

Fuzzy Random Facility Location Problems with Recourse

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Abstract—The objective of this paper is to study facility location problems under a hybrid uncertain environment involving randomness and fuzziness. A two-stage fuzzy random facility location model with recourse is developed in which the demands and the costs are assumed to be fuzzy random variables. As in general the fuzzy random parameters in the model can be regarded as continuous fuzzy random variables with infinite realizations, the computation of the recourse requires solving infinite second-stage programming problems. Owing to this fact, the recourse function cannot be calculated analytically, which implies that the model cannot benefit from the use of methods of classical mathematical programming. In order to solve the location problems of this nature, we first develop techniques of fuzzy random simulation. In the sequel, by combining the fuzzy random simulation, simplex algorithm and binary particle swarm optimization (BPSO), a hybrid algorithm is proposed to solve the two-stage fuzzy random facility location model. Finally, an illustrative numerical example is provided.

Index Terms—Facility location; Two-stage fuzzy random programming; Recourse; Fuzzy random variable; Binary particle swarm optimization.

I. INTRODUCTION

Facility location selection, as one of the most critical and strategic issues in supply chain design and management, exhibits a significant impact on market share and profitability. Stochastic facility location problems deal with the cases where the parameters are treated as random variables. Logendran and Terrell [14] developed a stochastic uncapacitated transportation plant location-allocation model with an objective of maximizing the expected profits, and proposed a heuristic solution algorithm; Louveaux and Peeters [15] discussed a dual-based procedure for a stochastic facility location problem with recourse. Schutz *et al.* [19] considered a stochastic facility location problem with general long-run costs and convex short-run costs, and solved the problem through a Lagrangian relaxation based method.

Recently, another class of facility location problems with uncertain parameters based on fuzzy set theory [3], [11], [17], [18], [25] has been developed, which aims to deal with the cases of imprecise or vague data. For instance, Bhattacharya *et al.* [1] considered facilities located under multiple fuzzy criteria, and proposed a fuzzy goal programming approach to deal with the problem. Ishii *et al.* [4] developed a location

model by considering the satisfaction degree with respect to the distance from the facility for each customer. Assuming that demands of customers are represented as fuzzy variables, Wen and Iwamura [24] presented a continuous α -cost facility location model employing Hurwicz criterion, while Zhou and Liu [27] presented three types of continuous capacitated location-allocation problem with different decision criteria.

In real-world applications, randomness and fuzziness may coexist in a facility location problem. On the one hand, due to the subjective judgement, imprecise human knowledge and perception in capturing statistical data, the parameters of the real location problems may embrace randomness and fuzziness at the same time. On the other hand, sometimes the historical data available for the parameters in location problems may be insufficient, therefore, the experience-based fuzzy information can be incorporated into the originally available statistic data. In both cases mentioned above, there is a genuine need to deal with a hybrid uncertainty containing simultaneously randomness and fuzziness.

The concept of fuzzy random variables [9], [10], [13] was introduced to quantify and deal with the phenomena in which vagueness and randomness appear at the same time. This formalism serves as a basic tool to construct a framework of decision making models operating in a fuzzy and random environment. By making use of fuzzy random variables, this paper aims to study facility location problems in the integrated fuzzy random environment. A two-stage fuzzy random facility location model with recourse is developed through expected profit maximization. In virtue of the fuzzy random parameters introduced, the two-stage facility location model proposed in the present paper can embrace more general cases than those captured by the stochastic and fuzzy facility location models separately, since it can cope with the randomness as well as fuzziness at the same time.

The study is organized as follows. Section II recalls some preliminaries on fuzzy random variables. In Section III, we formulate the problem and reveals its difficulties. In Section IV, we design the fuzzy random simulation of the recourse. Section V focuses on the solution of the model. In Section VI, a numerical example is provided. Section VII gives conclusions.

II. PRELIMINARIES

In this section, we briefly review the preliminaries on fuzzy random variables, which help the readers in understanding the proposed two-stage fuzzy random facility location model.

