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Solving Fuzzy Linear Programming Problems with Interval Type-2 RHS.

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***Abstract*—This paper presents two general methods to handle uncertainties in the Right Hand Side parameters of a Linear Programming (LP) model by means of Interval Type-2 Fuzzy Sets (IT2 FS).**

In this paper, a LP problem with uncertain Right Side parameters treated as Interval Type-2 Fuzzy sets is solved by two optimization strategies: The first one is a type-reduction method and the second one is a pre-defuzzified $\alpha - cut$ approach.

After the IT2 FS inference process, a real-valued solution must be found. In this way two methods based on classical optimization routines are presented to obtain optimal solutions when uncertain right hand side parameters exist.

I. Introduction and Motivation

FUZZY Linear Programming (FLP) is a suitable technique to involve uncertainty to some situations by finding optimal solutions by means of Type-1 Fuzzy Sets (T1 FS). To do so, three basic models can be identified: Fuzzy Objective parameters, Fuzzy Right Hand Side Parameters (RHS) and Fuzzy Joint parameters, all of them attemp to find optimal solutions in an $\alpha - cut$ sense.

Many Linear Programming (LP) and FLP applications as Production planning, Scheduling, Forecasting, Capacities planning, Facilities location, Inventory control, etc. use interval-valued RHS to get degrees of freedom, which implies both pre and post-optimal analysis in order to obtain a suitable interpretation of those intervals and its crisp optimal solution. An easy way to get these intervals from IT2 FS is by $\alpha - cuts$, so an $\alpha - cut$ optimization strategy is proposed.

The scope of this paper is to involve uncertainty in FLP problems by means of IT2 FS, obtaining an Interval Type-2 Fuzzy Linear Programming model (IT2 FLP) which is optimized by using classical algorithms. In addition, this paper presents an extension of the results presented by Figueroa in[1] in the sense that a pre-defuzzification method based on an $\alpha - cut$ method is proposed, turning the problem to an interval-valued approach.

The paper is divided into eight principal sections. In Section 1, a brief Introduction and Motivation is presented; the Section 2 presents basic definitions of LP models; in Section 3, the FLP with Fuzzy RHS model is presented; in Section 4, a discussion about sources of uncertainty in FLP is given; in Section 5, a IT2 FLP model with IT2 FS RHS parameters is presented; the Section 6 shows the proposed solution procedure; in Section 7 a small application example is solved and Section 8 contains the concluding remarks of the work.

II. Basic Definition of a LP problem.

The classical format of a LP model is a crisp set of equations which conforms a relational matrix A that solves i boolean row inequalities[1] represented by a vector named b where the optimal solution x^* is found by means of an objective function $c'x$. Its definition is given next.

$$\begin{aligned} &\textit{Opt}\,\{c'x + c_0\} \\ &\textit{Subject to:} \\ &Ax \leqslant b \\ &x \geqslant 0 \end{aligned} \tag{1}$$

Where $x \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$, $b \in \mathbb{R}^m$ and A is an *(n x m)* matrix, $A_{n\times m} \in \mathbb{R}^{n\times m}$.

This linear form assumes that c, c_0, A and b are deterministic parameters (constants) and they do not have uncertainty, this means that those parameters *"Are completely True"*. For further information about LP problems and their algorithms, see Bazaraa in [2] and Dantzig in [3].

III. Basic definitions for a LP with Fuzzy RHS.

The classical format of a LP problem with Fuzzy RHS is a deterministic set of equations which conforms a relational matrix A that solves i row inequalities represented by a fuzzy vector called $\breve{b}$ where the optimal solution x^* is found by means of an objective function $c'x$. Its definition is:

$$\begin{aligned} &\textit{Opt}\,\{c'x + c_0\} \\ &\textit{Subject to:} \\ &Ax \lesssim \breve{b} \\ &x \geqslant 0 \end{aligned} \tag{2}$$

Where $x \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$, $\breve{b} \in \mathcal{F}(S)$ and A is an *(n x m)* matrix, $A_{n\times m} \in \mathbb{R}^{n\times m}$. $\mathcal{F}(S)$ is the the set of all fuzzy sets.

