Proceedings of the 2009 IEEE International Conference on Systems, Man, and Cybernetics
San Antonio, TX, USA - October 2009

# SOS-based Stability Analysis of Polynomial Fuzzy Control Systems via Polynomial Membership Functions

Mohammand Narimani
Division of Engineering
Kings College London
Strand, London, WC2R 2LS, UK
E-mail: mohammad.narimani@kcl.ac.uk

H.K. Lam
Division of Engineering
Kings College London
Strand, London, WC2R 2LS, UK
E-mail: hak-keung.lam@kcl.ac.uk

***Abstract*—This paper presents stability analysis of polynomial fuzzy control systems using Sum-Of-Squares (SOS) approach. To take continuous form of membership functions into the stability analysis, based on the Lyapunov stability theory, stability conditions in the form of fuzzy summations are derived where each term contains product of polynomial fuzzy model and polynomial fuzzy controller membership functions. Then each product term is approximated by polynomials in the partitioned operating domain of membership functions. Regarding to the derived conditions in all sub-regions, SOS-based stability conditions are formed. The proposed approach can be utilized for stability analysis of polynomial fuzzy control system in which fuzzy model and fuzzy controller do not share the same membership functions named non-PDC design technique. The solution of the SOS-based stability conditions can be found numerically using the SOSTOOLS which is a free third-party MATLAB Toolbox. Numerical example is given to illustrate the effectiveness of the proposed stability conditions.**

***Index Terms*—Polynomial fuzzy control system, stability, sum of squares, membership functions shape-dependent stability conditions.**

## I. Introduction

Fuzzy Model-Based (FMB) control system offers a systematic way in handling control problems for nonlinear control systems. Considering the most common stability analysis approach for nonlinear control systems that is the Lyapunov stability theory [1], sufficient conditions to guarantee stability of the T-S FMB control system are investigated. Convex optimization techniques, such as Linear Matrix Inequality (LMI) approach [2], have played an important role in obtaining numerical solutions for the derived stability conditions. The well-known design technique named *Parallel Distributed Compensation* (PDC) [3], plays an important role to relax the stability analysis results. For the non-PDC approach, the membership functions of T-S fuzzy model and fuzzy controller are not necessarily the same and even it can be possible to employ different number of fuzzy controller rules from those of fuzzy model. Compared with PDC approach, design flexibility and robustness property for controller can be achieved [4]–[6]. Consequently, the cost of implementation would be reduced.

One of the sources of conservativeness is lack of membership functions in the stability analysis. When membership functions are not considered in the stability analysis, then the stability conditions are valid for any arbitrary membership functions. In [4]–[8], it was shown that the shape of the membership functions plays an important role for relaxation of stability conditions. Some relaxed shape-dependent stability conditions for non-PDC cases have been addressed in [4]–[6]. However, due to the limitation of the LMI MATLAB toolbox, which can only deal with the LMIs with constant matrices and variables, in the aforementioned approaches full information of membership functions have not been brought into the stability analysis. Then, finding solution for LMI systems containing continuous functions means setting infinite number of LMIs (in all points of continuous functions) which is not possible practically using convex programming techniques. This defect of the LMI-based approaches, motivates us to investigate the system stability via alternative computation approach so that it can hurdle the existing drawback.

Recently some attempts have been done to employ SOS approach for stability analysis and guaranteed cost control of the polynomial fuzzy control systems in [9], [10] respectively, for which the system and input matrices can be polynomial functions of system states. In both addressed works polynomials are allowed to be taken in the system and input matrices. Additionally, some preliminary attempts have been made to introduce shape-dependent SOS-based approaches [11]–[13].

In this paper, to investigate the system stability analysis of the polynomial fuzzy control system, the continuous form of membership functions are brought into stability analysis. Based on the Lyapunov stability theory, stability conditions in the form of fuzzy summations are derived. To construct SOS-based stability conditions, product terms of the fuzzy model and fuzzy controller membership functions are approximated with some polynomials. In order to obtain better approximation, the operating domain of membership functions is partitioned to sub-regions. Furthermore, the proposed approach can be employed to relax stability conditions for polynomial fuzzy control systems in which fuzzy model and fuzzy controller do not share the same membership functions under the non-PDC design technique. The solution of the SOS-based stability



978-1-4244-2794-9/09/$25.00 

conditions for the polynomial FMB control systems can be found numerically using the SOSTOOLS which is a free third-party MATLAB Toolbox [14].

The rest of this paper is organized as follows. In section II, the polynomial FMB control system are presented. In section III, stability analysis for polynomial FMB control system is carried out. In section IV, simulation example is given to verify the analysis results. In section V, a conclusion is drawn.

## II. Polynomial Fuzzy Control Systems

In this section, state dependent linear matrix inequality approach used as a computational method called *Some of Squares* and a recently introduced type of fuzzy control system containing polynomial fuzzy model and polynomial fuzzy controller whose consequent parts can be polynomials are recalled [9].

