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An Extended Power Index to Evaluate Coalition Influence Based on Blockability Relations on Simple Games

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***Abstract*—This paper extends the framework of power indices and proposes a notion of coalition power indices to evaluate influence of each coalition in simple games. A coalition power index is defined as a map that assigns a real number to each pair of a simple game and a coalition in the game. The blockability index is then defined as an example of coalition power indices based on the blockability relations on simple games. It is verified that the blockability index satisfies Dummy Coalition and Coalition Symmetry axioms, which are generalized from Dummy Player and Symmetry axioms. It is also shown that the blockability index coincides with the Banzhaf index except their constant coefficients on one-player coalitions.**

***Index Terms*—game theory, simple game, blockability relation, power index, coalition, group decision making.**

I. Introduction

Group decision situations such as meetings and elections are mathematically modeled as simple games [9], [10] in the framework of the theory of cooperative games. A simple game is defined as a special case of games in characteristic function form [10], and represents a decision situation by specifying the decision makers and the decision rule of the situation such as majority and unanimity. The decision makers are described as a finite set of players and the decision rule is described as a set of winning coalitions, which are coalitions that have enough power to control the decision as a whole.

It is very important for analysis of group decision situations to compare influence of each coalition. If such comparison becomes possible, we will be able to estimate what decision will be made and which coalitions will form. It may also be helpful information for players to make decisions.

Relations on the set of all coalitions are methods to enable such comparison of coalition influence on simple games. The blockability relations are proposed in [4] as another type of relations than the desirability relations [3], [9]. The blockability relations compare coalitions with respect to how much they can make winning coalitions losing, that is, how blockable they are, while the desirability relations compare coalitions in terms of how close they are to be winning. Differences between the blockability relations and the desirability relations are investigated in [4], [5]. The blockability relation is always transitive, while the desirability relation is not, which are proven in [4]. The blockability relation on a simple game is complete if and only if the game is S-unanimous for a coalition S as shown in [5], while the desirability relation on a simple game is complete if the game is a weighted majority game as presented in [3].

Power indices are methods to measure influence of players on simple games by assigning real numbers to them. Many types of power indices have been proposed. The most important one is the Shapley-Shubik index [8]. It is a direct application of the Shapley value [7], which was originally defined as a solution to games in characteristic function form. Specifically, the Shapley value is defined as expectation of each player's marginal contribution when each order of players arises with equal probability in formation of the grand coalition, and then, the Shapley-Shubik index is defined as the restriction of the Shapley value to simple games to indicate the probability of each player's turning a losing coalition into winning. Shapley [7] showed that the Shapley value is uniquely derived from the four axioms that are desirable properties for a solution to games in characteristic function form.

Another well-known power index is the Banzhaf index [1], [6], which measures players' probability of turning a losing coalition into winning under the assumption that each coalition forms with equal probability. In contrast to the case of the Shapley-Shubik index, the Banzhaf value [6] is derived from the Banzhaf index as its extension to games in characteristic function form. Dubey and Shapley [2] characterized the Banzhaf value as the unique value that satisfies another system of four axioms.

As far as the authors know, power indices developed in all the previous works are devoted to measure power of players. In this context, they can be said *player power indices*. However, as mentioned above, it is important to evaluate influence of each coalition, as well as of each player. The aim of this paper is to extend conventional player power indices to propose the notion of *coalition power indices*, which assign to each coalition a real number that indicates its power.

Then, the blockability index is defined as an example of coalition power indices based on the blockability relations. It measures influence of coalition S according to the concept of the blockability of S, that is, how many winning coalitions S can block in all winning coalitions in a simple game. Some properties of the blockability index are investigated in this paper. It is verified that the blockability index satisfies the two axioms, which are generalized from axioms that characterize the Shapley-Shubik index and the Banzhaf index. It is also confirmed that the blockability index coincides with the Banzhaf index except their constant coefficients on one-player coalitions.

The structure of this paper is as follows: Section II presents the framework of simple games, blockability relations and power indices. In Section III, the concept of coalition power indices is proposed and the definition and properties of the blockability index is provided. Section IV is devoted for concluding remarks.

II. Framework

This section presents the framework that is employed in this paper. Specifically, we see definitions of simple games and other relevant notions in Section II-A, the definition of the blockability relations in Section II-B, and the definitions of two types of conventional player power indices, the Shapley-Shubik index and the Banzhaf index, and their axiomatic characterizations in Section II-C.

