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NEW FORMULA FOR IMAGE RECONSTRUCTION FROM PROJECTIONS BY THE 2-D PAIRED TRANSFORMATION

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ABSTRACT

To solve the problem of image reconstruction from projections, we describe an effective representation of the image by the paired transform which is a frequency and time representation of the image. We present a new effective formula for the inverse 2-D paired transform, which can be used for solving the algebraic system of equations with measurement data for image reconstruction without calculating the Fourier transforms. The image is reconstructed directly from the paired image-signals which can be calculated from projection data. A new concept of direction images is introduced, that define the decomposition of the image by directions. The proposed method of image reconstruction has been implemented and applied for modeled images of sizes up to 256×256.

Key words – Fourier transform, reconstruction from projections, tensor and paired transforms.

1. INTRODUCTION

For practical applications of image reconstruction from projections, the Fourier transform based methods are used widely, in particular, the method of filtered back projection (FBP). The solution of this complex problem is very important in medicine diagnoses, where projections data for reconstructing two- and three-dimensional images are obtained by means of the roentgen radiation with an investigated part of the body [1].

In assumption that the complete set of linear integrals (measurements) is available, we can consider the mathematical problem of determination of functions $f(x, y)$ (in the rectangular system of coordinates (x, y)) from infinite set of their line integrals that was solved by Radon [2]. The linear integrals are described by the Radon transformation, $\mathcal{R}$, of the function $\bar{f}(r, \phi) = f(x, y)$

$$(\mathcal{R} \circ \bar{f})(l, \theta) = \int_{-\infty}^{+\infty} \bar{f}\big(\sqrt{l^2 + z^2}, \theta - arctg\,(\frac{z}{l})\big) dz, \ \text{ if } l \neq 0,$$
$$(\mathcal{R} \circ \bar{f})(0, \theta) = \int_{-\infty}^{+\infty} \bar{f}\big(z, \theta - \frac{\pi}{2}\big) dz, \qquad (1)$$

for each value (l, θ), in the system of polar coordinates (r, ϕ) that is connected with the rectangular system of coordinates (x, y) by the following expressions: $r = (x^2 + y^2)^{\frac{1}{2}}$ and $\phi = \text{arctg}(y/x)$. We can consider $\bar{f}(r, \phi)$ as the function or image of finite size, let say $L \times L$ with the center at $(0, 0)$, and therefore, consider equation (1) only for the polar coordinates (l, θ) such that $l \in [0, L/\sqrt{2}]$ and $\theta \in [0, \pi)$. The main formula obtained in computerized tomography allows us to express the image in the form of the Fourier slice theorem [3]:

$$\bar{f} = \mathcal{F}_{2D}^{-1}\big[\mathcal{F}_{1D;l}\big(\mathcal{R} \circ \bar{f}\big)\big]. \qquad (2)$$

One-dimensional Fourier transforms are calculated over all Radon linear integrals and then the inverse two-dimensional Fourier transform is applied to obtain the reconstructed image. This method requires the interpolation when transforming the Fourier projections from the polar grid (r_k, ϕ_l) to the traditional rectangular lattice (n, m). And because the assumption does not hold in typical medical imaging systems, the sampled version of the Radon transform is used [4]. Another methods to solve this problem for practical applications are the series expansion methods (SEM) for solving the system of line integrals in the discrete case [5, 6]. In this approach, Gordon and others researches proposed the iterative procedures of approaches of the reconstructed image.

In this paper, we generalize and simplify the Fourier transform based method of reconstruction and consider the discrete model which is used in the SEM for solving the system of line integrals. Our focus is the paired representation that reduces the 2-D image to the set of independent 1-D signals of different lengths [7, 8]. We present a new effective formula for the inverse 2-D paired transform, which can be used for solving the algebraic system of equations of the discrete model without using the Fourier transforms. For that a new concept of the direction image is introduced, with respect to the paired representation.