Here, all parameters are constant except the RHS vector $\breve{b}$ which is considered as a T1 FS or a *Soft Constraint* as Zimmermann defined in [4] and [5]. To reduce its complexity, only linear membership functions with parameters $\check{b}$ and $\hat{b}$ are considered (See (3) and (4)). For further information about FLP problems, see Klir & Yuan in [6], Lai & Hwang in [7], Kacprzyk & Orlovski in [8], Pandiant in [9] and Zimmermann in [4] and [5].

[1] Also known as restrictions.

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Two possible partial orders can be identified: $\lesssim$ and $\gtrsim$. For the $=$ case, its membership function could vary depending on the original objective. For the first and second instances, their membership functions are:

$$\mu_f(x;\check{b},\hat{b}) = \begin{cases} 1, & x \leqslant \check{b} \\ \dfrac{\hat{b}-x}{\hat{b}-\check{b}}, & \check{b} \leqslant x \leqslant \hat{b} \\ 0, & x \geqslant \hat{b} \end{cases} \quad (3)$$

$$\mu_f(x;\check{b},\hat{b}) = \begin{cases} 0, & x \leqslant \check{b} \\ \dfrac{x-\check{b}}{\hat{b}-\check{b}}, & \check{b} \leqslant x \leqslant \hat{b} \\ 1, & x \geqslant \hat{b} \end{cases} \quad (4)$$

A. Solution Procedure.

Zimmermann in [4] and [5] presented the most used method to solve this problem. He defined a Fuzzy Set of solutions $\check{\tilde{z}}(x^*)$ to find a joint-optimal $\alpha - cut$ for $\check{\tilde{z}}(x^*)$ and $\check{b}$. The method is summarized next:

1) Calculate an inferior bound called *Z Minimum* $(\check{z})$ by using $\check{b}$ as a frontier of the model.
2) Calculate a superior bound called *Z Maximum* $(\hat{z})$ by using $\hat{b}$ as a frontier of the model.
3) Define a Fuzzy Set $\check{\tilde{z}}(x^*)$ with bounds $\check{z}$ and $\hat{z}$ and trapezoidal membership function. This set represents the degree that any feasible solution has regarding the optimization objective.
4) If the objective is to Maximize, then its membership function is:

$$\mu_{\check{\tilde{z}}}(x;\check{z},\hat{z}) = \begin{cases} 1, & c'x \geqslant \hat{z} \\ \dfrac{c'x-\check{z}}{\hat{z}-\check{z}}, & \check{z} \leqslant c'x \leqslant \hat{z} \\ 0, & c'x \leqslant \check{z} \end{cases} \quad (5)$$

5) If the objective is to Minimize, then its membership function is:

$$\mu_{\check{\tilde{z}}}(x;\check{z},\hat{z}) = \begin{cases} 1, & c'x \leqslant \check{z} \\ \dfrac{\check{z}-c'x}{\hat{z}-\check{z}}, & \check{z} \leqslant c'x \leqslant \hat{z} \\ 0, & c'x \geqslant \hat{z} \end{cases} \quad (6)$$

Now, the principal idea is to maximize the fulfillment degree that a set of optimal solutions x^* has regarding an optimal fuzzy set $\check{\tilde{z}}(x^*)$ obtained from A and $\check{b}$. As in LP problems where the intersection among restrictions generates a polyhedric convex set of feasible solutions, the problem is focused in finding an optimal $\alpha - cut$ overall degree of satisfaction of all fuzzy, "Boolean"[2] restrictions and $\check{\tilde{z}}(x^*)$. To that effect, the following definitions are given.

Definition 3.1: The fuzzy set conformed by all possible values which satisfy a fuzzy restriction with parameters $\check{b}$, and $\hat{b}$ is called $\check{b}$. Its membership function is defined in (3) or (4).

[2]A Boolean restriction b_i is considered as the set of possible values content in $[-\infty, b_i]$, each one with one as its belonging degree, otherwise, all other values have zero belonging degree.

Definition 3.2 (Fuzzy Feasible Region): Be:

$$D_i(x) = {}^{\alpha}\check{b}_i\left(\sum_{j=1}^{n} a_{ij}x_j\right);\ \forall\ i \in \mathbb{N}_n \quad (7)$$

The $\alpha - cut$ degree that the "i_{th}" fuzzy restriction has regarding the $\check{b}_i$ set, a_{ij} is the $(i,j)_{th}$ element of A, $(i,j) \in \mathbb{N}_n$.