A. Simple Games

A simple game is defined as a special case of a game in characteristic function form in the following way in the framework of the theory of cooperative games:

Let $N = \{1, 2, \ldots, n\}$ be a set of players. Each subset of N is called a *coalition* and a coalition $\{i_1, i_2, \ldots, i_m\}$ is often denoted by $i_1 i_2 \cdots i_m$ for simplicity. For example, $\{1, 2, 3\}$ is simply expressed by 123. Let 2^N denote the family of all subsets of N and $\mathbf{R}$ denote the set of all real numbers.

Definition II.1 (Game in characteristic function form [10])**.** A *game in characteristic function form* is a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbf{R}$ is a characteristic function where $v(\emptyset) = 0$.

The number $v(S)$ is interpreted as the payoff that coalition S can obtain by cooperation of members in S. Note that a game in characteristic function form is sometimes denoted by just v, instead of (N, v), when the set N of players is clear.

Definition II.2 (Simple game [9], [10])**.** A game (N, v) in characteristic function form is said to be a *simple game* when it satisfies the following conditions:

1) $v(S) \in \{0, 1\}$ for all $S \subset N$
2) $v(\emptyset) = 0$, $v(N) = 1$
3) For $S, T \subset N$, if $S \subset T$, then $v(S) \leq v(T)$ (called *monotonicity*)

The first condition divides coalitions into two types: $2^N = W(v) \cup L(v)$, where $W(v) = \{S \subset N \mid v(S) = 1\}$ and $L(v) = \{S \subset N \mid v(S) = 0\}$. A coalition in $W(v)$ is called a *winning coalition*, and in $L(v)$ a *losing coalition*. Because giving a characteristic function v is equivalent to giving a set of winning coalitions $W(v)$, a simple game is sometimes defined as a pair $(N, W(v))$ of a set N of players and a family $W(v)$ of winning coalitions.

Example II.1. Let $N = \{1, 2, 3\}$. For a characteristic function $v : 2^N \to \{0, 1\}$ such that $W(v) = \{S \subset N \mid v(S) = 1\} = \{12, 13, 23, 123\}$, the pair (N, v) form a simple game. For $v' : 2^N \to \{0, 1\}$ such that $W(v') = \{123\}$, (N, v') is another example of simple games. Simple games (N, v) and (N, v') represent three-player decision situations with the majority and the unanimous rule, respectively.

Two special types of simple games are identified by the concepts of properness and S-unanimity for a coalition S. In a proper simple game, the complement of a winning coalition is always losing. In an S-unanimous simple game, every winning coalition includes S. Each member of S is a vetoer.

Definition II.3 (properness)**.** A simple game (N, v) is said to be *proper*, if and only if it is satisfied that for all $T \subset N$, if $v(T) = 1$ then $v(N \backslash T) = 0$.

Definition II.4 (S-unanimity)**.** Consider a simple game (N, v) and a coalition $S \subset N$. (N, v) is said to be *S-unanimous*, if and only if it is satisfied that $v(T) = 1$ if and only if $T \supset S$.

B. Blockability Relations

The blockability relation [4], [5] on a simple game is defined as a binary relation on the set 2^N of all coalitions in N. Let $W(v) = \{S \subset N \mid v(S) = 1\}$ be the set of all winning coalitions of simple game (N, v).

Definition II.5 (Blockability relation [4])**.** Consider a simple game (N, v). For coalitions S and S', $S \succeq^b S'$ is defined as: for all $T \in W(v)$, if $T \backslash S' \notin W(v)$ then $T \backslash S \notin W(v)$. $\succeq^b$ is called the *blockability relation* on (N, v).

This definition is interpreted as follows: coalition S is said to be equally or more *blockable* than coalition S' when it is satisfied that for all winning coalitions T, if S' can block T, that is, S' can make T losing by deviating from T, then coalition S can also block T.