### A. Sum of Squares

Here, sum of squares decomposition and its application in solving state dependent LMIs and some related concepts addressed in [9] are recalled. A multivariate polynomial $f(\mathbf{x}(t))$, is a sum of squares (SOS), if there exist polynomials $f_1(\mathbf{x}(t)), ..., f_m(\mathbf{x}(t))$ such that $f(\mathbf{x}(t)) = \sum_{i=1}^{m} f_i^2(\mathbf{x}(t))$ where $\mathbf{x}(t) \in \mathbb{R}^n$. Furthermore, a monomial in $\mathbf{x}(t)$ is a function of the form $x_1^{\alpha_1}(t)x_2^{\alpha_2}(t)...x_n^{\alpha_n}(t)$ , where $\alpha_1, \alpha_2, ..., \alpha_n$ are nonnegative integers. In this case, the degree of the monomial is given by $\alpha_1 + \alpha_2 + ... + \alpha_n$. To utilize the concepts of SOS in the quadratic form, the following proposition has been proposed.

**Proposition 1:** [9] Let $\mathbf{F}(\mathbf{x}(t))$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $\mathbf{x}(t) \in \mathbb{R}^n$. Furthermore, let $\hat{\mathbf{x}}(\mathbf{x}(t))$ be a column vector whose entries are monomials in $\mathbf{x}(t)$ with degree no more than $d$. If $\upsilon^T(t)\mathbf{F}(\mathbf{x}(t))\upsilon(t)$ is a sum of squares, where $\upsilon(t) \in \mathbb{R}^N$, then $\mathbf{F}(\mathbf{x}(t)) \geq 0$ for all $\mathbf{x}(t) \in \mathbb{R}^n$.

### B. Polynomial Fuzzy Model

Consider a nonlinear system described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t)) + \mathbf{g}(\mathbf{x}(t))\mathbf{u}(t), \tag{1}$$

in which $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector. According to the sector nonlinearity concept [15], the nonlinear system (1) can be represented by the following polynomial fuzzy model [9],

**Plant Rule *i*** :

IF $z_1(\mathbf{x}(t))$ is $M_1^i$ AND ... AND $z_p(\mathbf{x}(t))$ is $M_p^i$
THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)$ (2)

for which $z_\alpha(\mathbf{x}(t))$ is the premise variable and $M_\alpha^i$ is a fuzzy term corresponding to $i^{th}$ rule and $\mathbf{A}_i(\mathbf{x}(t)) \in \mathbb{R}^{n\times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \mathbb{R}^{n\times m}$ are the polynomial systems and polynomial input matrices respectively, for $i = 1, ..., r$, $\alpha = 1, ..., p$. In addition, $\hat{\mathbf{x}}(t) = (\hat{x}_1(t), \hat{x}_2(t), ..., \hat{x}_N(t)) \in \mathbb{R}^N$ is a vector of monomials in $\mathbf{x}(t)$ with assumption that $\hat{\mathbf{x}}(t) = 0$ *iff* $\mathbf{x}(t) = 0$. The overall system dynamics is represented by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \frac{\sum_{i=1}^{r} w_i(\mathbf{z}(\mathbf{x}(t)))\{\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\}}{\sum_{i=1}^{r} w_i(\mathbf{z}(\mathbf{x}(t)))} \\ &= \sum_{i=1}^{r} \mu_i(\mathbf{z}(\mathbf{x}(t)))\{\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\},\end{aligned} \tag{3}$$

### C. Polynomial Fuzzy Controller

The structure of the polynomial fuzzy controller is similar to the T-S fuzzy controller, with the consequent parts as polynomials represented as in the following.

**Control Rule *j*** :

IF $s_1(\mathbf{x}(t))$ is $N_1^j$ AND ... AND $s_q(\mathbf{x}(t))$ is $N_q^j$
THEN $\mathbf{u}(t) = \mathbf{F}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) \quad j = 1, ...c$ (4)

where $s_\theta(\mathbf{x}(t))$ is the premise variable and $N_\theta^j$ is a fuzzy term corresponding to $j^{th}$ control rule $j = 1, ..., c$, $\theta = 1, ..., q$. $\mathbf{F}_j(\mathbf{x}(t)) \in \mathbb{R}^{m\times N}$ is the polynomial feedback gain of the $j^{th}$ rule to be determined.

The overall polynomial fuzzy controller is defined as

$$\begin{aligned}\mathbf{u}(t) &= \frac{\sum_{j=1}^{c} v_j(\mathbf{s}(\mathbf{x}(t)))\mathbf{F}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t))}{\sum_{j=1}^{c} v_j(\mathbf{s}(\mathbf{x}(t)))} \\ &= \sum_{j=1}^{c} \lambda_j(\mathbf{s}(\mathbf{x}(t)))\mathbf{F}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)),\end{aligned} \tag{5}$$

Considering the polynomial fuzzy model of (3) and the polynomial fuzzy controller of (5) connected in a closed loop, the FMB control system is obtained as follows.

$$\begin{aligned}\dot{\mathbf{x}}(t) = &\sum_{i=1}^{r}\sum_{j=1}^{c} \mu_i(\mathbf{z}(\mathbf{x}(t)))\lambda_j(\mathbf{s}(\mathbf{x}(t))) \\ &\times \Big(\mathbf{A}_i(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{F}_j(\mathbf{x}(t))\Big)\hat{\mathbf{x}}(\mathbf{x}(t)).\end{aligned} \tag{6}$$

**Remark 1:** Considering $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{x}(t)$, $\mathbf{A}_i(\mathbf{x}(t))$, $\mathbf{B}_i(\mathbf{x}(t))$ and $\mathbf{F}_j(\mathbf{x}(t))$ as constant matrices for all $i$, the polynomial FMB control system represented by (6), is reduced to the traditional T-S FMB control system [16] represented as

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{c} \mu_i(\mathbf{z}(\mathbf{x}(t)))\lambda_j(\mathbf{s}(\mathbf{x}(t)))\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{F}_j\right)\mathbf{x}(t). \tag{7}$$

In addition, if $r = c$ and $\mu_i(\mathbf{z}(\mathbf{x}(t))) = \lambda_j(\mathbf{s}(\mathbf{x}(t)))$, for $i = 1, ...r$, $j = 1, ..., c$, then (7) becomes the T-S FMB control system with PDC design technique [3], [17].