Let the triplet $(\Gamma, \mathcal{P}(\Gamma), \text{Pos})$ be a possibility space, where $\mathcal{P}(\Gamma)$ is the power set of Γ , X be a fuzzy variable defined on $(\Gamma, \mathcal{P}(\Gamma), \text{Pos})$ whose membership function is μ_X , and r be a real number. The possibility, necessity, and credibility of the event $X \leq r$ are expressed as follows:

$$\begin{aligned} \text{Pos}\{X \leq r\} &= \sup_{t \leq r} \mu_X(t), \\ \text{Nec}\{X \leq r\} &= 1 - \sup_{t > r} \mu_X(t), \text{ and} \\ \text{Cr}\{X \leq r\} &= \frac{1}{2} \left(\sup_{t \leq r} \mu_X(t) + 1 - \sup_{t > r} \mu_X(t) \right). \end{aligned} \quad (1)$$

Suppose that $(\Omega, \mathcal{A}, \text{Pr})$ is a probability space, \mathcal{F}_v is a collection of fuzzy variables defined on possibility space $(\Gamma, \mathcal{P}(\Gamma), \text{Pos})$. A fuzzy random variable (see [13]) is a map $\xi : \Omega \rightarrow \mathcal{F}_v$ such that for any Borel subset B of \mathfrak{R} , $\text{Pos}\{\xi(\omega) \in B\}$ is a measurable function of ω .

From the above definition, for fuzzy random variable ξ on $(\Omega, \mathcal{A}, \text{Pr})$, we know that for each $\omega \in \Omega$, $\xi(\omega)$ is a fuzzy variable. Furthermore, for a fuzzy random variable ξ defined on a probability space $(\Omega, \mathcal{A}, \text{Pr})$, the expected value of ξ is defined as (see [13])

$$E[\xi] = \int_{\Omega} \left[\int_0^{\infty} \text{Cr}\{\xi(\omega) \geq r\} dr - \int_{-\infty}^0 \text{Cr}\{\xi(\omega) \leq r\} dr \right] \text{Pr}(d\omega).$$

For more detailed discussion on fuzzy random variables, one may refer to [9], [10], [16], [22], [23], [26].

III. PROBLEM STATEMENT

A. Mathematical modelling

When modelling a fuzzy random facility location problem with recourse, we encounter the following settings: a firm intends to open a facility in n potential sites, the cost of each facility consists of fix opening and operating cost and variable operating cost, the latter being a fuzzy random variable. There are m customers having fuzzy random demands for some commodity. Each customer can be supplied from an open facility where the commodity is made available. The distribution pattern from facilities to customers is not predetermined but adapted to the realization of the fuzzy random event with respected to the demands and variable operating costs. The objective of the firm is to maximize the expected profit by choosing the optimal number of facilities to open and their locations in market areas. To proceed with detailed discussion, it would be advantageous to introduce some useful notation.

Notation

i	the index of facilities, $1 \leq i \leq n$
j	the index of clients, $1 \leq j \leq m$
D_j	fuzzy demand of client j
r_j	the unit price charged to client j
s_i	the capacity of facility i
c_i	the fixed cost for opening and operating facility i
V_i	the unit variable operating cost of facility i , which is a fuzzy random variable
ξ	fuzzy random demand-cost vector
x_i	decision variable which is a binary variable equal to one if facility i , is open and zero otherwise
\mathbf{x}	decision vector which is $\mathbf{x} = (x_1, x_2, \dots, x_n)$
y_{ij}	the quantity supplied to client j from i
t_{ij}	unit transportation cost from i to j

Assumptions

1. Each customer's demand cannot be over served, but it is possible that not all demand is served.

2. The total supply from one facility to all clients cannot exceed the capacity of the facility.

3. Fuzzy random demand-cost vector $\xi = (D_1, \dots, D_m, V_1, \dots, V_n)$ is defined from a probability space $(\Omega, \mathcal{A}, \text{Pr})$ to a collection of fuzzy vectors on possibility space $(\Gamma, \mathcal{P}(\Gamma), \text{Pos})$, and the demand vector $D = (D_1, \dots, D_m)$ and cost vector $V = (V_1, \dots, V_n)$ are mutually independent.