The next proposition shows that $S \succeq^b S'$ is equivalent to $B(S) \supset B(S')$, where $B(S)$ is the set of all winning coalitions that are blocked by coalition S. This notation of $B(S)$ is employed for the definition of the blockability index in Section III-A

Proposition II.1 ([5])**.** *Consider a simple game (N, v) and the blockability relation $\succeq^b$ on (N, v). Then, it is satisfied that for all coalitions S and S', $S \succeq^b S'$ is equivalent to $B(S) \supset B(S')$, where for $S \subset N$,*

$$B(S) = \{T \subset N \mid v(T) = 1 \text{ and } v(T \backslash S) = 0\}.$$

Completeness of a relation is an important property for a better comparison. In [5], the completeness of the blockability relation on a simple game is characterized with S-unanimity of the game for a coalition S.

Definition II.6 (Completeness of blockability relations)**.** The blockability relation $\succeq^b$ on a simple game (N, v) is said to be *complete*, if and only if for all coalitions S and S', $S \succeq^b S'$ or $S' \succeq^b S$ holds.

Proposition II.2 ([5])**.** *The blockability relation $\succeq^b$ on a simple game (N, v) is complete, if and only if (N, v) is S-unanimous for a coalition S.*

C. Power Indices

A power index is defined as the restriction of a value on games in characteristic function form to simple games. Power indices and values are methods to evaluate each player by assigning to him a real number whose magnitude represents his power in the decision situation. In the succeeding arguments in this paper, let the set $N = \{1, 2, \ldots, n\}$ of players be fixed and a game (N, v) in characteristic function form (including the case of a simple game) be denoted simply by v.

Definition II.7 (Value)**.** Let G^N be the set of all games v in characteristic function form on N. A *value* on G^N is a map $\varphi : G^N \to \mathbf{R}^n$, $v \mapsto (\varphi_1(v), \varphi_2(v), \ldots, \varphi_n(v))$. The ith component $\varphi_i(v)$ is called the value of player i in v.

Definition II.8 (Power index)**.** Let SG^N be the set of all simple games v on N. A *power index* on SG^N is a map $\varphi : SG^N \to \mathbf{R}^n$, $v \mapsto (\varphi_1(v), \varphi_2(v), \ldots, \varphi_n(v))$. The ith component $\varphi_i(v)$ is called the power index of player i in v.

Many power indices have been proposed based on various ways of thinking. This subsection presents, among others, the Shapley-Shubik index and the Banzhaf index, which are in most important power indices.

The Shapley-Shubik index [8] is defined as the restriction of the Shapley value [7].

Definition II.9 (Shapley value [7])**.** The *Shapley value* is the value $\phi : G^N \to \mathbf{R}^n$, $v \mapsto (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$ where for all $i \in N$,

$$\phi_i(v) = \sum_{S \subset N, i \notin S} \frac{s!(n-s-1)!}{n!} \big[v(S \cup \{i\}) - v(S)\big].$$

Here, s is the number of members in S.

The Shapley value $\phi_i(v)$ of player i in game v is interpreted as expectation of i's marginal contribution $v(S \cup \{i\}) - v(S)$ to the payoff $v(N)$ of grand coalition N when each order of n players has equal probability with respect to formation of the grand coalition.

Definition II.10 (Shapley-Shubik index [8])**.** The *Shapley-Shubik index* is the power index $\phi : SG^N \to \mathbf{R}^n$, $v \mapsto (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$ where for all $i \in N$,

$$\phi_i(v) = \sum_{\substack{v(S)=0,\\ v(S \cup \{i\})=1}} \frac{s!(n-s-1)!}{n!}.$$

The Shapley value, and consequently the Shapley-Shubik index, are characterized as the unique value and the unique power index, respectively, that satisfy the following axioms. Consider a value $\phi : G^N \to \mathbf{R}^n$, $v \mapsto (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$ and take $v \in G^N$ arbitrary.

Axiom II.1 (Efficiency [7])**.** $\sum_{i \in N} \phi_i(v) = v(N)$.

Axiom II.2 (Dummy Player [7])**.** If $i \in N$ is a *dummy player* in game v, that is, $v(S \cup \{i\}) = v(S)$ for any coalition S, then $\phi_i(v) = 0$.

Axiom II.3 (Symmetry [7])**.** If player i and j are *symmetric* in game v, that is, $v(S \cup \{i\}) = v(S \cup \{j\})$ for any coalition S such that $i, j \notin S$, then $\phi_i(v) = \phi_j(v)$.