2. MODEL OF RECONSTRUCTION

We consider the simple process of the projection data collection in the computerized X-tomography. Suppose that a reconstruction image $f(x, y)$ occupies the quadratic domain of dimension L by L, i.e., $x, y \in [0, L]$, on which the quadratic lattice $N \times N$ of image elements (IE) are marked. We assume that the absorption function of the (n, m)th image element, where $n, m = 0 : (N - 1)$, takes a constant value $f_{n,m}$. We also assume that the radiation source and detector represent the points, and that the rays spreading between them are straight. When using such a discrete model, the measured value of the total attenuation energy along the lth ray, denoted via y_l, $l \in \{1, ..., M\}$, can be represented in the form of a finite series of the unknown image $\{f_{n,m}\}$ along this

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ray:

$$y_l = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} a_{n,m}^l f_{n,m}, \quad (3)$$

where $a_{n,m}^l$ is the length way of lth ray along the (n,m)th IE. y_l are attenuation measurements that are also called the summation coefficients along the lth ray. The set of measurements y_l taken at a fixed direction is called a *projection*. We assume that the size of the image elements is small and the following condition holds:

$$a_{n,m}^l = \begin{cases} 1, & \text{if the } l\text{th ray intersects the } (n,m)\text{th IE} \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

for all $l = 1 : M$ and $n,m = 0 : (N-1)$. The notation $n = 0 : (N-1)$ denotes an integer number n that runs numbers from 0 to $(N-1)$. We also assume that the rays pass along knots of the discrete lattice and values of $f_{n,m}$ correspond to the samples of the discrete image at points with coordinates (n,m). We now describe the solution of system (3) relative to the image $\{f_{n,m}\}$.

3. RECONSTRUCTION BY THE 2-D PAIRED TRANSFORM

Let $f = \{f_{n,m}\}$ be an image of size $N \times N$, where $N = 2^r$ and $r > 1$. The $N \times N$-point 2-D DFT of the image, accurate to the normalizing factor $1/N$, is defined by

$$F_{p,s} = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} f_{n,m} W^{np+ms}, \quad W = \exp(-2\pi j/N).$$

Frequency-points are from the square lattice $X_{N,N} = \{(p,s);\ p, s = 0 : (N-1)\}$. This lattice can be divided by a set of subsets in such a way, that the 2-D DFT of the image at frequency-points of each subset is represented by the 1-D DFT of a corresponding signal. In other words, such signal represents the image in a subset of frequency-points. To show such a splitting of the lattice, we consider the concept of the paired representation. Given a triplet (p,s,t), where (p,s) is a frequency-point and t is an integer time-point from the interval $[0, N-1]$, we define the following set of points (n,m) of the lattice in the image plane

$$V_{p,s,t} = \{(n,m);\ n,m = 0 : (N-1),\ np + ms = t \bmod N\}$$

and its characteristic function

$$\chi_{p,s,t}(n,m) = \begin{cases} 1, & \text{if } np + ms = t \bmod N \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

The sets $V_{p,s,t}$ consist of points (n,m) allocated along a maximum of $M = p + s$ parallel straight lines at an angle of $\psi = \pi - \text{arctg}(p/s)$ to the horizontal axis. In the image domain the equations for these parallel lines are

$$xp + ys = t + kN, \quad k = 0 : (p + s - 1). \quad (6)$$

As a partition of the lattice $X_{N,N}$, we consider the following family of $(3N-2)$ disjoint subsets:

$$\sigma' = \left(\left(\left(T'_{2^k p, 2^k s}\right)_{(p,s)\in G_{r-k}} \right)_{k=0:(r-1)}, \{(0,0)\} \right),$$

where the subsets equal

$$T'_{p,s} = \left\{ (\overline{(2m+1)p}, \overline{(2m+1)s});\ m = 0 : (N/2-1) \right\}$$