Consider a constrained mathematic programming problem where any RHS coefficient is defined as a fuzzy number, then its feasible fuzzy region *(FFR)* is defined as the fuzzy polyhedric set generated by the elements $x \in \mathbb{R}^n$ which satisfy all fuzzy and boolean frontiers, that is:

$$\textit{FFR} = \bigcap_{i=1}^{m} D_i(x), b_i \quad (8)$$

It means that all elements of $x \in \mathbb{R}^n$ which satisfy both crisp b_i and fuzzy D_i, $i \in \mathbb{N}_n$ restrictions define the set of solutions.

Now, the *Fuzzy Feasible Set* (FFS) is represented by all values which satisfy all fuzzy and crisp restrictions with non-negative membership degrees, that is:

$$\textit{FFS} = \{X \mid x \in \bigcap_{i=1}^{m} D_i(x), b_i\} \quad (9)$$

Where $D_i(x)$ is the $\alpha - cut$ membership degree regarding the "i_{th}" Fuzzy restriction, b_i is the "i_{th}" crisp restriction and $\check{b}_i$ is the "i_{th}" Fuzzy restriction, $i \in \mathbb{N}_n$.

In the domain of x, the above is currently the problem of finding a vector of solutions $\{x \in \mathbb{R}^n\}$ such that:

$$\max{}^{\alpha}\left\{\left(\bigcap_{i=1}^{m} D_i(x)\right)\bigcap \check{\tilde{z}}(x^*)\right\} \quad (10)$$

In this way the problem is focused in finding an $\alpha - cut$ which maximizes the model by using (6) or (7) according to the objective. To do so, an auxiliary variable namely α is created to represent the $\alpha - cut$ overall degree.

Thus, if the objective is to Maximize, then the FLP is:

$$\begin{gathered} \max\ \{\alpha\} \\ \textit{Subject to:} \\ c'x + c_0 - \alpha(\hat{z}-\check{z}) = \check{z} \\ Ax + \alpha(\hat{b}-\check{b}) \leqslant \hat{b} \\ x \geqslant 0 \end{gathered} \quad (11)$$

Where $x \in \mathbb{R}^n$, $\check{z} \in \check{\tilde{z}}$, $\hat{z} \in \check{\tilde{z}}$, $c_0 \in \mathbb{R}$, $\check{b} \in \mathcal{F}(S)$, $\hat{b} \in \mathcal{F}(S)$, $\alpha \in [0,1]$ and A is an *(n x m)* matrix, $A_{n\times m} \in \mathbb{R}^{n\times m}$. And if the objective is to Minimize, then the FLP is:

$$\begin{gathered} \max\ \{\alpha\} \\ \textit{Subject to:} \\ c'x + c_0 + \alpha(\hat{z}-\check{z}) = \hat{z} \\ Ax - \alpha(\check{b}-\hat{b}) \geqslant \check{b} \\ x \geqslant 0 \end{gathered} \quad (12)$$

Where $x \in \mathbb{R}^n$, $\breve{z} \in \breve{z}$, $\hat{z} \in \breve{z}$, $c_0 \in \mathbb{R}$, $\breve{b} \in \mathcal{F}(S)$, $\hat{b} \in \mathcal{F}(S)$, $\alpha \in [0,1]$ and A is an *(n x m)* matrix, $A_{n\times m} \in \mathbb{R}^{n\times m}$

So, a linear form which maximizes the joint $\alpha - cut$ degree for both $\breve{z}$ and $\breve{b}$ is obtained and later it is optimized by using classical algorithms.

IV. Uncertainty Sources.

Data analysis and its measurement could become into a low confidence process, getting biased calculations and estimations in both statistical and mathematical contexts, being a source of uncertainty which leads to bad post-optimal solutions. For further information see [10] and [11].

As always, the estimation of the RHS vector could contain bias from reality because many estimates could be considered as the *"Best Linear Unbiased Estimate"*. Moreover, several analysts could define different estimates as "correct", knowingly that they are obtained by different methods. All these situations are *Uncertainty Sources*.

According to Mendel in [12] pp. 66-78, Klir & Yuan in [6] and Klir & Folger in [13], many uncertainty sources on a FLP context exist. Several parameters used in LP may contain uncertainty depending on its intrinsic properties (See Lodwick & Jamison in [14] and Inuiguchi & Sakawa [15]). Specifically, the following sources of uncertainty are proposed:

- Poor variables measurement.
- Lack input information (Information Retrieval).
- Bad statistical analysis and estimation process for the system parameters.
- Selection of a worst criterion for data analysis.
- Ambiguity in data estimation.
- Bad definition of the system restrictions.