Axiom II.4 (Additivity [7])**.** For all games $v_1, v_2 \in G_N$, $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$, where the game $v_1 + v_2$ is defined as: $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for all $S \subset N$.

Efficiency axiom means that players share all the obtained payoff $v(N)$. Dummy Player axiom says that the value for a player who makes no contribution should be zero. Symmetry axiom guarantees the players in the same situations with respect to the function v should have the same value. Additivity is the axiom that guarantees linearity of the value.

Theorem II.1 (Shapley [7])**.** *The Shapley value is the unique value that satisfies Efficiency, Dummy Player, Symmetry and Additivity axioms.*

In contrast to the case in which the Shapley-Shubik index is derived from the Shapley value, the Banzhaf index [1], [6] is first defined and then the Banzhaf value [6] is defined as an extension of it.

Definition II.11 (Swing [1])**.** Consider a simple game v and take $i \in N$ and $S \subset N \backslash \{i\}$. The pair of coalitions $(S \cup \{i\}, S)$ is said to be a *swing* for player i if and only if $v(S \cup \{i\}) = 1$ and $v(S) = 0$. The player i is called a $swinger$. The number of swings for i in game v is denoted by $\eta_i(v)$.

Dividing $\eta_i(v)$ by 2^{n-1}, the number of all possible pairs, we obtain the Banzhaf index.

Definition II.12 (Banzhaf index [6])**.** The *Banzhaf index* is the power index $\beta : SG^N \to \mathbf{R}^n$, $v \mapsto (\beta_1(v), \beta_2(v), \ldots, \beta_n(v))$ where for all $i \in N$,

$$\beta_i(v) = \frac{\eta_i(v)}{2^{n-1}}.$$

Note that what Banzhaf [1] originally defined as player i's *voting power* was simply $\eta_i(v)$. The Banzhaf index is interpreted as expectation of player i to be a swinger, when every coalition is equally probable.

Definition II.13 (Banzhaf value [6])**.** The *Banzhaf value* is the value $\beta : G^N \to \mathbf{R}^n$, $v \mapsto (\beta_1(v), \beta_2(v), \ldots, \beta_n(v))$ where for all $i \in N$,

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S \subset N, i \notin S} \big[v(S \cup \{i\}) - v(S)\big].$$

Dubey and Shapley [2] characterized the uniqueness of the Banzhaf index and Banzhaf value with another system of four axioms including Symmetry and Dummy Player axioms.

III. Coalition Power Indices

This section presents the notion of a coalition power index (CPI) on simple games, which is extended from the conventional framework of a player power index (PPI) to evaluates each coalition's power. In Section III-A, the blockability index is defined as an example of CPIs based on the blockability relations. In Section III-B, properties of the blockability index are investigated. It is proven that the blockability index satisfies *Dummy Coalition* and *Coalition Symmetry* axioms, which are generalized from Dummy Player and Symmetry axioms. It also turns out that the blockability index coincides with the Banzhaf index except their constant coefficients when it is restricted to one-player coalitions.

A CPI on simple games is defined as a map that assigns a real number to each pair of a simple game and a coalition of the game.

Definition III.1 (Coalition power index)**.** A *coalition power index (CPI)* is a map $c : SG^N \times 2^N \to \mathbf{R}$.

We usually write $c_S(v)$ instead of $c\big((v, S)\big)$ with $v \in SG^N$ and $S \in 2^N$, and we call $c_S(v)$ the power index of coalition S in simple game v.

A. Blockability Index

Consider a simple game v, and let $W(v) = \{S \subset N \mid v(S) = 1\}$ be the set of all winning coalitions in v. For a coalition S, $B(S) = \{T \subset N \mid v(T) = 1 \text{ and } v(T\backslash S) = 0\}$ is the set of all winning coalitions that are blocked by coalition S, as seen in section II-B. It is natural to say that the larger the number $|B(S)|$ of winning coalitions that S can block is, the more "blocking power" coalition S have as a whole. Based on this point of view, we define the blockability index as follows:

Definition III.2 (Blockability index)**.** The *blockability index* is a CPI $\hat{b} : SG^N \times 2^N \to \mathbf{R}$ where for $S \subset N$,

$$\hat{b}_S(v) = \frac{|B(S)|}{|W(v)|}.$$

The blockability index of coalition S indicates the ratio of the number of winning coalitions that S can block to the number of all winning coalitions. In fact, for any simple game v and any coalition S, $0 \leq \hat{b}_S(v) \leq 1$ holds. If the game v is proper, we have that $\hat{b}_T(v) = 1$ for every winning coalition T since a winning coalition can block all the winning coalitions in proper games.