In the rest for brevity, $\mathbf{x}(t)$, $\hat{\mathbf{x}}(\mathbf{x}(t))$, $\mu_i(\mathbf{z}(\mathbf{x}(t)))$, $\lambda_j(\mathbf{s}(\mathbf{x}(t)))$ are denoted as $\mathbf{x}$, $\hat{\mathbf{x}}(\mathbf{x})$, $\mu_i(\mathbf{x})$, $\lambda_j(\mathbf{x})$ respectively.

## III. Stability Analysis of polynomial Fuzzy Control Systems

To investigate the quadratic stability of the polynomial fuzzy control system (6) with the Lyapunov stability theory, the following polynomial quadratic Lyapunov function candidate addressed in [9] is employed

$$V(\mathbf{x}) = \hat{\mathbf{x}}(\mathbf{x})^T \mathbf{X}^{-1}(\tilde{\mathbf{x}})\hat{\mathbf{x}}(\mathbf{x}), \tag{8}$$

where $\mathbf{X}^{-1}(\tilde{\mathbf{x}}) \in \mathbb{R}^{n\times n}$ is a symmetric positive definite polynomial matrix and $\tilde{\mathbf{x}} = (x_{k_1}, x_{k_2}, ..., x_{k_d})$ is a vector to be chosen such that $\mathbf{K} = \{k_1, k_2, ..., k_d\}$ denotes the indices of the corresponding zero rows in $\mathbf{B}_i(\mathbf{x})$s for all $i$. As a result

$$\frac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial x_k} = \mathbf{0} \qquad \text{for } k \notin \mathbf{K}, \tag{9}$$

also considering $\mathbf{A}_i^k(\mathbf{x})$, $\mathbf{B}_i^k(\mathbf{x})$ denoted the $k^{th}$ row in $\mathbf{A}_i(\mathbf{x})$, $\mathbf{B}_i(\mathbf{x})$ respectively we have

$$\mathbf{B}_i^k(\mathbf{x}) = \mathbf{0} \qquad \text{for } k \in \mathbf{K}. \tag{10}$$

The aforementioned assumption are considered to avoid introducing non-convex conditions to the stability conditions. For more details reader is referred to [9], [18]. According to the Lyapunov stability theory, if $\mathbf{X}(\tilde{\mathbf{x}})$ and $\mathbf{X}^{-1}(\tilde{\mathbf{x}})$ are positive definite matrices in $\tilde{\mathbf{x}}$ and $\dot{V}(t) < 0$, then the polynomial fuzzy control system (6) is asymptotically stable. Then we have

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \dot{\hat{\mathbf{x}}}^T(\mathbf{x})\mathbf{X}^{-1}(\tilde{\mathbf{x}})\hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}^T(\mathbf{x})\mathbf{X}^{-1}(\tilde{\mathbf{x}})\dot{\hat{\mathbf{x}}}(\mathbf{x}) \\ &\quad + \hat{\mathbf{x}}^T(\mathbf{x})\dot{\mathbf{X}}^{-1}(\tilde{\mathbf{x}})\hat{\mathbf{x}}(\mathbf{x}) \\ &= \dot{\mathbf{x}}^T\mathbf{\Phi}^T(\mathbf{x})\mathbf{X}^{-1}(\tilde{\mathbf{x}})\hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}^T(\mathbf{x})\mathbf{X}^{-1}(\tilde{\mathbf{x}})\mathbf{\Phi}(\mathbf{x})\dot{\mathbf{x}} \\ &\quad + \hat{\mathbf{x}}^T(\mathbf{x})\left(\sum_{k=1}^{n}\frac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial x_k}\dot{x}_k\right)\hat{\mathbf{x}}(\mathbf{x}),\end{aligned} \tag{11}$$

where $\mathbf{\Phi}(\mathbf{x}) \in \mathbb{R}^{N\times n}$ is a polynomial matrix defined as follows $\mathbf{\Phi}_{ij}(\mathbf{x}) = \dfrac{\partial \hat{x}_i}{\partial x_j}$. Regarding to (6), (9) and (10)

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \hat{\mathbf{x}}^T\sum_{i=1}^{r}\sum_{j=1}^{c}\mu_i(\mathbf{x})\lambda_j(\mathbf{x})\ \Big(\big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{F}_j(\mathbf{x})\big)^T \\ &\quad \times \mathbf{\Phi}^T(\mathbf{x})\mathbf{X}^{-1}(\tilde{\mathbf{x}}) + \mathbf{X}^{-1}(\tilde{\mathbf{x}})\mathbf{\Phi}(\mathbf{x})\big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{F}_j(\mathbf{x})\big)\Big)\hat{\mathbf{x}} \\ &\quad + \hat{\mathbf{x}}^T(\mathbf{x})\left(\sum_{k\in\mathbf{K}}\frac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})\right)\hat{\mathbf{x}}(\mathbf{x}).\end{aligned} \tag{12}$$