In this way a Type-2 Fuzzy Sets approach is a suitable way to find an optimal solution of an uncertain LP problem.

A. Elements of Uncertain RHS.

In this paper, some definitions about IT2 FS and IT2 FLP are taken from Figueroa in [1] & [16], Mendel in [12], [17], [18], [19], [20] & [21] and Melgarejo in [22] & [23].

Definition 4.1 (IT2 RHS): Consider a set of RHS parameters of a FLP problem defined as an IT2 FS $\tilde{b}$ defined on the closed interval $\tilde{b_i} \in [\underline{b}_i, \bar{b_i}]$, $\{\underline{b}_i, \bar{b_i}\} \in \mathbb{R}$ and $i \in \mathbb{N}_n$. The membership function which represents the fuzzy space[3] $\tilde{b}_i$ is:

$$\tilde{b}_i = \int_{b_i\in\mathbb{R}} \left[\int_{u\in J_{b_i}} 1/u\right] \Big/ b_i;\ \ i \in \mathbb{N}_n,\ J_{b_i} \subseteq [0,1] \quad (13)$$

Definition 4.2 ($\alpha - cut$ for IT2 FS RHS): First, it is presented the classical definition of a Type-1 $\alpha - cut$:

$$^{\alpha}b = \{x|\mu_b(x) \geqslant \alpha\} \quad (14)$$

Now, an extension for a Type-2 fuzzy set is:

$$^{\alpha}\tilde{b} = \int_{x\in X} \left[\int_{u\in J_x \geqslant \alpha} f_x(u)/u\right] \Big/ x\ ;\ \alpha \subseteq [0,1] \quad (15)$$

$$^{\alpha}\tilde{b} = \int_{x\in X} \int_{u\in f_x} \{(x,u)|J_x \geqslant \alpha\}\ ;\ \alpha, f_x \subseteq [0,1] \quad (16)$$

[3]A Fuzzy Space is defined by the interval $b_i \subseteq \tilde{b}$.

Or in a wive slice representation:

$$^{\alpha}\tilde{b} = \bigcup_{x\in J_x} \{\mu_{\tilde{b}}(x,u)|J_x \geqslant \alpha\} \quad (17)$$

$$^{\alpha}FOU(\tilde{b}) = \bigcup_{x\in X} \{J_x|J_x \geqslant \alpha\} \quad (18)$$

In this way, the crisp interval of an $\alpha - cut$ of the FOU of a Type-2 fuzzy set is defined as $\alpha^I - cut$:

$$^{\alpha^I}\tilde{b} = \int_{x\in X} \int_{u\in f_x} \{(x,u)|J_x = \alpha\}\ ;\ \alpha, f_x \subseteq [0,1] \quad (19)$$

$$^{\alpha^I}\tilde{b} = \bigcup_{x\in J_x} \{\mu_{\tilde{b}}(x,u)|J_x = \alpha\} \quad (20)$$

Now, $\tilde{b}$ is bounded by two *Lower* and *Upper* primary membership functions[4] called $\underline{\mu}_{\tilde{b}}(x)$ with parameters $\underline{\check{b}}$ and $\underline{\hat{b}}$ and $\bar{\mu}_{\tilde{b}}(x)$ with parameters $\bar{\check{b}}$ and $\bar{\hat{b}}$ respectively, so the crisp bounds of $^{\alpha^I}\tilde{b}$ are:

$$[^{\alpha^c}\underline{\mu}_{\tilde{b}}(x), ^{\alpha^c}\bar{\mu}_{\tilde{b}}(x)] = \left[\inf\{x \in X : x \in ^{\alpha^I}\tilde{b}\}, \sup\{x \in X : x \in ^{\alpha^I}\tilde{b}\}\right] \quad (21)$$

Its graphical representation is shown in the Figure 1.

An IT2 RHS is an uncertain restriction composed by an IT2 FS with four parameters: $\bar{\check{b}}$, $\underline{\check{b}}$, $\bar{\hat{b}}$ and $\underline{\hat{b}}$ so two distances $\triangle$ and ∇ are defined as follows.