Example III.1. Consider a proper simple game (N, v) where

$$\begin{aligned} N &= \{1, 2, 3, 4\} \text{ and} \\ W(v) &= \{12, 123, 124, 234, 1234\}. \\ (L(v) &= \{\emptyset, 1, 2, 3, 4, 13, 14, 23, 24, 34, 134\}) \end{aligned}$$

To calculate the blockability index $\hat{b}_S(v)$ of all coalitions S, we first find out $B(S)$ for all coalitions S:

$$B(S) = \begin{cases} \{234\} & \text{for } S = 3, 4, 34, \\ \{12, 123, 124\} & \text{for } S = 1, \text{ and} \\ W(v) & \text{otherwise.} \end{cases}$$

From this, we obtain the result:

$$\hat{b}_S(v) = \begin{cases} 0.2 & \text{for } S = 3, 4, 34, \\ 0.6 & \text{for } S = 1, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

B. Properties of Blockability Index

Referring to the axiomatic characterizations of the Shapley-Shubik index and the Banzhaf index seen in Section II-C, we generalize Dummy Player and Symmetry axioms to define Dummy Coalition and Coalition Symmetry axioms. It is verified that the blockability index satisfies both of the two generalized axioms.

Consider a CPI $c : SG^N \times 2^N \to \mathbf{R}$ and take $v \in SG^N$ arbitrary.

Axiom III.1 (Dummy Coalition)**.** If coalition D is a *dummy coalition* in simple game v, that is, $v(S \cup D) = v(S)$ for any coalition S, then $c_D(v) = 0$.

Axiom III.2 (Coalition Symmetry)**.** If coalitions T and T' are *symmetric* in simple game v, that is, there exists a bijection $f : T \to T'$ such that for all $i \in T$ players i and $f(i)$ are symmetric in v, then $c_T(v) = c_{T'}(v)$.

Note that a coalition becomes a dummy coalition if and only if every player in the coalition is a dummy player, which is shown in the next proposition:

Proposition III.1. *For all $v \in SG^N$, it is satisfied that a coalition D is a dummy coalition if and only if every player $i \in D$ is a dummy player.*

Proof: Assume that D is a dummy coalition, that is, for all $S \subset N$, $v(S \cup D) = v(S)$. Take $i \in D$. We have that $v(S \cup D) \geq v(S \cup \{i\}) \geq v(S)$ from monotonicity. Now since $v(S \cup D) = v(S)$, we have that $v(S \cup \{i\}) = v(S)$, which implies i is a dummy player.

Assume that every player $i_j \in D = \{i_1, i_2, \ldots, i_d\}$ is a dummy player and take $S \subset N$ arbitrary. We have that

$$\begin{aligned} & v\big(S \cup D\big) \\ &= v\big(S \cup (D\backslash\{i_1\}) \cup \{i_1\}\big) = v\big(S \cup (D\backslash\{i_1\})\big) \\ &= v\big(S \cup (D\backslash\{i_1, i_2\}) \cup \{i_2\}\big) = v\big(S \cup (D\backslash\{i_1, i_2\})\big) \\ &\ \vdots \\ &= v\big(S \cup (D\backslash\{i_1, i_2, \ldots, i_d\}) \cup \{i_d\}\big) \\ &= v\big(S \cup (D\backslash\{i_1, i_2, \ldots, i_d\})\big) \\ &= v(S), \end{aligned}$$

which implies that D is a dummy coalition. □

To prove that the blockability index satisfies Coalition Symmetry axioms, the following sequence of four lemmas is needed. Let (N, v) be a simple game and $W = \{S \subset N \mid v(S) = 1\}$ be the set of all winning coalitions in (N, v).

Lemma III.1. *Let $i, j \in N$ be such players that i and j are symmetric. For the transposition $\tau = (i \;\; j)$, $S \in W$ if and only if $\tau(S) \in W$.*

Proof: Assume $S \in W$, that is, $v(S) = 1$. We have four cases.