$\bar{l} = l \bmod N$, and the sets G_k of generators are defined by

$$G_k = \{(p,1), p = 0 : (2^k - 1)\} \cup \{(1,2s), s = 0 : (2^{k-1} - 1)\}.$$

We denote by $J'_{N,N}$ the set of generators of all subsets $T'_{p,s}$ of the partition σ'. The number of frequency-points of subset $T'_{p,s}$ equals $N/2^{k+1}$, where $2^k = g.c.d(p,s)$, $k \geq 0$. The number m in the definition of the subset $T'_{p,s}$ runs from 0 to $(N/2-1)$ with the step 2^k. With each subset $T'_{p,s}$, we associate a 1-D short signal $f_{T'_{p,s}}$ of length $N/2^{k+1}$, which is called the *paired image-signal*,

$$f_{T'_{p,s}} = \left\{ f'_{p,s,0}, f'_{p,s,2^k}, f'_{p,s,2\cdot 2^k}, \ldots, f'_{p,s,N/2-2^k} \right\}. \quad (7)$$

This is the unique 2-D frequency plus 1-D time representation of the 2-D image. The components of this signal are calculated by

$$f'_{p,s,t} = \sum_{V_{p,s,t}} f_{n,m} - \sum_{V_{p,s,t+N/2}} f_{n,m} = f_{p,s,t} - f_{p,s,t+N/2}$$

where $t = 0, 2^k, 2\cdot 2^k, \ldots, N/2 - 2^k$. Here we remind that the representation of the image in the form of the $3N/2$ image-signals of length N each,

$$\chi_{N,N} : \{f_{n,m}\} \to \{f_{p,s,t};\ (p,s) \in G_r,\ t = 0 : (N-1)\},$$

is called *the tensor transformation* which is redundant. Given a frequency-point (p,s) of the lattice X, the following holds [7]:

$$F_{\overline{(2m+1)p}, \overline{(2m+1)s}} = \sum_{t=0}^{N/2^{k+1}-1} \left(f'_{p,s,2^k t} W^t_{N/2^k} \right) W^{mt}_{N/2^{k+1}} \quad (8)$$

where $m = 0 : (N/2^{k+1} - 1)$. The 2-D DFT at frequency-points of the subset $T'_{p,s}$ is defined by the $N/2^{k+1}$-point DFT of the splitting-signal $f_{T'_{p,s}}$ modified by the vector of twiddle coefficients $\{W^t_{N/2^k};\ t = 0 : (N/2^{k+1} - 1)\}$. As an example, Figure 1 shows the tree image of size 256×256 in part a, along with the image-signal of length 128, which is generated by the frequency $(p,s) = (1,4)$ in b, the magnitude of the 1-D 128-point DFT of this signal in c, the 2-D DFT of the image with marked locations of the frequency-points of the subset $T'_{1,4}$, where the 2-D DFT coincides with the 1-D DFT, in d.

The transformation

$$\chi'_{N,N} : \{f_{n,m}\} \to \{f'_{p,s,t};\ (p,s,t) \in U\}$$

is called the 2-D *paired* transformation [7]. U is the set of all number-triplets (p,s,t) of components of the image-signals generated by frequencies $(p,s) \in J'_{N,N}$. The paired transform does not require multiplications. The paired transformation is not separable and is not defined by an abstract system of basic functions, but the system that is extracted from the mathematical structure of the 2-D DFT. The complete system of paired functions is defined as

$$\chi'_{p,s,t}(n,m) = \chi_{p,s,t}(n,m) - \chi_{p,s,t+N/2}(n,m), \ \ (p,s,t) \in U.$$

Each paired function represents itself a 2-D plane wave, and the decomposition of the image by the paired functions is the decomposition of the image by plane waves. All samples of the set $V_{p,s,t}$ lie on a family of parallel rays passing along samples of the discrete lattice $X_{N,N}$ traced on the initial image. All "1"s in the mask of the 2-D function $\chi_{p,s,t}(n,m)$ lie on the parallel lines that we denote by $r(p,s,t)_1, r(p,s,t)_2, \ldots, r(p,s,t)_M$,