Definition 4.3: Consider an IT2 FLP problem with restrictions in the form $\precsim$[5]. Then $\triangle$ is defined as the distance between $\underline{\check{b}}$ and $\bar{\check{b}}$, $\triangle = \bar{\check{b}} - \underline{\check{b}}$ and ∇ is defined as the distance between $\underline{\hat{b}}$ and $\bar{\hat{b}}$, $\nabla = \bar{\hat{b}} - \underline{\hat{b}}$.

Henceforth, three cases can be identified on an IT2 RHS environment: Uncertain ∇, uncertain $\triangle$ and joint uncertain ∇ & $\triangle$. Uncertain ∇ is a case in which $\nabla > 0$ with $\triangle = 0$, uncertain $\triangle$ is a case in which $\triangle > 0$ with $\nabla = 0$, uncertain $\triangle$ is a case in which $\triangle > 0$ with $\triangle = \nabla$ and joint uncertain $\triangle$ & ∇ is a case in which $\triangle > 0$ and $\nabla > 0$ where $\triangle \neq \nabla$. Its graphical representations are displayed below[6].

In this Figure only exist two $^{\alpha^c}\tilde{b}$, that is, $^{\lambda^c}\underline{\mu}_{\tilde{b}_i}$ and $^{\lambda^c}\bar{\mu}_{\tilde{b}_i}$, but other fuzzy sets can have 2 or more intervals of $^{\alpha^c}\tilde{A}$.

Note that the following statement is mandatory: $\{[\underline{\check{b}_i}, \underline{\hat{b}_i}] \cap [\bar{\check{b}}_i, \bar{\hat{b}}_i]\} > 0\ \forall\ i \in \mathbb{N}_n$.

The special case where $\{[\underline{\check{b}_i}, \underline{\hat{b}_i}] \cap [\bar{\check{b}}_i, \bar{\hat{b}}_i]\} = 0$ for any $i \in \mathbb{N}_n$ converges to a T1FS and it is not treated in this work, being an interesting case to be used in a IT2 FS approach.

V. FLP with IT2 RHS Parameters.

The difference between the T1 and IT2 FLP lies in the handle of uncertainty. While the T1 FLP uses a single membership function for each RHS, the IT2 FLP approach uses an infinite amount of T1 FS embedded on the FOU of each RHS, involving the opinions and perceptions of many analysts.

[4]For simplicity effects, only trapezoidal fuzzy sets with parameters a_1 and a_2 are considered.

[5]For the $\succsim$ case it is easy to use a symmetric reasoning to solve the problem.

[6]FOU is the acronym for *Foot of Uncertainty.*

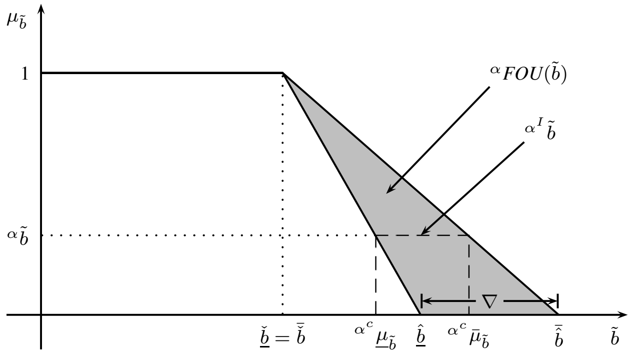

Fig. 1. IT2 FS RHS with Uncertain ∇

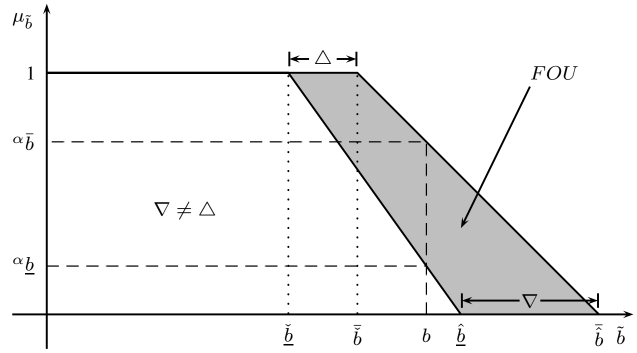

Fig. 3. IT2 FS RHS with Joint Uncertain $\triangle$ & ∇.