Denoting $\mathbf{y}(\mathbf{x}) = \mathbf{X}^{-1}\hat{\mathbf{x}}(\mathbf{x})$, we have

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \mathbf{y}^T(\mathbf{x})\sum_{i=1}^{r}\sum_{j=1}^{c}\mu_i(\mathbf{x})\lambda_j(\mathbf{x})\ \Big(\mathbf{X}(\tilde{\mathbf{x}})\big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \\ &\quad \times \mathbf{F}_j(\mathbf{x})\big)^T\mathbf{\Phi}^T(\mathbf{x}) + \mathbf{\Phi}(\mathbf{x})\big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{F}_j(\mathbf{x})\big)\mathbf{X}(\tilde{\mathbf{x}})\Big) \\ &\quad \times \mathbf{y}(\mathbf{x}) + \mathbf{y}^T(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})\left(\sum_{k\in\mathbf{K}}\frac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})\right) \\ &\quad \times \mathbf{X}(\tilde{\mathbf{x}})\mathbf{y}(\mathbf{x}).\end{aligned} \tag{13}$$

In order to cast equation (13) so that the existing numerical computation technique would be employed, it can be shown that $\mathbf{X}(\tilde{\mathbf{x}})\dfrac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{X}(\tilde{\mathbf{x}}) = -\dfrac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}$ for details reader is referred to [9]. In addition, $\mathbf{F}_j(\mathbf{x})\mathbf{X}(\tilde{x})$ is replaced with $\mathbf{M}_j(\mathbf{x})$. Then (13) is rewritten as follows.

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \mathbf{y}^T(\mathbf{x})\sum_{i=1}^{r}\sum_{j=1}^{c}\Big(\mu_i(\mathbf{x})\lambda_j(\mathbf{x})\Big(\big(\mathbf{A}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{B}_i(\mathbf{x}) \\ &\quad \times \mathbf{M}_j(\mathbf{x})\big)^T\mathbf{\Phi}^T(\mathbf{x}) + \mathbf{\Phi}(\mathbf{x})\big(\mathbf{A}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{B}_i(\mathbf{x})\mathbf{M}_j(\mathbf{x})\big)\Big) \\ &\quad - \sum_{k\in\mathbf{K}}\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})\Big)\mathbf{y}(\mathbf{x}),\end{aligned} \tag{14}$$

in which the local feedback gains are

$$\mathbf{F}_j = \mathbf{M}_j(\mathbf{x})\mathbf{X}^{-1}(\tilde{x}) \qquad\qquad j = 1, ..., c \tag{15}$$

and (14) is denoted as

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^{r}\sum_{j=1}^{c}\mu_i(\mathbf{x})\lambda_j(\mathbf{x})\mathbf{y}^T(\mathbf{x})\mathbf{T}_{ij}(\mathbf{x})\mathbf{y}(\mathbf{x}), \tag{16}$$

where

$$\begin{aligned}\mathbf{T}_{ij}(\mathbf{x}) &= \mathbf{y}^T(\mathbf{x})\sum_{i=1}^{r}\sum_{j=1}^{c}\Big(\mu_i(\mathbf{x})\lambda_j(\mathbf{x})\Big(\big(\mathbf{A}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{B}_i(\mathbf{x}) \\ &\quad \times \mathbf{M}_j(\mathbf{x})\big)^T\mathbf{\Phi}^T(\mathbf{x}) + \mathbf{\Phi}(\mathbf{x})\big(\mathbf{A}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{B}_i(\mathbf{x})\mathbf{M}_j(\mathbf{x})\big)\Big) \\ &\quad - \sum_{k\in\mathbf{K}}\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})\Big)\mathbf{y}(\mathbf{x}).\end{aligned} \tag{17}$$

To take the continuous form of product terms $\mu_i(\mathbf{x})\lambda_j(\mathbf{x})$, $i = 1, ...r,\ j = 1, ..., c$, into stability analysis, the membership functions should be represented in the form of polynomials. Hence each product term $\mu_i(\mathbf{x})\lambda_j(\mathbf{x})$ is approximated with polynomials. However rewriting the product terms with one polynomial in the whole operating range may not result in an accurate estimation. To hurdle this problem, the operating domain of membership functions is partitioned to sub-regions so that corresponding to each sub-region we can find a suitable polynomial as the approximation term for each $\mu_i(\mathbf{x})\lambda_j(\mathbf{x})$ in that sub-region. Hence we define

$$\mu_i(\mathbf{x})\lambda_j(\mathbf{x}) = \eta_{ij,s_\tau}(\mathbf{x}) + \triangle\eta_{ij,s_\tau}(\mathbf{x}), \tag{18}$$

in which $\eta_{ij,s_\tau}(\mathbf{x})$ is the approximated polynomial for $\mu_i(\mathbf{x})\lambda_j(\mathbf{x})$ in the sub-region $s_\tau$ and $\triangle\eta_{ij,s_\tau}(\mathbf{x})$ is the error term, for $i = 1, ..., r,\ j = 1, ..., c,\ \tau = 1, ..., D$, where $D$ denotes the number of sub-regions.

