.443024 & 1.118644 & -0.727439 \\ 0.327416 & -1 & 0.594259 & -0.081434 \\ 0.308052 & 0.558899 & -1 & 0.183647 \\ 0.099534 & -0.069677 & -0.073516 & -1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.983308 x_1^{(1)}(k-1) - 2.654874 \\ 0.162035 x_2^{(1)}(k-1) - 0.411193 \\ 0.326721 x_3^{(1)}(k-1) - 0.177290 \\ 0.955054 x_4^{(1)}(k-1) - 1.136202 \end{bmatrix}.$$

Substituting the 1-AGO series into matrix D respectively, we could obtain the simulated values of the 1-AGO series, and then carrying out inverse accumulating restoration, so we can get the simulated values (see table 1).

TABLE I. THE SIMULATION ERROR

Original values	GMSP(1, 4)		DGMSP(1, 4)	
	Simulated values	Relative errors (%)	Simulated values	Relative errors (%)
$x_1^{(0)}(2)$	2.01258852	56.232613	1.28820781	0.000606
$x_1^{(0)}(3)$	2.27024388	0.001933	2.27020518	0.000228
$x_1^{(0)}(4)$	2.99094825	0.001613	2.99090653	0.000218
$x_1^{(0)}(5)$	3.97297396	0.001862	3.97290839	0.000211
$x_2^{(0)}(2)$	1.04523859	-2.578191	1.07290251	0.000234
$x_2^{(0)}(3)$	1.20200192	-0.008159	1.20210166	0.000138
$x_2^{(0)}(4)$	1.68697855	-0.007199	1.68710189	0.000112
$x_2^{(0)}(5)$	2.18905711	-0.006527	2.18920236	0.000108
$x_3^{(0)}(2)$	1.05755859	-2.547126	1.08520135	0.000124
$x_3^{(0)}(3)$	1.20552451	-0.006262	1.20560068	0.000056
$x_3^{(0)}(4)$	1.66141026	-0.005401	1.66150101	0.000061
$x_3^{(0)}(5)$	2.10319582	-0.004953	2.10330137	0.000065
$x_4^{(0)}(2)$	1.00491533	-1.602369	1.02128118	0.000116
$x_4^{(0)}(3)$	1.02894113	-0.000862	1.02895037	0.000036
$x_4^{(0)}(4)$	1.04070690	0.000663	1.04070059	0.000057
$x_4^{(0)}(5)$	1.08221688	0.001560	1.08220081	0.000075

Note: GMSP (1, 4) is the original grey model of systematic prediction (see Ref [4]); DGMSP (1, 4) is the discrete model in this paper.

We can further get their predicted values as follows:

TABLE II. PREDICTED VALUES OF EACH VARIABLE

Predicted values	GMSP(1, 4)	DGMSP(1, 4)
$x_1^{(0)}(6)$	5.1336869175	5.1101614455
$x_1^{(0)}(7)$	6.5613508171	6.4906208858
$x_1^{(0)}8)$	8.3093915477	8.1560887635
$x_2^{(0)}(6)$	2.8258710153	2.8110176402
$x_2^{(0)}(7)$	3.5977049174	3.5549372082
$x_2^{(0)}(8)$	4.5471342087	4.4570696141
$x_3^{(0)}(6)$	2.6758140017	2.6692928377
$x_3^{(0)}(7)$	3.3704045731	3.3439155800
$x_3^{(0)}(8)$	4.2284045177	4.1662345438
$x_4^{(0)}(6)$	1.1511638048	1.1500948917
$x_4^{(0)}(7)$	1.2545187543	1.2509110528
$x_4^{(0)}(8)$	1.3982760722	1.3896549771

As can be seen from Table I, the simulation accuracy of the discrete systematic model are higher than that of the original systematic model, the reason lies in that the discrete model don't refer to the background value, avoiding to optimize the background value. When simulating the second data, GMSP (1, 4) brought about a comparatively large error, the reason lies in that we deduced the GMSP (1, N) at K>2. And fitting the coefficients of the GMSP (1, N), we don't use the first data. So it also shows that the original systematic model exists defects.

Example 2 Select the example 6.6.1 in reference [13] as another example to explain it again. The original sequence of behavior variables are as follows:

$$x_1^{(0)} = (2.874,3.278,3.307,3.390,3.679),$$
$$x_2^{(0)} = (7.040,7.645,8.075,8.530,8.774).$$

Utilize the original data to set up the discrete grey model of systematic prediction:

$$x_1^{(1)}(k)+0.2038x_1^{(1)}(k-1)=0.4877x_2^{(1)}(k)-0.4220$$
$$x_2^{(1)}(k)-1.4879x_2^{(1)}(k-1)=-1.0316x_1^{(1)}(k)+10.6148.$$

By means of the above method, we could get the simulated values and the predicted values; see Table III and Table IV respectively.

TABLE III. COMPARISON OF SIMULATION ERROR

original values	GM(1, 2)		GM(0, 2)		DGMSP(1, 2)	
	simulated values	relative errors (%)	simulated values	relative errors (%)	simulated values	relative errors (%)
$x_1^{(0)}(2)$	2.7696	15.5105	3.2701	0.2418	3.2986	0.6273
$x_1^{(0)}(3)$	3.5477	7.2794	3.2922	0.4482	3.2464	1.8328
$x_1^{(0)}(4)$	3.5345	4.2626	3.4777	2.5865	3.4501	1.7715
$x_1^{(0)}(5)$	3.5824	2.6245	3.5772	2.7681	3.6585	0.5581
$x_2^{(0)}(2)$	6.7502	11.7039	7.6650	0.2611	7.6824	0.4898
$x_2^{(0)}(3)$	9.0316	11.8468	8.1101	0.4345	8.0263	0.6025
$x_2^{(0)}(4)$	8.6743	1.6913	8.3136	2.5365	8.4561	0.8668
$x_2^{(0)}(5)$	9.1372	4.1396	9.0224	2.7681	8.9181	1.6421

Note: GM (1, 2) and GM (0, 2) are established in reference [13], but the simulated values of GM (0, 2) are wrong, here the simulated values are re-calculated in the above table.

TABLE IV. PREDICTED VALUES OF EACH VARIABLE

	$x_1^{(0)}(6)$	$x_1^{(0)}(7)$	$x_1^{(0)}(8)$
Predicted values of x_1	3.7371	3.9634	4.1398
	$x_2^{(0)}(6)$	$x_2^{(0)}(7)$	$x_2^{(0)}(8)$
Predicted values of x_2	9.2000	9.6881	10.1447

As can be seen from Table III, the simulation results of the discrete grey model of systematic prediction are much better than that of GM (1, 2) and GM (0, 2).

V. CONCLUSIONS

This paper constructed the discrete grey model of systematic prediction, which consists of a set of discrete GM (1, N). Through the identical transformation, it can be transformed into linear equations. Using the knowledge of linear algebra theory, the predicted values can be obtained. It is easy to realize through the use of computer. By Proposition 1, we can see the original grey model of systematic prediction is a special case of this model. The model thus extends the application scope of the original grey model of systematic prediction. In this paper, from the calculation example, we can find out the model has very good simulation results. From Table II and Table IV, the prediction results show that the predicted values accord with the development trend of the original sequence.

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