Given the above assumptions and making use of the introduced notation, a two-stage fuzzy random facility location model with recourse (FR-FLMR) can be formulated concisely as follows

$$\begin{cases} \max & Q(\mathbf{x}) - \sum_{i=1}^n c_i x_i \\ \text{subject to} & x_i \in \{0, 1\}, i = 1, 2, \dots, n, \end{cases} \quad (2)$$

where $Q(\mathbf{x}) = E[Q(\mathbf{x}, \xi)]$, and

$$\begin{cases} Q(\mathbf{x}, \xi(\omega, \gamma)) \\ = \max & \sum_{i=1}^n \sum_{j=1}^m (r_j - V_i(\omega, \gamma) - t_{ij}) y_{ij} \\ \text{subject to} & \sum_{i=1}^n y_{ij} \leq D_j(\omega, \gamma), j = 1, 2, \dots, m, \\ & \sum_{j=1}^m y_{ij} \leq s_i x_i, i = 1, 2, \dots, n, \\ & y_{ij} \geq 0, i = 1, 2, \dots, n, j = 1, 2, \dots, m. \end{cases} \quad (3)$$

Here, $D_j(\omega, \gamma)$ and $V_i(\omega, \gamma)$ are the realizations of fuzzy random demand D_j and fuzzy random cost V_i , respectively, for any $(\omega, \gamma) \in \Omega \times \Gamma$.

In the two-stage fuzzy random facility location problem (2)-(3) with recourse, a distinction is made between the first stage and the second stage of processing. The location decision vector \mathbf{x} is the first-stage decision which must be considered

before the realizations of fuzzy random demand-cost vector

$$\xi(\omega, \gamma) = (D_1(\omega, \gamma), \dots, D_m(\omega, \gamma), V_1(\omega, \gamma), \dots, V_n(\omega, \gamma)) \quad (4)$$

coming out. At the second stage, the fuzzy random demands and costs are known (the realizations $\xi(\omega, \gamma)$ are observed), therefore, the second-stage decision variables (also called recourse decisions) $y_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$, which represent the distribution pattern form facilities to customers, are determined to maximize the return

$$\sum_{i=1}^n \sum_{j=1}^m (r_j - V_i(\omega, \gamma) - t_{ij}) y_{ij}$$

corresponding to the current outcome of the fuzzy random event (ω, γ) . The decisions y'_{ij} s are determined as soon as \mathbf{x} and ξ are known by solving a linear programming (3), which is called a *second-stage problem*. Since $y_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$ are completely determined by the selection of \mathbf{x} and the realized value $\xi(\omega, \gamma)$ of ξ , the only real decision of problem (2)-(3) is the location decision \mathbf{x} .

Furthermore, given \mathbf{x} and $\xi(\omega, \gamma)$, the optimal value of the second-stage problem (3), i.e., $Q(\mathbf{x}, \xi(\omega, \gamma))$, is usually referred to as second-stage value function, and the expected value of $Q(\mathbf{x}, \xi)$, $Q(\mathbf{x}) = E[Q(\mathbf{x}, \xi)]$ is called the *recourse function* of the two-stage fuzzy random facility location problem.

B. Difficulties

From the above descriptions, we can observe that in order to obtain the objective value $Q(\mathbf{x}) - \sum_{i=1}^n c_j x_j$ at fixed decision \mathbf{x} of problem (2)-(3), we first need to calculate the recourse function $Q(\mathbf{x})$, which involves solving the second-stage problems (3) to obtain $Q(\mathbf{x}, \xi(\omega, \gamma))$ for all realizations $\xi(\omega, \gamma), (\omega, \gamma) \in \Omega \times \Gamma$. Nevertheless, this is not an easy task. Since the recourse function $Q(\mathbf{x})$ is the expected value of $Q(\mathbf{x}, \xi)$, to obtain $Q(\mathbf{x})$ we need to compute $Q(\mathbf{x}, \xi(\omega, \gamma))$ by solving a second (linear) programming (3) for each (ω, γ) . In the case that the fuzzy random demand-cost vector ξ comes with finitely discrete realizations, $Q(\mathbf{x}, \xi)$ also takes on finite values, then we can obtain the recourse function $Q(\mathbf{x})$ through finite calculations. However, generally speaking, for each $\omega, \xi(\omega)$ can be a continuous fuzzy vector which has infinite numbers of realizations. In such a case, calculating $Q(\mathbf{x})$ becomes an infinite-dimensional optimization problem which can not be solved analytically, since it requires solving infinite linear programming problems. Furthermore, the complexity of the problem rapidly increases if the random parameter involved in ξ is a continuous random vector at the same time, which means we have infinite ω 's to deal with in addition to the infinite γ 's.