Consequently (16) in the sub-region $s_\tau$ can be written as

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \sum_{\tau=1}^{D}\sum_{i=1}^{r}\sum_{j=1}^{c}\zeta_\tau(\mathbf{x})\Big(\eta_{ij,s_\tau}(\mathbf{x}) + \triangle\eta_{ij,s_\tau}(\mathbf{x})\Big) \\ &\quad \times \mathbf{y}^T(\mathbf{x})\mathbf{T}_{ij}(\mathbf{x})\mathbf{y}(\mathbf{x}),\end{aligned} \tag{19}$$

in which $\zeta_\tau(\mathbf{x})$ is a scalar function which takes 1 if $\mathbf{x}$ is inside the sub-region $\tau$, and otherwise 0.

Assume that the lower and upper bounds of the error terms are defined as

$$\gamma_{ij,s_\tau} \leq \triangle\eta_{ij,s_\tau}(\mathbf{x}) \leq \beta_{ij,s_\tau}, \tag{20}$$

then consider the following equation

$$(\triangle\eta_{ij,s_\tau}(\mathbf{x}) - \gamma_{ij,s_\tau})(\boldsymbol{\Lambda}_{ij}(\mathbf{x}) - \boldsymbol{\Lambda}_{ij}(\mathbf{x})) = \mathbf{0}, \tag{21}$$
$$(\beta_{ij,s_\tau} - \triangle\eta_{ij,s_\tau}(\mathbf{x}))(\boldsymbol{\Omega}_{ij}(\mathbf{x}) - \boldsymbol{\Omega}_{ij}(\mathbf{x})) = \mathbf{0}, \tag{22}$$

where $\boldsymbol{\Lambda}_{ij}(\mathbf{x})$ and $\boldsymbol{\Omega}_{ij}(\mathbf{x}) \in \mathbb{R}^{n\times n}$ are polynomial matrices and

$$\boldsymbol{\Lambda}_{ij}(\mathbf{x}) = \boldsymbol{\Lambda}_{ij}^T(\mathbf{x}) \geq 0, \tag{23}$$
$$\boldsymbol{\Omega}_{ij}(\mathbf{x}) = \boldsymbol{\Omega}_{ij}^T(\mathbf{x}) \geq 0. \tag{24}$$

From (19), (21) and (22), we have

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= \sum_{\tau=1}^{D}\sum_{i=1}^{r}\sum_{j=1}^{c}\zeta_\tau(\mathbf{x})\Bigg(\Big(\eta_{ij,s_\tau}(\mathbf{x}) + \triangle\eta_{ij,s_\tau}(\mathbf{x})\Big)\\
&\times \mathbf{y}^T(\mathbf{x})\mathbf{T}_{ij}(\mathbf{x})\mathbf{y}(\mathbf{x})\\
&+ \Big(\triangle\eta_{ij,s_\tau}(\mathbf{x}) - \gamma_{ij,s_\tau}\Big)\mathbf{y}^T(\mathbf{x})(\boldsymbol{\Lambda}_{ij}(\mathbf{x}) - \boldsymbol{\Lambda}_{ij}(\mathbf{x}))\mathbf{y}(\mathbf{x})\\
&+ \Big(\beta_{ij,s_\tau} - \triangle\eta_{ij,s_\tau}(\mathbf{x})\Big)\mathbf{y}^T(\mathbf{x})(\boldsymbol{\Omega}_{ij}(\mathbf{x}) - \boldsymbol{\Omega}_{ij}(\mathbf{x}))\mathbf{y}(\mathbf{x})\Bigg)\\
&= \sum_{\tau=1}^{D}\sum_{i=1}^{r}\sum_{j=1}^{c}\zeta_\tau(\mathbf{x})\Bigg(\eta_{ij,s_\tau}(\mathbf{x})\mathbf{y}^T(\mathbf{x})\mathbf{T}_{ij}(\mathbf{x})\mathbf{y}(\mathbf{x})\\
&+ \frac{1}{2}\Big(\triangle\eta_{ij,s_\tau}(\mathbf{x}) + \gamma_{ij,s_\tau} - \gamma_{ij,s_\tau}\Big)\mathbf{y}^T(\mathbf{x})\mathbf{T}_{ij}(\mathbf{x})\mathbf{y}(\mathbf{x})\\
&+ \Big(\triangle\eta_{ij,s_\tau}(\mathbf{x}) - \gamma_{ij,s_\tau}\Big)\mathbf{y}^T(\mathbf{x})(\boldsymbol{\Lambda}_{ij}(\mathbf{x}) - \boldsymbol{\Lambda}_{ij}(\mathbf{x}))\mathbf{y}(\mathbf{x})\\
&+ \frac{1}{2}\Big(\triangle\eta_{ij,s_\tau}(\mathbf{x}) + \beta_{ij,s_\tau} - \beta_{ij,s_\tau}\Big)\mathbf{y}^T(\mathbf{x})\mathbf{T}_{ij}(\mathbf{x})\mathbf{y}(\mathbf{x})\\
&+ \Big(\beta_{ij,s_\tau} - \triangle\eta_{ij,s_\tau}(\mathbf{x})\Big)\mathbf{y}^T(\mathbf{x})(\boldsymbol{\Omega}_{ij}(\mathbf{x}) - \boldsymbol{\Omega}_{ij}(\mathbf{x}))\mathbf{y}(\mathbf{x})\Bigg)\\
&= \sum_{\tau=1}^{D}\sum_{i=1}^{r}\sum_{j=1}^{c}\zeta_\tau(\mathbf{x})\Bigg(\Big(\eta_{ij,s_\tau}(\mathbf{x}) + \frac{1}{2}\gamma_{ij,s_\tau} + \frac{1}{2}\beta_{ij,s_\tau}\Big)\\
&\times \mathbf{y}^T(\mathbf{x})\mathbf{T}_{ij}(\mathbf{x})\mathbf{y}(\mathbf{x})\\
&+ \Big(\triangle\eta_{ij,s_\tau}(\mathbf{x}) - \gamma_{ij,s_\tau}\Big)\mathbf{y}^T(\mathbf{x})\boldsymbol{\Lambda}_{ij}(\mathbf{x})\mathbf{y}(\mathbf{x})\\
&+ \Big(\beta_{ij,s_\tau} - \triangle\eta_{ij,s_\tau}(\mathbf{x})\Big)\mathbf{y}^T(\mathbf{x})\boldsymbol{\Omega}_{ij}(\mathbf{x})\mathbf{y}(\mathbf{x})\\
&+ \Big(\triangle\eta_{ij,s_\tau}(\mathbf{x}) - \gamma_{ij,s_\tau}\Big)\mathbf{y}^T(\mathbf{x})\Big(\frac{1}{2}\mathbf{T}_{ij}(\mathbf{x}) - \boldsymbol{\Lambda}_{ij}(\mathbf{x})\Big)\mathbf{y}(\mathbf{x})\\
&+ \Big(\beta_{ij,s_\tau} - \triangle\eta_{ij,s_\tau}(\mathbf{x})\Big)\mathbf{y}^T(\mathbf{x})\Big(-\frac{1}{2}\mathbf{T}_{ij}(\mathbf{x}) - \boldsymbol{\Omega}_{ij}(\mathbf{x})\Big)\\
&\times \mathbf{y}(\mathbf{x})\Bigg).
\end{aligned} \tag{25}$$