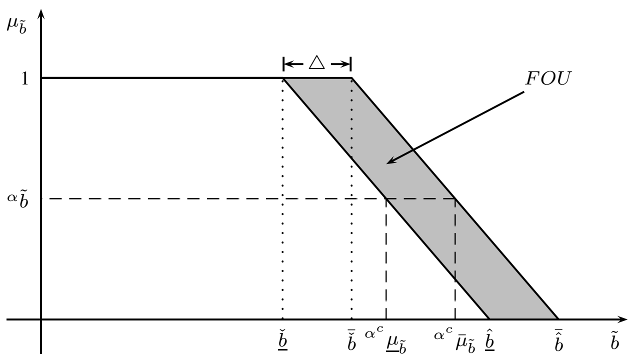

Fig. 2. IT2 FS RHS with Uncertain $\triangle = \nabla$

In the same way as is presented in (2), (3) and (4) a FLP with IT2 RHS can be defined as follows:

$$
\begin{aligned}
&\textit{Opt}\,\{c'x + c_0\}\\
&\textit{Subject to:}\\
&Ax \precsim \tilde b\\
&x \geqslant 0
\end{aligned}
\tag{22}
$$

Where $x \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$, $\tilde b \in \mathcal{F}(S)$ and A is an *(n x m)* matrix, $A_{n\times m} \in \mathbb{R}^{n\times m}$. Note that $\tilde b$ is an IT2 FS defined by two primary membership functions $\underline{\mu}_b(x)$ and $\bar\mu_b(x)$.

Two possible partial orders $\precsim$ and $\succsim$ with IT2 fuzzy RHS exist. The lower membership function for $\precsim$ is:

$$
\underline{\mu}_b(x; \check{\underline b}, \hat{\underline b}) = \begin{cases} 1, & x \leqslant \check{\underline b} \\ \dfrac{\hat{\underline b} - x}{\hat{\underline b} - \check{\underline b}}, & \check{\underline b} \leqslant x \leqslant \hat{\underline b} \\ 0, & x \geqslant \hat{\underline b} \end{cases}
\tag{23}
$$

And its upper membership function is:

$$
\bar\mu_b(x; \check{\bar b}, \hat{\bar b}) = \begin{cases} 1, & x \leqslant \check{\bar b} \\ \dfrac{\hat{\bar b} - x}{\hat{\bar b} - \check{\bar b}}, & \check{\bar b} \leqslant x \leqslant \hat{\bar b} \\ 0, & x \geqslant \hat{\bar b} \end{cases}
\tag{24}
$$

The Lower membership function for $\succsim$ is:

$$
\underline{\mu}_b(x; \check{\underline b}, \hat{\underline b}) = \begin{cases} 0, & x \leqslant \check{\underline b} \\ \dfrac{x - \check{\underline b}}{\hat{\underline b} - \check{\underline b}}, & \check{\underline b} \leqslant x \leqslant \hat{\underline b} \\ 1, & x \geqslant \hat{\underline b} \end{cases}
\tag{25}
$$

And its upper membership function is:

$$
\bar\mu_b(x; \check{\bar b}, \hat{\bar b}) = \begin{cases} 0, & x \leqslant \check{\bar b} \\ \dfrac{x - \check{\bar b}}{\hat{\bar b} - \check{\bar b}}, & \check{\bar b} \leqslant x \leqslant \hat{\bar b} \\ 1, & x \geqslant \hat{\bar b} \end{cases}
\tag{26}
$$

VI. Solution procedure for an IT2 RHS problem.

In this instance the main goal is to design a suitable method to obtain optimal solutions for an IT2 RHS problem by using classical algorithms. A first approach is to decompose the problem in two stages: A first one which only uses the lower membership functions $\underline{\mu}_b(x; \check{\underline b}, \hat{\underline b})$ and the second one uses the upper membership functions $\bar\mu_b(x; \check{\bar b}, \hat{\bar b})$. After individual solutions are obtained, the best one should be selected.

The second method selects an optimal T1 RHS embedded on the FOU of the problem[7] to find only one $\alpha - cut$ which maximizes the problem. This proposal could be viewed as an Optimal Type-Reducer since it finds only one IT1 FS for each restriction. On next subsections, both proposals are explained.