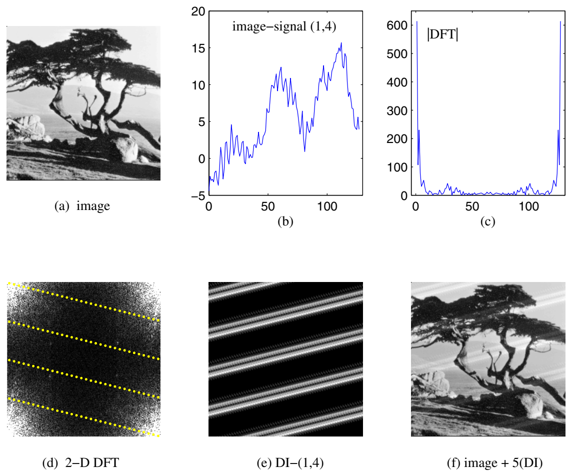

Fig. 1. (a) Original image 256×256, (b) the image-signal generated by the frequency $(1,4)$, (c) the magnitude of the 1-D DFT of this signal, (d) the 2-D DFT of the image and marked locations of the frequency-points of $T'_{1,4}$, (e) the direction image, and (f) the tree image with the amplified direction image.

$M \geq 1$. The linear integrals, or the sums of the image along these parallel lines define the component $f_{p,s,t}$ of the image-signal $\{f_{p,s,0}, f_{p,s,1}, \ldots, f_{p,s,N-1}\}$ in the tensor representation. This image-signal is calculated by the linear integrals which compose one projection, whose angle is defined by the frequency (p,s). Therefore, the following statement holds [9]:

Statement 1 *Given a frequency-point (p,s), components $f_{p,s,t}$, $t = 0 : (N-1)$, of the image-signal of a discrete image $f_{n,m}$ in tensor representation can be calculated directly from the projection data. These components are equal to the following sums of summation coefficients:*

$$f_{p,s,t} = y(p,s,t)_1 + y(p,s,t)_2 + \ldots + y(p,s,t)_M \tag{9}$$

where $M = (p+s)$ and

$$y(p,s,t)_k = \sum_{(n,m)\in r(p,s,t)_k} f_{n,m}, \quad k = 1 : M. \tag{10}$$

All "1"s and "-1"s in the masks of paired functions $\chi'_{p,s,t}(n,m)$ lie on the parallel lines. As an example, Figure 2 shows the gray scale images of three basic paired functions $\chi'_{1,5,2}, \chi'_{2,2,8}$, and $\chi'_{4,4,8}$ for the $N = 32$ case in parts a, b, and c, respectively. The components of the paired image-signals are defined by subtracting components of the tensor image-signals.

Statement 2 *Given a frequency-point (p,s), the components $f'_{p,s,t}$ of the corresponding image-signal $f_{T'_{p,s}}$ of a discrete image $f_{n,m}$ can be calculated directly from the projection data. These components are equal to the following sums of summation coefficients:*

$$f'_{p,s,t} = y(p,s,t)_1 + y(p,s,t)_2 + \ldots + y(p,s,t)_M -$$
$$-y(p,s,t+\tfrac{N}{2})_1 - y(p,s,t+\tfrac{N}{2})_2 - \ldots - y(p,s,t+\tfrac{N}{2})_M$$

where $t = 0, 2^k, 2\cdot 2^k, \ldots, N/2 - 2^k$, $2^k = g.c.d(p,s)$, and $M = (p+s)$.

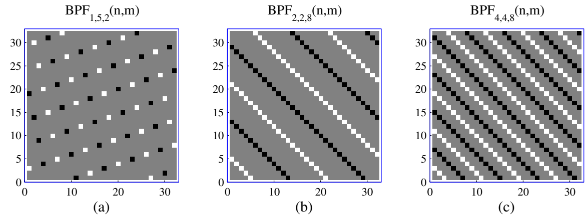

Fig. 2. Images 32×32 of the paired functions (a) $\chi'_{1,5,2}$, $(b)\chi'_{2,2,8}$, and (c) $\chi'_{4,4,8}$. (Values $1, -1$, and 0 correspond to the white, black, and gray levels, respectively.)