1) $i \in S$ and $j \in S$:
$v(\tau(S)) = v(S) = 1$
2) $i \in S$ and $j \notin S$:
$v(\tau(S)) = v((S\backslash\{i\}) \cup \{j\}) = v((S\backslash\{i\}) \cup \{i\}) = v(S) = 1$
3) $i \notin S$ and $j \in S$:
$v(\tau(S)) = v((S\backslash\{j\}) \cup \{i\}) = v((S\backslash\{j\}) \cup \{j\}) = v(S) = 1$
4) $i \notin S$ and $j \notin S$:
$v(\tau(S)) = v(S) = 1$

In all cases, we have $v(\tau(S)) = 1$, which means $\tau(S) \in W$.

Next, assume $\tau(S) \in W$. Taking the same procedure given above, we have $\tau(\tau(S)) \in W$, which implies $S \in W$. □

Lemma III.2. *Let $\pi : N \to N$ be such a permutation that for any $i \in N$, i and $\pi(i)$ are symmetric. Then $\pi(W) = W$, where $\pi(W) = \{\pi(S) \mid S \in W\}$.*

Proof: For some positive integer k, the permutation π can be written as a product $\pi = \tau_1\tau_2\cdots\tau_k$ of k transpositions $\tau_l = (i_l \;\; j_l)$ $(l = 1, 2, \ldots, k)$ that transpose symmetric players i_l and j_l, which is proven as follows:

Define a binary relation R on N as: iRj if and only if i and j are symmetric. We can easily verify that R is an equivalence relation on N. Hence, for some positive integer m, N is divided into a partition $N = C_1 \cup C_2 \cup \cdots \cup C_m$ with m equivalence classes C_p $(p = 1, 2, \ldots, m)$ by the relation R. Then π is closed on every C_p, that is, $\pi(C_p) = C_p$, by the definitions of π and R, and is described as $\pi = \pi_1\pi_2\cdots\pi_m$ where π_p is such the permutation that maps players in C_p with π, and fixes players not in C_p:

$$\pi_p(i) = \begin{cases} \pi(i) & \text{if } i \in C_p, \\ i & \text{otherwise.} \end{cases}$$

From a theorem in algebra, the restriction $\pi_p|_{C_p}$ of π_p to C_p can be written as a product of transpositions in C_p. Because every two players in C_p are symmetric, all of those transpositions in C_p transpose symmetric players. Consequently, each π_p can be written as a product of transpositions all of which transpose symmetric players, and thus, so can π.

Now prove that $\pi(W) = W$. For $S \subset N$, by using Lemma III.1 repeatedly, it is satisfied that

$$\begin{aligned} S \in W &\iff \tau_1(S) \in W \\ &\iff \tau_2\tau_1(S) \in W \\ &\;\;\vdots \\ &\iff \tau_k\cdots\tau_2\tau_1(S) \in W \\ &\iff \pi^{-1}(S) \in W. \end{aligned}$$

Note that $\pi^{-1} = (\tau_1\tau_2\cdots\tau_k)^{-1} = \tau_k^{-1}\cdots\tau_2^{-1}\tau_1^{-1} = \tau_k\cdots\tau_2\tau_1$. Since $\pi^{-1}(S) \in W$ is equivalent to $\pi(\pi^{-1}(S)) = S \in \pi(W)$, it is satisfied that $S \in W$ if and only if $S \in \pi(W)$, and thus, $\pi(W) = W$. □

Lemma III.3. *For a permutation $\pi : N \to N$ and coalitions $S, T \subset N$, $\pi(S\backslash T) = \pi(S)\backslash\pi(T)$.*

Proof: For all $i \in N$, it is satisfied that

$$\begin{aligned} i \in \pi(S\backslash T) &\iff \pi^{-1}(i) \in S\backslash T \\ &\iff \pi^{-1}(i) \in S \text{ and } \pi^{-1}(i) \notin T \\ &\iff i \in \pi(S) \text{ and } i \notin \pi(T) \\ &\iff i \in \pi(S)\backslash\pi(T). \end{aligned}$$

Therefore, $\pi(S\backslash T) = \pi(S)\backslash\pi(T)$ holds. □

Lemma III.4. *Let $T, T' \subset N$ be such coalitions that T and T' are symmetric. Then $|B(T)| = |B(T')|$.*