A. First Method.

This first method was proposed by Figueroa in [1]. It consists on a Type-reduction method based on finding a T1 FS embedded on the FOU of the IT2 RHS parameters of the problem which is summarized as follows:

1) Calculate an inferior bound called *Z Minimum* $(\check z)$ by using $\underline{b} + \triangle$ as a frontier of the model, where $\triangle$ is an auxiliary set of variables which represents the lower uncertainty interval subject to the following statement:

$$
\triangle \leqslant \check{\bar b} - \check{\underline b}
\tag{27}
$$

[7]The T1 RHS shape depends if the IT2 RHS has Uncertain $\triangle$ or ∇.

2) Calculate a superior bound called *Z maximum* $(\hat{z})$ by using $\bar{\hat{b}} + \nabla$ as a frontier of the model, where ∇ is an auxiliary set of variables which represents the upper uncertainty interval subject to the following statement:

$$\nabla \leqslant \bar{\hat{b}} - \hat{\underline{b}} \tag{28}$$

3) Find the final solution using the third and subsequent steps of the algorithm presented in III-A, (11) and (12).

B. Second Method.

This method is a pseudo-optimal proposal which is a pre-defuzzification procedure to find a solution by means of a $\alpha - cut$ of the IT2FS which generate an interval which is used to optimize the problem by means of classical methods.

1) Select an overall pre-deffuzification level named λ for all fuzzy RHS parameters.
2) Compute the $\alpha^I - cut$ for all fuzzy RHS sets, ${}^{\lambda^c}\mu_{\tilde{b}_i}(b)$. It generates an interval in the form $[{}^{\lambda^c}\underline{\mu}_{\tilde{b}_i}(b_i),\ {}^{\lambda^c}\bar{\mu}_{\tilde{b}_i}(b_i)]$[8].
3) Solve the problem by using an interval valued LP approach in the form:

$$\begin{gathered} Opt\ \{c'x + c_0\} \\ \textit{Subject to:} \\ Ax - b \leqslant 0 \\ {}^{\lambda^c}\underline{\mu}_{\tilde{b}}(b) \leqslant b \leqslant {}^{\lambda^c}\bar{\mu}_{\tilde{b}}(b) \\ x \geqslant 0 \end{gathered} \tag{29}$$

Where $x, c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $c_0 \in \mathbb{R}$, and A is an *(n x m)* matrix, $A_{n\times m} \in \mathbb{R}^{n\times m}$.

The existence of a solution is ensured in the sense that all restrictions are defined on a feasible set as is shown in (9), so $\nabla, \triangle$ and all α^I have a feasible solution.

Remark 6.1: Due to the conditions given in (27) and (28), these methods could lead to multiple solutions. To avoid this inconvenient, it is recommended to assign an appropriate weight to each variable, according to either unitary cost or benefit of increase a specifical resource.

Note that the first method builds an optimal T1 FS and then it is solved as a T1 FLP problem, being a T1 FLP embedded on the FOU, meanwhile the second method finds an interval of solutions for each $\tilde{b}_i$ IT2 RHS as a function of λ, ${}^{\lambda^c}\mu_{\tilde{b}_i}(b_i)$ and b_i as a decision variable. Both approaches can be used but their results could not converge to same optimal solution due to the second method gets larger $z(x^*)$ values when $\lambda \approx 0$ while it gets smaller $z(x^*)$ values when $\lambda \approx 1$.

VII. Application Example.

The following application example is solved by both proposals. To use the second method, it is necessary to define λ, ${}^{\lambda^c}\mu_{\tilde{b}_i}(b_i) = 0.75$.

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 10 & 4 & 9 \\ 4 & 6 & 3 \\ 2 & 7 & 7 \\ 5 & 6 & 11 \end{bmatrix};\ \check{\underline{b}} = \begin{bmatrix} 50 \\ 70 \\ 40 \\ 60 \\ 40 \end{bmatrix};\ \hat{\underline{b}} = \begin{bmatrix} 72 \\ 104 \\ 65 \\ 95 \\ 80 \end{bmatrix}$$

$$\bar{\check{b}} = \begin{bmatrix} 60 \\ 80 \\ 55 \\ 75 \\ 57 \end{bmatrix};\ \bar{\hat{b}} = \begin{bmatrix} 95 \\ 110 \\ 77 \\ 102 \\ 98 \end{bmatrix};\ c = \begin{bmatrix} 12 \\ 17 \\ 9 \end{bmatrix}$$

$${}^{\lambda^c}\underline{\mu}_b(b) = \begin{bmatrix} 66.5 \\ 95.5 \\ 58.75 \\ 86.25 \\ 70 \end{bmatrix};\ {}^{\lambda^c}\bar{\mu}_b(b) = \begin{bmatrix} 86.25 \\ 102.5 \\ 71.5 \\ 95.25 \\ 87.75 \end{bmatrix}$$