We denote by $\mathcal{L}_2$ the stated linear operation of composition of components $f'_{p,s,t}$ of the paired transform from projection data y_l. Each subset $T'_{p,s}$ of frequency-points generates the image-signal $f_{T'_{p,s}}$ which carries the spectral information of the image $f_{n,m}$ in these frequencies. The image-signal is thus the set-frequency characteristics of the image. The paired representation similar to the tensor representation allows for reconstructing the image through the 2-D DFT in the Cartesian grid,

$$f = \mathcal{F}_{2D}^{-1}\left[\mathcal{F}_{1D;p,s}\left(\mathcal{L}_2 \circ y_l\right)\right]. \tag{11}$$

This formula can be referred to as the Fourier slice theorem in the Cartesian grid.

However, the paired representation allows for inverting directly the projection data from the frequency and time domain, without calculating the Fourier transforms, both 1-D and 2-D. The paired image-signal defines the corresponding direction image-component of $f_{n,m}$. Indeed, let D_{p_1,s_1} be the following 2-D DFT composed only from the components of the 2-D DFT which lie on the given subset $T'_{p,s}$:

$$D_{p_1,s_1} = \begin{cases} F_{p_1,s_1}; & \text{if } (p_1,s_1) \in T'_{p,s}, \\ 0; & \text{otherwise.} \end{cases} \tag{12}$$

We first consider the case when g.c.d.$(p,s) = 1$, i.e. $(p,s) \in J_{2^r,2^r}$. The inverse transform of the defined 2-D DFT, or the direction image of the 1st series can be calculated as follows:

$$\begin{aligned} d_{n,m} &= \frac{1}{N^2}\sum_{p_1=0}^{N-1}\sum_{s_1=0}^{N-1} D_{p_1,s_1} W^{-(np_1+ms_1)} \\ &= \frac{1}{N^2}\sum_{(p_1,s_1)\in T'_{p,s}} F_{p_1,s_1} W^{-(np_1+ms_1)} \\ &= \frac{1}{N^2}\sum_{k=0}^{N/2-1} F_{\overline{(2k+1)p},\overline{(2k+1)s}} W^{-(n\overline{(2k+1)p}+m\overline{(2k+1)s})} \\ &= \frac{1}{N^2}\sum_{k=0}^{N/2-1} F_{\overline{(2k+1)p},\overline{(2k+1)s}} W^{-(2k+1)\overline{(np+ms)}} \\ &= \frac{1}{2N}\left(\frac{2}{N}\sum_{k=0}^{N/2-1} F_{\overline{(2k+1)p},\overline{(2k+1)s}} W_{N/2}^{-kt}\right) W^{-t} \\ &= \frac{1}{2N}\left(f'_{p,s,t}W^t\right) W^{-t} = \frac{1}{2N} f'_{p,s,t} \end{aligned} \tag{13}$$

where we denote $t = (np+ms) \bmod N$ and consider that the components $f'_{p,s,t+N/2} = -f'_{p,s,t}$ when $t = 0 : (N/2-1)$. Thus the direction image of the 1st series is defined as

$$d_{n,m} = d_{n,m;p,s} = \frac{1}{2N} f'_{p,s,(np+ms) \bmod N}, \quad (n,m) \in X_{N,N}.$$

In other words, the direction image $N \times N$ is composed of $N/2$ values (or N values if we consider their signs) of the image-signal $f_{T'_{p,s}}$, which are placed along the parallel lines $np+ms = t \bmod N$, $t = 0 : (N-1)$. The direction of these lines, or the projection is defined by coordinates of the frequency-point (p,s).