Additionally, from the above discussion, we see that in general the recourse function $Q(\mathbf{x})$ cannot be expressed analytically, therefore, the problem (2)-(3) cannot be solved by the methods of classical mathematical programming.

IV. COMPUTING RECOURSE FUNCTION

This section focuses on the computation of the recourse function $Q(\mathbf{x})$. Incorporating the fuzzy simulation technique [12] into the random simulation, we propose the fuzzy random simulation to approximate the $Q(\mathbf{x})$ in the following two cases, i.e., discrete and continuous fuzzy random demand-cost vector ξ , respectively.

A. Case I: Discrete Demand-Cost Vector

For discrete case, the fuzzy random demand-cost vector ξ involved in the problem (2)-(3) is a discrete one such that ω is a discrete random vector taking on a finite number of values ω_i with probability $p_i, i = 1, 2, \dots, N$, respectively; and for each $i, \xi(\omega_i)$ is a discrete fuzzy vector which takes on the following values

$$\begin{aligned} \hat{\xi}^{i1} &= (\hat{D}_1^{i1}, \dots, \hat{D}_m^{i1}, \hat{V}_1^{i1}, \dots, \hat{V}_n^{i1}) \\ \hat{\xi}^{i2} &= (\hat{D}_1^{i2}, \dots, \hat{D}_m^{i2}, \hat{V}_1^{i2}, \dots, \hat{V}_n^{i2}) \\ &\dots \\ \hat{\xi}^{iN_i} &= (\hat{D}_1^{iN_i}, \dots, \hat{D}_m^{iN_i}, \hat{V}_1^{iN_i}, \dots, \hat{V}_n^{iN_i}) \end{aligned}$$

with membership degrees $\mu_{i1}, \mu_{i2}, \dots, \mu_{iN_i}$, where for each $i, \max_{j=1}^{N_i} \mu_{ij} = 1$. Without any loss of generality, we assume that for each i and fixed \mathbf{x} , the second-stage value function satisfies the condition $Q(\mathbf{x}, \hat{\xi}^{i1}) \leq Q(\mathbf{x}, \hat{\xi}^{i2}) \leq \dots \leq Q(\mathbf{x}, \hat{\xi}^{iN_i})$, then the value of the recourse function $Q(\mathbf{x})$ at \mathbf{x} is computed as follows

$$Q(\mathbf{x}) = \sum_{i=1}^N p_i Q(\mathbf{x}, \omega_i) \quad (5)$$

where

$$Q(\mathbf{x}, \omega_i) = \sum_{j=1}^{N_i} q_{ij} Q(\mathbf{x}, \hat{\xi}^{ij}), \quad (6)$$

for each pair (i, j) . Here, the second-stage value $Q(\mathbf{x}, \hat{\xi}^{ij})$ is obtained by solving the second-stage programming (3) via Simplex Algorithm, and the corresponding weights q_{ij} 's are given in the form

$$q_{ij} = \frac{1}{2} \left(\max_{k=1}^j \mu_{ik} - \max_{k=0}^{j-1} \mu_{ik} \right) + \frac{1}{2} \left(\max_{k=j}^{N_i} \mu_{ik} - \max_{k=j+1}^{N_i+1} \mu_{ik} \right) \quad (7)$$

($\mu_{i0} = 0, \mu_{i, N_i+1} = 0$) for $i = 1, 2, \dots, N; j = 1, 2, \dots, N_i$, and satisfy the following constraints

$$q_{ij} \geq 0, \text{ and } \sum_{j=1}^{N_i} q_{ij} = \max_{j=1}^{N_i} \mu_{ij} = 1, i = 1, 2, \dots, N.$$

The computation procedure for the recourse function $Q(\mathbf{x})$ with discrete fuzzy random demand-cost vector can be summarized in the following manner.

Algorithm 1. [Discrete case]

- Step 1. Set i from 1 to N , repeat the following Steps 2-5.
- Step 2. Compute the $Q(\mathbf{x}, \hat{\xi}^{ij})$ by solving the second-stage programming (3) for $j = 1, 2, \dots, N_i$.