The boundary information of the error terms in (20) is represented in two parts as $\triangle\eta_{ij,s_\tau}(\mathbf{x}) \geq \gamma_{ij,s_\tau}$, $\triangle\eta_{ij,s_\tau}(\mathbf{x}) \leq \beta_{ij,s_\tau}$. Regarding to (23) and (24), $-\triangle\eta_{ij,s_\tau}(\mathbf{x})\boldsymbol{\Omega}_{ij} \leq -\gamma_{ij,s_\tau}\boldsymbol{\Omega}_{ij}$ , $\triangle\eta_{ij,s_\tau}(\mathbf{x})\boldsymbol{\Lambda}_{ij} \leq \beta_{ij,s_\tau}\boldsymbol{\Lambda}_{ij}$. Then

$$\begin{aligned}
\dot{V}(\mathbf{x}) &\leq \mathbf{y}^T(\mathbf{x})\sum_{\tau=1}^{D}\sum_{i=1}^{r}\sum_{j=1}^{c}\zeta_\tau(\mathbf{x})\Bigg(\Big(\eta_{ij,s_\tau}(\mathbf{x}) + \frac{1}{2}\gamma_{ij,s_\tau}\\
&+ \frac{1}{2}\beta_{ij,s_\tau}\Big)\mathbf{T}_{ij}(\mathbf{x}) + \Big(\beta_{ij,s_\tau} - \gamma_{ij,s_\tau}\Big)\Big(\boldsymbol{\Lambda}_{ij}(\mathbf{x})\\
&+ \boldsymbol{\Omega}_{ij}(\mathbf{x})\Big) + (\triangle\eta_{ij,s_\tau}(\mathbf{x}) - \gamma_{ij,s_\tau})\Big(\frac{1}{2}\mathbf{T}_{ij}(\mathbf{x}) - \boldsymbol{\Lambda}_{ij}(\mathbf{x})\Big)\\
&+ \Big(\beta_{ij,s_\tau} - \triangle\eta_{ij,s_\tau}(\mathbf{x})\Big)\Big(-\frac{1}{2}\mathbf{T}_{ij}(\mathbf{x}) - \boldsymbol{\Omega}_{ij}(\mathbf{x})\Big)\Bigg)\mathbf{y}(\mathbf{x}).
\end{aligned} \tag{26}$$

According to the Lyapunov stability theory, the FMB control system is asymptotically stable if the following inequalities are satisfied

$$\begin{aligned}
&\sum_{i=1}^{r}\sum_{j=1}^{c}\Bigg((\eta_{ij,s_\tau}(\mathbf{x}) + \frac{1}{2}\gamma_{ij,s_\tau} + \frac{1}{2}\beta_{ij,s_\tau})\mathbf{T}_{ij}(\mathbf{x})\\
&\quad + \Big(\beta_{ij,s_\tau} - \gamma_{ij,s_\tau}\Big)\Big(\boldsymbol{\Lambda}_{ij}(\mathbf{x}) + \boldsymbol{\Omega}_{ij}(\mathbf{x})\Big)\Bigg) < 0,
\end{aligned} \tag{27}$$

for all sub-regions $\tau = 1, ..., D$ and $\frac{1}{2}\mathbf{T}_{ij}(\mathbf{x}) - \boldsymbol{\Lambda}_{ij}(\mathbf{x}) < 0$, $-\frac{1}{2}\mathbf{T}_{ij}(\mathbf{x}) - \boldsymbol{\Omega}_{ij}(\mathbf{x}) < 0$, for $i = 1, ..., r,\ j = 1, ..., c$. The achieved results can be proposed as the following theorem.