By using the first method, the obtained IT2 FS $\tilde{z}(x^*)$ is displayed in the Figure 4. The optimal solution is provided by $x_1 = 4.9365$, $x_2 = 6.8545$ and $x_3 = 0$, $\triangle_1 = \triangle_2 = \triangle_4 = 0$, $\triangle_3 = 15$ and $\triangle_5 = 17$, $\nabla_1 = \nabla_4 = 0$, $\nabla_2 = 6$, $\nabla_3 = 12$ and $\nabla_5 = 18$, $\underline{z} = 157.16$, $\bar{z} = 193.59$, $\breve{z}^* = 175.76$, ${}^{\alpha}\breve{z} = 0.5106$, $b_1 = 61.233$, $b_2 = 90.296$, $b_3 = 60.424$, $b_4 = 77.871$ and $b_5 = 77.913$ by evaluating the optimal $\breve{z}(x^*)$ fuzzy set.

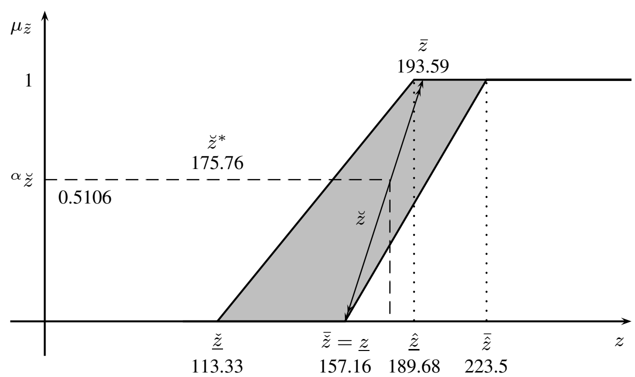

Fig. 4. Obtained embedded T1 FS $\breve{z}^*$ on the FOU of $\tilde{z}$.

By using the second method, the optimal solution is provided by $x_1 = 7.47734$, $x_2 = 6.9318$ and $x_3 = 0$, $b_1 = 66.5$, $b_2 = 102.5$, $b_3 = 71.5$, $b_4 = 86.25$, $b_5 = 78.977$ and $z^* = 138.25$.

It is clear that both models does not provide a same solution since they find optimal solutions on IT2 regions by using different linear approximations of the problem.

VIII. Concluding Remarks.

Some conclusions can be suggested:

1) Two new methods are presented, the first one gets an IT1 FS embedded on the FOU of each IT2 RHS and then it is optimized. The second one is a pseudo-optimal approach which reduces the complexity of the problem by using $\alpha - cuts$ and interval optimization.
2) These procedures can vary on their results because they are optimal in different ways. Moreover, the analyst

[8]The order of this interval is defined for fuzzy sets as is shown in the Figures 1, 2 and 3. For their complement, the correct interval is $[{}^{\lambda^c}\bar{\mu}_{\tilde{b}_i}(b_i),\ {}^{\lambda^c}\underline{\mu}_{\tilde{b}_i}(b_i)]$.

should select the best choice depending on the computational efforts, amount of restrictions and variables.
3) A sensibility analysis must be done to get more robustness in the study. It will performs its interpretability.
4) The reader should keep in mind that the first method solves three LP problems before finding a solution while the second method solve only one LP problem. It could become into a disadvantage for large-scale problems.
5) The second method is conditioned to the selection of the λ value, given by an expert of the system so it does not lead to the same solution than the first method.

Finally, an infinite amount of T1 FS are embedded on the FOU of the IT2 FS where the proposed methods get a solution of the LP problem. A Type-reduction method and an $\alpha - cut$ method are proposed to obtain a solution of the problem.

A. Further Topics.

The use of non-linear primary membership functions is an interesting topic to be treated via $\alpha - cuts$. Additionally, the secondary membership function $f_x(u)/u$ of a Type-2 fuzzy set induces to new directions and uncertain problems.

DEDICATORY

Juan Carlos Figueroa García dedicates his work to the memory of Maria Ninfa García Bautista (*Rest in peace my angel*).

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