Step 3. Rearrange the subscript j of $Q(\mathbf{x}, \hat{\xi}^{ij})$ such that

$$Q(\mathbf{x}, \hat{\xi}^{i1}) \leq Q(\mathbf{x}, \hat{\xi}^{i2}) \leq \dots \leq Q(\mathbf{x}, \hat{\xi}^{iN_i}).$$

Step 4. Determine the weight q_{ij} of $Q(\mathbf{x}, \hat{\xi}^{ij})$ by (7) for
 $j = 1, \dots, N_i.$

Step 5. Calculate $Q(\mathbf{x}, \omega_i)$ via (6).

Step 6. Return the value of $Q(\mathbf{x})$ through (5).

B. Case II: Continuous Demand-Cost Vector

In the case of continuous demand-cost vector, for the continuous ξ , the involved random parameter is a continuous random vector and, simultaneously, for any $\omega \in \Omega$, fuzzy vector $\xi(\omega)$ is also a continuous one with an infinite support

$$\Xi = \prod_{j=1}^{m+n} [a_j, b_j]. \quad (8)$$

In this case, we make use of the random simulation together with the discretization method [12] so as to calculate the recourse function $Q(\mathbf{x})$.

To deal with the infinite random realizations, we first generate randomly M values $\hat{\omega}_i, 1 \leq i \leq M$ from the distribution of the continuous random vector, and for each $\hat{\omega}_i$, in order to handle the infinite fuzzy realizations of $\xi(\hat{\omega}_i)$, we use the discretization method as follows to generate uniformly $N_{\hat{\omega}_i}$ sample vectors $\hat{\zeta}_l^{ij}, 1 \leq j \leq N_{\hat{\omega}_i}$ from the support Ξ of $\xi(\hat{\omega}_i)$.

Given some integer l , we construct the discrete fuzzy random vector $\hat{\zeta}_l^{ij} = (\hat{\mathcal{D}}_{l,1}^{ij}, \dots, \hat{\mathcal{D}}_{l,m}^{ij}, \hat{\mathcal{V}}_{l,1}^{ij}, \dots, \hat{\mathcal{V}}_{l,n}^{ij})$ as follows: For each $1 \leq j \leq m$, we define $\mathcal{D}_{l,j} = g_{l,j}(D_j)$ and $\mathcal{V}_{l,j} = g_{l,j}(V_{j-m})$ for $m+1 \leq j \leq n$, where $g_{l,j}(\cdot)$'s are given as follows

$$g_{l,j}(u_j) = \sup \left\{ \frac{k}{l} \mid k \in Z, \text{ s.t. } \frac{k}{l} \leq u_j \right\}, u_j \in [a_j, b_j] \quad (9)$$

Z is the set of integers.

Then applying the method of determination of $Q(\mathbf{x}, \omega_i)$ as presented in Subsection 4.1 to $\hat{\zeta}_l^{ij}, 1 \leq j \leq N_{\hat{\omega}_i}$, we can obtain the $Q(\mathbf{x}, \hat{\omega}_i)$. After that, making use of the random simulation, we obtain the recourse function $Q(\mathbf{x})$ by taking the expression of $\sum_{i=1}^M Q(\mathbf{x}, \hat{\omega}_i)/M$.

The detailed computing procedures is outlined as follows.

Algorithm 2. [Continuous case]

Step 1. Set $Q = 0$ and $l = L$, where L is a sufficiently large integer.

Step 2. Randomly generate a simple point $\hat{\omega}$ from the distribution of the continuous random vector.

Step 3. Generate sample points

$$\hat{\zeta}_l^j = (\hat{\mathcal{D}}_{l,1}^j, \dots, \hat{\mathcal{D}}_{l,m}^j, \dots, \hat{\mathcal{V}}_{l,1}^j, \hat{\mathcal{D}}_{l,n}^j)$$

uniformly via (9) from the support Ξ of $\xi(\hat{\omega})$ for $j = 1, 2, \dots, N_{\hat{\omega}}$.

Step 4. Compute the $Q(\mathbf{x}, \hat{\zeta}_l^j)$ through solving the second-stage programming (3) for $j = 1, 2, \dots, N_{\hat{\omega}}$.

Step 5. Rearrange the subscript j of $Q(\mathbf{x}, \hat{\zeta}_l^j)$ such that

$$Q(\mathbf{x}, \hat{\zeta}_l^1) \leq Q(\mathbf{x}, \hat{\zeta}_l^2) \leq \dots \leq Q(\mathbf{x}, \hat{\zeta}_l^{N_{\hat{\omega}}}).$$

Step 6. Determine the weight q_j of $Q(\mathbf{x}, \hat{\zeta}_l^j)$ by (7) for
 $j = 1, \dots, N_{\hat{\omega}}.$

Step 7. Calculate $Q(\mathbf{x}, \hat{\omega})$ by (6).