**Theorem 1:** Consider pre-defined polynomial $\eta_{ij,s_\tau}(\mathbf{x})$ as the approximation for product term $\mu_i(\mathbf{x})\lambda_j(\mathbf{x})$ in the sub-region $s_\tau$ for all $i = 1, ..., r,\ j = 1, ..., c,\ \tau = 1, ..., D$, satisfying (18) in which $\triangle\eta_{ij,s_\tau}(\mathbf{x})$ is the error term where according to (20), the lower and upper bounds are represented by $\gamma_{ij,s_\tau}$, $\beta_{ij,s_\tau}$ respectively. The polynomial FMB control system of (6) is asymptotically stable if there exist symmetric polynomial matrices $\mathbf{X}(\tilde{\mathbf{x}}) \in \mathbb{R}^{N\times N}$, $\boldsymbol{\Lambda}_{ij}(\mathbf{x}) \in \mathbb{R}^{N\times N}$, $\boldsymbol{\Omega}_{ij}(\mathbf{x}) \in \mathbb{R}^{N\times N}$, and polynomial matrices $\mathbf{M}_j(\mathbf{x}) \in \mathbb{R}^{N\times N}$, such that the following SOS conditions are satisfied.

$$\upsilon^T(\mathbf{X}(\tilde{\mathbf{x}}) - \epsilon_1(\mathbf{x})\mathbf{I})\upsilon \quad \text{is SOS}, \tag{28}$$
$$\upsilon^T\boldsymbol{\Lambda}_{ij}(\mathbf{x})\upsilon \quad \text{is SOS}, \tag{29}$$
$$\upsilon^T\boldsymbol{\Omega}_{ij}(\mathbf{x})\upsilon \quad \text{is SOS}, \tag{30}$$
$$\begin{aligned}
&-\upsilon^T\sum_{i=1}^{r}\sum_{j=1}^{c}\Bigg(\Big(\eta_{ij,s_\tau}(\mathbf{x}) + \frac{1}{2}\gamma_{ij,s_\tau} + \frac{1}{2}\beta_{ij,s_\tau}\Big)\mathbf{T}_{ij}(\mathbf{x})\\
&+ \Big(\beta_{ij,s_\tau} - \gamma_{ij,s_\tau}\Big)\Big(\boldsymbol{\Lambda}_{ij}(\mathbf{x}) + \boldsymbol{\Omega}_{ij}(\mathbf{x})\Big) + \epsilon_{2,s_\tau}(\mathbf{x})\boldsymbol{I}\Bigg)\upsilon\\
&\text{is SOS}, \qquad \forall s_\tau = 1, ..., D,
\end{aligned} \tag{31}$$
$$-\upsilon^T\Big(\frac{1}{2}\mathbf{T}_{ij}(\mathbf{x}) - \boldsymbol{\Lambda}_{ij}(\mathbf{x}) + \epsilon_{3ij}(\mathbf{x})\boldsymbol{I}\Big)\upsilon \quad \text{is SOS}, \tag{32}$$
$$\begin{aligned}
&-\upsilon^T\Big(-\frac{1}{2}\mathbf{T}_{ij}(\mathbf{x}) - \boldsymbol{\Omega}_{ij}(\mathbf{x}) + \epsilon_{4ij}(\mathbf{x})\boldsymbol{I}\Big)\upsilon \quad \text{is SOS}\\
&\qquad \forall i = 1, ..., r, \quad j = 1, ..., c,
\end{aligned} \tag{33}$$

where $\upsilon(t) \in \mathbb{R}^n$ is a vector independent of $\mathbf{x}$ and $\epsilon_1(\mathbf{x})$, $\epsilon_{2,s_\tau}(\mathbf{x})$, $\epsilon_{3ij}(\mathbf{x})$, $\epsilon_{4ij}(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$ are pre-defined polynomial scalars and the local feedback gains can be derived from (15).

**Remark 2:** In the proposed approach, to approximate each product term of membership functions $\mu_i(\mathbf{x})\lambda_j(\mathbf{x})$ by polynomials, the operating domain of membership functions is divided to sub-regions $s_1$ to $s_\tau,\ \tau = 1, ..., D$, such that it can facilitate better approximation. Hence each stability condition in (31), should be only satisfied in the sub-region where $\eta_{ij,s_\tau}(\mathbf{x})$ is presented as the approximated membership function. To release conservativeness resulting from consideration global operating range for polynomials during finding numerical solutions via SOSTOOLS, concepts of $S$-procedure [2] is considered to introduce some slack expressions denoted as $L_{s_\tau}(\mathbf{x})$ where $L_{s_\tau}(\mathbf{x}) \leq 0,\ \forall \mathbf{x} \in s_\rho$ then we have