Proof: Let $f : T \to T'$ be such a bijection that for all $i \in T$, player i and player $f(i)$ are symmetric. Here we construct another bijection $g : T \to T'$ from f, according to the following four cases:

1) For $i \in T\backslash T'$ such that $f(i) \in T'\backslash T$;
we define $g(i) = f(i)$. Then players i and $g(i)$ are symmetric.
2) For $i \in T\backslash T'$ such that $f(i) \in T' \cap T$;
because f is a injection and $T' \cap T$ is a finite set, there exist a positive integer k such that $f^k(i) \in T'\backslash T$ and $f^m(i) \in T' \cap T$ for $m = 1, 2, \ldots k-1$. We define $g(i) = f^k(i)$. Then players i and $g(i)$ are symmetric from transitivity of symmetric property of two players.
3) For $i \in T \cap T'$ such that $f(i) \in T'\backslash T$;
because the inverse $f^{-1} : T' \to T$ of f is a injection and $T' \cap T$ is a finite set, there exist a positive integer k' such that $f^{-k'}(i) \in T\backslash T'$ and $f^{-m'}(i) \in T' \cap T$ for $m' = 1, 2, \ldots k'-1$. We define $g(i) = f^{-(k'-1)}(i)$. Then players i and $g(i)$ are symmetric from transitivity of symmetric property of two players.
4) For $i \in T \cap T'$ such that $f(i) \in T' \cap T$;
we define $g(i) = f(i)$. Then players i and $g(i)$ are symmetric.

Defined as above, $g : T \to T'$ is such a bijection that i and $g(i)$ are symmetric for all $i \in T$, and that induces two bijective restrictions:

$$\begin{aligned} g|_{T\backslash T'} &: T\backslash T' \to T'\backslash T \quad \text{and} \\ g|_{T\cap T'} &: T \cap T' \to T \cap T'. \end{aligned}$$

Using the map $g : T \to T'$, we define a permutation $\pi : N \to N$ as

$$\pi(i) = \begin{cases} g(i) & \text{if } i \in T, \\ g^{-1}(i) & \text{if } i \in T'\backslash T, \\ i & \text{otherwise.} \end{cases}$$

Then it is satisfied that for all $i \in N$, i and $\pi(i)$ are symmetric, and that $\pi(T) = T'$.

Now we have that for all $S \in W$,

$$\begin{aligned}
&S \in B(T)\\
&\quad\iff S\backslash T \notin W\\
&\quad\iff \pi(S\backslash T) \notin \pi(W)\\
&\quad\iff \pi(S)\backslash\pi(T) \notin \pi(W) \quad \text{(from Lemma III.3)}\\
&\quad\iff \pi(S)\backslash\pi(T) \notin W \quad \text{(from Lemma III.2)}\\
&\quad\iff \pi(S)\backslash T' \notin W\\
&\quad\iff \pi(S) \in B(T').
\end{aligned}$$

Therefore, it is satisfied that $B(T') = \{\pi(S) \mid S \in B(T)\}$, which implies $|B(T)| = |B(T')|$. □

Now we have the next theorem.

Theorem III.1. *The blockability index $\hat{b}$ satisfies Dummy Coalition and Coalition Symmetry axioms.*

Proof: Let $v \in SG^N$ be a simple game and D be a dummy coalition. To verify $B(D) = \emptyset$, assume $B(D) \neq \emptyset$. Then there exists a winning coalition $S \in W(v)$ such that $v(S) = 1$ and $v(S\backslash D) = 0$. From monotonicity, it is satisfied that $v(S) \leq v(S \cup D)$, which implies $v(S \cup D) = 1$. Since $S \cup D = (S\backslash D) \cup D$, $v((S\backslash D) \cup D) = 1$ holds. For coalition $S\backslash D$, it is satisfied that $v(S\backslash D) = 0$ and $v((S\backslash D) \cup D) = 1$, which contradicts that D is a dummy coalition. Therefore, we have that $B(D) = \emptyset$ and

$$\hat{b}_D(v) = \frac{|B(D)|}{|W(v)|} = 0.$$

Hence, the blockability index $\hat{b}$ satisfies Dummy Coalition axiom.