Step 8. $Q \leftarrow Q + Q(\mathbf{x}, \hat{\omega})$.

Step 9. Repeat the Steps 2-8 M times.

Step 10. Return the value of $Q(\mathbf{x}) = Q/M$.

V. THE ALGORITHM

In this section, we incorporate the fuzzy random simulation with the Simplex Algorithm into a binary particle swarm optimization algorithm to produce a hybrid algorithm to solve the two-stage fuzzy random facility location problem (2)-(3).

Particle swarm optimization (PSO), a nature-inspired evolutionary computation algorithm, was originally developed by Kennedy and Eberhart [6] in 1995. Standard PSO is a real-coded algorithm, in order to deal with binary-coded optimization problems, a discrete binary PSO (BPSO) was introduced by Kennedy and Eberhart [7] in 1997. BPSO has been found to be robust in solving optimization problems featuring discrete decision variables [20], [21].

In the BPSO, an n -dimensional potential solution to a problem is represented as a particle i having current position \mathbf{x}_i which is an integer vector in $\{0, 1\}^n$ and the current velocity \mathbf{v}_{id} of the particle which represents the probability of \mathbf{x}_i taking the value 1. Each particle i maintains a record of the position of its previous best performance in a vector called \mathbf{p}_{id} . The variable g is the index of the particle with best performance so far in the population. An iteration comprises evaluation of each particle, then stochastic adjustment of \mathbf{v}_{id} in the direction of particle i 's best previous position \mathbf{p}_{id} and the best previous position \mathbf{p}_{gd} of any particle in the population. For each particle i , the velocity vector \mathbf{v}_{id} is updated through following formula:

$$\mathbf{v}_{id} = \mathcal{W} * \mathbf{v}_{id} + c_1 * \text{rand}() * (\mathbf{p}_{id} - \mathbf{x}_i) + c_2 * \text{rand}() * (\mathbf{p}_{gd} - \mathbf{x}_i) \quad (10)$$

where c_1 and c_2 are learning rates generated in the interval [0,4], $\text{rand}()$ is a uniform random number in the interval [0,1], \mathcal{W} is the inertia weight whose value decreases linearly as the number of iterations of the algorithm increases. After that, each particles $\mathbf{x}_i, i = 1, 2, \dots, P_{size}$ can be updated according to the following rule [7]

$$\text{if}(\text{rand}() < S(v_{id,j})) \text{ then } x_{ij} = 1; \text{ else } x_{ij} = 0 \quad (11)$$

for $j = 1, 2, \dots, n$, where $v_{id,j}$ and x_{ij} are the components of the vector \mathbf{v}_{id} and \mathbf{x}_i , respectively, and $S(\cdot)$ is a sigmoid function $S(x) = 1/(1 + e^{-x})$.

Incorporating the fuzzy random simulation into the BPSO, a hybrid algorithm for solving two-stage fuzzy random facility allocation problem is summarized as follows.

Algorithm 3. [Hybrid BPSO Algorithm]

Step 1. Initialize randomly a population of particles $\mathbf{x}_1, \dots, \mathbf{x}_{P_{size}}$.

- Step 2. Calculate the fitness $FIT(\mathbf{x}) = Q(\mathbf{x}) - \mathbf{c}^T \mathbf{x}$ for all particles through Algorithm 1 or 2, and evaluate each particle according to the fitness;
- Step 3. Determine the \mathbf{p}_{id} for each particle, and the \mathbf{p}_{gd} for the population;
- Step 4. Update all the particles by formulae (10) and (11);
- Step 5. Repeat Step 2 to Step 4 for a given number of generations;
- Step 6. Return the particle \mathbf{p}_{gd} as the optimal solution of the problem (3)-(2), and $FIT(\mathbf{p}_{gd})$ the optimal value.