As an example, we consider the tree image of size 256×256 shown in Figure 3(a). Fifteen direction images $d_{n,m;p,s}$, for the generators $(p,s) = (0,1), (1,1), \ldots, (10,1), (1,0), (1,2), (1,4)$, and $(1,6)$ are shown in parts (b)-(p), respectively. The direction

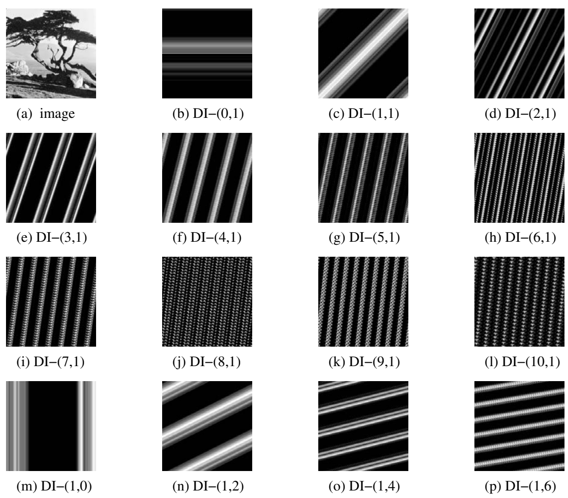

Fig. 3. (a) Original image and (b)–(p) 15 direction image components of the image. (All images have been scaled.)

image for $(p,s) = (1,4)$ is shown also in Figure 1 in part e, together with the image $f_{n,m} + 5d_{n,m;1,4}$ in f.

In the case when g.c.d.$(p,s) = 2^k$, $k \geq 1$, the calculations similar to that in (13) result in the following direction image of the kth series (or the 2-D signal with positive and negative values)

$$d_{n,m} = d_{n,m;p,s} = \frac{1}{2^{k+1}N} f'_{p,s,(np+ms) \bmod N}, \quad (n,m) \in X_{N,N}. \tag{14}$$

All $(3N-2)$ subsets $T'_{p,s}$, with generators (p,s) from the set $J'_{N,N}$ compose the partition of the grid $X_{N,N}$. Therefore, the sum of all direction images $d_{n,m} = d_{n,m;p,s}$ equals the original image $f_{n,m}$. In other words, we obtain the following decomposition of the image by $3N-2$ direction images:

$$\begin{aligned} f_{n,m} &= \sum_{(p,s)\in J'_{N,N}} d_{n,m;p,s} \\ &= \frac{1}{2N} \sum_{k=0}^{r} \frac{1}{2^k} \sum_{(p,s)\in 2^k G_{r-k}} f'_{p,s,(np+ms) \bmod N} \end{aligned}$$

where the $k = r$ case corresponds to the set $G_0 = \{(0,0)\}$ and normalize coefficient $1/2^{k-1}$ instead of $1/2^k$. This is the formula of reconstruction of the image by its paired transform, by using operations of addition/subtraction and division by powers of two.

Formula of Image Reconstruction:

$$\begin{aligned} f_{n,m} &= \frac{1}{2N} \sum_{k=0}^{r} \frac{1}{2^k} \sum_{(p,s)\in 2^k G_{r-k}} \Big[y(p,s,\overline{np+ms})_1 \\ &+ y(p,s,\overline{np+ms})_2 + \ldots + y(p,s,\overline{np+ms})_M - \\ &- y(p,s,\overline{np+ms} + \tfrac{N}{2})_1 - y(p,s,\overline{np+ms} + \tfrac{N}{2})_2 \\ &- \ldots - y(p,s,\overline{np+ms} + \tfrac{N}{2})_M \Big] \end{aligned}$$

where $M = p + s$. Thus, we solve the problem of the image reconstruction from the projections on the Cartesian grid by using the inverse 2-D paired transform

$$f = (\chi'_{N,N})^{-1} \big[(\mathcal{L}_2 \circ y_l)_{(p,s)\in J'} \big]. \tag{15}$$

This is the paired transform slice theorem.

4. CONCLUSION

Based on the paired representation of the image, the problem of image reconstruction from its projections in the framework of the discrete model has been solved. A new effective formula has been derived for the inverse 2-D paired transform, which can be used for solving the algebraic system described the measurement data of image reconstruction without using the 2-D Fourier transform technique. The image can be reconstructed directly from the image-signals which are calculated from projection data. The proposed formula can be effectively implemented in practical application when reconstructing images from their projections.

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