Let T, $T' \subset N$ be such coalitions that T and T' are symmetric. Since $|B(T)| = |B(T')|$ from Lemma III.4, we have that

$$\hat{b}_T(v) = \frac{|B(T)|}{|W(v)|} = \frac{|B(T')|}{|W(v)|} = \hat{b}_{T'}(v).$$

Hence, the blockability index $\hat{b}$ satisfies Coalition Symmetry axiom. □

Finally, we consider the relationship between the blockability index and the Banzhaf index. Although they seem to be defined in completely different ways, they coincide except their constant coefficients when the blockability index is restricted to one-player coalitions and viewed as a PPI.

Theorem III.2. *For all $v \in SG^N$ and all $i \in N$,*

$$\hat{b}_{\{i\}}(v) = \frac{2^{n-1}}{|W(v)|}\beta_i(v)$$

Proof: Tracing back to the original definitions of the blockability index and the Banzhaf index, we have that

$$\hat{b}_{\{i\}}(v) = \frac{|B(\{i\})|}{|W(v)|} \quad \text{and} \quad \beta_i(v) = \frac{\eta_i(v)}{2^{n-1}}.$$

To prove the theorem, we should verify that $|B(\{i\})| = \eta_i(v)$. It is satisfied that for all $S \subset N$, $S \in B(\{i\})$ if and only if $v(S) = 1$ and $v(S\backslash\{i\}) = 0$, which is equivalent to that the pair $(S, S\backslash\{i\})$ is a swing for player i. Thus, the number $\eta_i(v)$ of swings for i equals to $|B(\{i\})|$. □

IV. Conclusion

As a method to evaluate coalition influence in a simple game, the notion of a coalition power index (CPI) was proposed in this paper (Definition III.1). A CPI was defined as a map that assigns a real number to each pair of a simple game and a coalition in the game. The blockability index was then defined as an example of coalition power indices, based on the blockability relation on a simple game (Definition III.2). It measures power of each coalition with respect to how many winning coalitions it can turn losing by deviation. Some properties of the blockability index were investigated (Theorems III.1 and III.2). It turned out that the blockability index satisfies Dummy Coalition and Coalition Symmetry axioms and that the blockability index coincides with the Banzhaf index except their constant coefficients on one-player coalitions.

One of the most important topics for future research opportunities is to provide an axiomatic characterization of the blockability index. It is desired to identify a system of axioms that characterizes the blockability index as the unique CPI that satisfies the axioms. Another topic is to apply the blockability index to analysis of group decisions in reality. Voting situations in national and regional parliaments, for example, are suitably modeled as simple games. The blockability index can be an effective tool to evaluate each coalition of parties.

References

[1] J. F. Banzhaf, "Weighted voting doesn't work: a mathematical analysis," Rutgers Law Review, vol. 19, pp. 317-343, 1965

[2] P. Dubey and L. S. Shapley, "Mathematical Properties of the Banzhaf Power Index," Mathematics of Operations Research, vol. 4, pp. 99-131, 1979.

[3] E. Einy, "The desirability relation of simple games," Mathematical Social Sciences, vol. 10 pp. 155-168, 1985.

[4] K. Ishikawa and T. Inohara, "A method to compare influence of coalitions on group decision other than desirability relation," Applied Mathematics and Computation, vol. 188, no. 1, pp. 838-849, 2007.

[5] M. Kitamura and T. Inohara, "A characterization of the completeness of blockability relation with respect to unanimity," Applied Mathematics and Computation, vol. 197, pp. 715-718, 2008.

[6] G. Owen, "Multilinear extensions and the Banzhaf value," Naval Research Logistics Quarterly, vol. 22, pp. 741-750, 1975.

[7] L. S. Shapley, "A value for n-person games," in: H. W. Kuhn and A. W. Tucker (Eds.), Contributions to the Theory of Games vol. II, Princeton, NJ: Princeton University Press, pp. 307-317, 1953.

[8] L. S. Shapley and M. Shubik, "A method for evaluating the distribution of power in a committee system," American Political Science Review, vol. 48, pp. 787-792, 1954.

[9] A. D. Taylor and W. S. Zwicker, Simple Games: Desirability Relation, Trading, Pseudoweightings. Princeton, NJ: Princeton University Press, 1999.

[10] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior (Sixtieth Anniversary Edition). Princeton, NJ: Princeton University Press, 2004.