VI. A NUMERICAL EXAMPLE

A numerical example is provided to illustrate the solution of the proposed two-stage fuzzy random facility location problem (2)-(3), and offer some insights into the effectiveness of the designed hybrid algorithm. In the PSO, the learning rates are set $c_1 = c_2 = 2$, the inertia weight \mathcal{W} is set decreased linearly from about 0.9 to 0.4 by the following expression

$$\mathcal{W} = 0.5 * (G_{max} - G_n) / G_{max} + 0.4,$$

where G_n is the index of the current generation, those settings come as a result of intensive experimentation (see [8]).

TABLE I
CAPACITIES, FIXED COSTS USED IN EXAMPLE 1

Facility site i	Capacity s_i	Fixed cost c_i
1	80	6
2	70	2
3	60	5
4	60	3
5	70	2
6	80	8
7	90	2
8	60	4
9	80	3
10	50	9

TABLE II
VARIABLE COSTS USED IN EXAMPLE 1

Facility site i	Variable cost V_i	Random parameter Y_i
1	$(2 + Y_1, 4 + Y_1, 6 + Y_1)$	$Y_1 \sim \mathcal{U}(1, 2)$
2	$(3 + Y_2, 5 + Y_2, 6 + Y_2)$	$Y_2 \sim \mathcal{U}(0, 2)$
3	$(2 + Y_3, 4 + Y_3, 5 + Y_3)$	$Y_3 \sim \mathcal{U}(1, 3)$
4	$(4 + Y_4, 6 + Y_4, 8 + Y_4)$	$Y_4 \sim \mathcal{U}(2, 3)$
5	$(3 + Y_5, 5 + Y_5, 6 + Y_5)$	$Y_5 \sim \mathcal{U}(1, 2)$
6	$(2 + Y_6, 4 + Y_6, 6 + Y_6)$	$Y_6 \sim \mathcal{U}(1, 3)$
7	$(5 + Y_7, 7 + Y_7, 8 + Y_7)$	$Y_7 \sim \mathcal{U}(2, 4)$
8	$(4 + Y_8, 6 + Y_8, 7 + Y_8)$	$Y_8 \sim \mathcal{U}(0, 2)$
9	$(3 + Y_9, 5 + Y_9, 6 + Y_9)$	$Y_9 \sim \mathcal{U}(3, 4)$
10	$(2 + Y_{10}, 4 + Y_{10}, 5 + Y_{10})$	$Y_{10} \sim \mathcal{U}(2, 3)$

Example 1. We consider a firm which plans to open new facilities in 10 potential sites, the capacities s_i , fixed costs c_i and fuzzy random operating costs V_i of each site $i, i = 1, 2, \dots, 10$ are given in Table 1. We suppose that there are 5 customers and their fuzzy random demands $D_j, j = 1, 2, \dots, 5$ are given in Table III. In addition, the unit price r_j charged to each

TABLE III
FUZZY RANDOM DEMANDS USED IN EXAMPLE 1

Customer j	Demand D_j	Random parameter Z_j
1	$(16 + Z_1, 18 + Z_1, 20 + Z_1)$	$Z_1 \sim \mathcal{U}(1, 2)$
2	$(8 + Z_2, 12 + Z_2, 14 + Z_2)$	$Z_2 \sim \mathcal{U}(1, 3)$
3	$(13 + Z_3, 16 + Z_3, 18 + Z_3)$	$Z_3 \sim \mathcal{U}(2, 4)$
4	$(12 + Z_4, 15 + Z_4, 17 + Z_4)$	$Z_4 \sim \mathcal{U}(2, 3)$
5	$(10 + Z_5, 12 + Z_5, 15 + Z_5)$	$Z_5 \sim \mathcal{U}(3, 4)$

customer and the unit transportation cost t_{ij} are listed in the form of $r_j - t_{ij}$ for the simplicity in Table IV.

TABLE IV
THE VALUES OF $r_j - t_{ij}$

$r_j - t_{ij}$	$i=1$	2	3	4	5	6	7	8	9	10
$j=1$	6	3	5	4	5	6	8	4	6	4
2	5	7	5	7	4	6	5	4	5	3
3	4	5	3	6	5	6	4	6	4	6
4	7	4	8	6	5	7	4	5	4	8
5	6	8	3	6	5	8	6	4	5	6

From the above settings, we can formulate the two-stage fuzzy random facility location problem as follows

$$\begin{cases} \max & Q(\mathbf{x}) - 6x_1 - 2x_2 - 5x_3 - 3x_4 \\ & -2x_5 - 8x_6 - 2x_7 - 4x_8 - 3x_9 - 9x_{10} \\ \text{subject to} & x_1, x_2, \dots, x_{10} \in \{0, 1\}, \end{cases} \quad (12)$$

where $Q(\mathbf{x}) = E[Q(\mathbf{x}, \boldsymbol{\xi})]$, and

$$\begin{cases} Q(\mathbf{x}, \boldsymbol{\xi}(\omega, \gamma)) \\ = \max & \sum_{i=1}^{10} \sum_{j=1}^5 (r_j - t_{ij} - V_i(\omega, \gamma)) y_{ij} \\ \text{subject to} & \sum_{i=1}^n y_{ij} \leq D_j(\omega, \gamma), j = 1, 2, \dots, 5, \\ & \sum_{j=1}^m y_{ij} \leq s_i x_i, i = 1, 2, \dots, 10, \\ & y_{ij} \geq 0, i = 1, 2, \dots, 10, j = 1, 2, \dots, 5. \end{cases} \quad (13)$$

We note that in problem (12)-(13), the demand-cost vector $\boldsymbol{\xi} = (D_1, \dots, D_5, V_1, V_2, \dots, V_{10})$ is a continuous fuzzy random vector, since all the V_i and D_j are continuous triangular fuzzy random variables for $i = 1, 2, \dots, 10; j = 1, 2, \dots, 5$. Therefore, fuzzy random simulation in the continuous case, i.e., Algorithm 4, should be utilized to compute the recourse function $Q(\mathbf{x})$.

To solve this two-stage fuzzy random facility location problem, for any feasible solution \mathbf{x} , we first generate 6,000 random sample points $\hat{\omega}_i, i = 1, 2, \dots, 6,000$, for the random simulation (such sample size is sufficient for the simulation of random expected value), and generate 4,000 fuzzy sample points $\hat{\zeta}_j, j = 1, 2, \dots, 4,000$ for each ω_i . Based on the generated samples, the second-stage programming (13) is solved by the Simplex Algorithm for each each pair $(\hat{\omega}_i, \hat{\zeta}_j)$.

Furthermore, we compute the $Q(x)$ by the fuzzy random simulation (Algorithm 4), which requires solving the second-stage programming $6,000 \times 4,000$ times. After that, the fuzzy random simulation together with the simplex algorithm is embedded into a BPSO algorithm to produce a hybrid algorithm (Algorithm 5) to search for the optimal solutions.

Following [7] where the population size of the BPSO is around the value of particle dimension, which is found to be the best for the problem, we take the population size $P_{size} = 10$. A run of the hybrid algorithm with 100 populations, we shows that the optimal location solution is $x^* = (0, 1, 1, 1, 0, 1, 1, 0, 1, 0)$, and the corresponding optimal value is $Q(x^*) - 2x_2 - 5x_3 - 3x_4 - 8x_6 - 2x_7 - 3x_9 = 267.727436$.

VII. CONCLUSIONS

In real location problems, due to subjective judgement, imprecise human knowledge and perception in capturing statistic data, many parameters are of both fuzzily imprecise and probabilistically uncertain information. In this paper, we dealt with the facility location problems with hybrid uncertainty of fuzziness and randomness. Assuming the demands of the customers and the variable operating costs of the new facilities are fuzzy random variables, we established a two-stage fuzzy random facility location model with recourse.

The solution of the model is not an easy task. Firstly, since the continuous fuzzy random parameters in the model have infinite realizations, to obtain the value of the recourse function $Q(x)$ for each location decision x , we need to solve infinite second-stage programming problems. Secondly, due to the fact that the analytical expression of the objective function $Q(x) - c^T x$ is unavailable, the classical mathematical programming methods cannot be applied to the two-stage fuzzy random facility location problem.

To compute the recourse function, we embedded the fuzzy simulation techniques into the random simulation and designed the fuzzy random simulation approaches for the recourse function with continuous demand-cost vectors, respectively. Furthermore, we integrate the Simplex Algorithm, fuzzy random simulation and BPSO to produce a hybrid algorithm for solving the two-stage fuzzy random facility location model. In addition, a numerical experiment illustrated the hybrid algorithm.

ACKNOWLEDGMENT

This work was supported by the Research Fellowships of the Japan Society for the Promotion of Science (JSPS) for Young Scientists.

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