$$\begin{aligned}
&-\upsilon^T \sum_{i=1}^{r}\sum_{j=1}^{c}\bigg(\Big(\eta_{ij,s_\tau}(\mathbf{x}) + \frac{1}{2}\gamma_{ij,s_\tau} + \frac{1}{2}\beta_{ij,s_\tau}\Big)\mathbf{T}_{ij}(\mathbf{x}) \\
&+ \Big(\beta_{ij,s_\tau} - \gamma_{ij,s_\tau}\Big)\Big(\mathbf{\Lambda}_{ij}(\mathbf{x}) + \mathbf{\Omega}_{ij}(\mathbf{x})\Big) + \epsilon_{2,s_\tau}(\mathbf{x})\boldsymbol{I}\bigg)\upsilon \geq \\
&-\upsilon^T \sum_{i=1}^{r}\sum_{j=1}^{c}\bigg(\Big(\eta_{ij,s_\tau}(\mathbf{x}) + \frac{1}{2}\gamma_{ij,s_\tau} + \frac{1}{2}\beta_{ij,s_\tau}\Big)\mathbf{T}_{ij}(\mathbf{x}) \\
&+ \Big(\beta_{ij,s_\tau} - \gamma_{ij,s_\tau}\Big)\Big(\mathbf{\Lambda}_{ij}(\mathbf{x}) + \mathbf{\Omega}_{ij}(\mathbf{x})\Big) + \epsilon_{2,s_\tau}(\mathbf{x})\boldsymbol{I}\bigg)\upsilon \\
&+ \kappa(\mathbf{x})\ \upsilon^T L_{s_\tau}(\mathbf{x})\upsilon, \qquad \forall \mathbf{x} \in s_\rho \qquad (34)
\end{aligned}$$

where $\kappa(\mathbf{x}) \in \mathbb{R}$ is SOS. Positivity of the right hand side of the (34) satisfies that of (31).

For example suppose that the membership functions are function of $x_1$ and the operating domain for sub-region $s_\rho$ is $x_{min,s_\rho} \leq x_1 \leq x_{max,s_\rho}$. Then $L_{s_\rho}(x_1) = (x_1 - x_{min,s_\rho})(x_1 - x_{max,s_\rho})$. The relaxed conditions are proposed as the following theorem.

**Theorem 2:** Consider pre-defined polynomial $\eta_{ij,s_\tau}(\mathbf{x})$ as the approximation for $\mu_i(\mathbf{x})\lambda_j(\mathbf{x})$ in the sub-region $s_\tau$ for all $i = 1, ..., r,\ j = 1, ..., c,\ \tau = 1, ..., D$, satisfying (18) in which $\triangle\eta_{ij,s_\tau}(\mathbf{x})$ is the error term where according to (20), the lower and upper bounds are represented by $\gamma_{ij,s_\tau}$, $\beta_{ij,s_\tau}$ respectively. The polynomial FMB control system of (6) is asymptotically stable if there exist symmetric polynomial matrices $\mathbf{X}(\tilde{\mathbf{x}}) \in \mathbb{R}^{N\times N},\ \mathbf{\Lambda}_{ij}(\mathbf{x}) \in \mathbb{R}^{N\times N},\ \mathbf{\Omega}_{ij}(\mathbf{x}) \in \mathbb{R}^{N\times N}$, polynomial matrices $\mathbf{M}_j(\mathbf{x}) \in \mathbb{R}^{m\times N}$ and polynomial scaler $\kappa(\mathbf{x})$ such that the following SOS conditions are satisfied.

$$\upsilon^T(\mathbf{X}(\tilde{\mathbf{x}}) - \epsilon_1(\mathbf{x})\mathbf{I})\upsilon \quad \text{is SOS,} \qquad (35)$$

$$\upsilon^T\mathbf{\Lambda}_{ij}(\mathbf{x})\upsilon \quad \text{is SOS,} \qquad (36)$$

$$\upsilon^T\mathbf{\Omega}_{ij}(\mathbf{x})\upsilon \quad \text{is SOS,} \qquad (37)$$

$$\begin{aligned}
&-\upsilon^T \sum_{i=1}^{r}\sum_{j=1}^{c}\bigg(\Big(\eta_{ij,s_\tau}(\mathbf{x}) + \frac{1}{2}\gamma_{ij,s_\tau} + \frac{1}{2}\beta_{ij,s_\tau}\Big)\mathbf{T}_{ij}(\mathbf{x}) \\
&+ \Big(\beta_{ij,s_\tau} - \gamma_{ij,s_\tau}\Big)\Big(\mathbf{\Lambda}_{ij}(\mathbf{x}) + \mathbf{\Omega}_{ij}(\mathbf{x})\Big) + \epsilon_{2,s_\tau}(\mathbf{x})\boldsymbol{I}\bigg)\upsilon \\
&+ \kappa(\mathbf{x})\ \upsilon^T L_{s_\tau}(\mathbf{x})\upsilon \quad \text{is SOS,} \qquad \forall s_\tau = 1, ..., D, \quad (38)
\end{aligned}$$

$$\kappa(\mathbf{x}) \quad \text{is SOS,} \qquad (39)$$

$$-\upsilon^T\Big(\frac{1}{2}\mathbf{T}_{ij}(\mathbf{x}) - \mathbf{\Lambda}_{ij}(\mathbf{x}) + \epsilon_{3ij}(\mathbf{x})\boldsymbol{I}\Big)\upsilon \quad \text{is SOS,} \qquad (40)$$

$$-\upsilon^T\Big(-\frac{1}{2}\mathbf{T}_{ij}(\mathbf{x}) - \mathbf{\Omega}_{ij}(\mathbf{x}) + \epsilon_{4ij}(\mathbf{x})\boldsymbol{I}\Big)\upsilon \quad \text{is SOS,} \qquad (41)$$

$$\forall i = 1, ..., r, \quad j = 1, ..., c.$$

where $\upsilon(t) \in \mathbb{R}^n$ is a vector independent of $\mathbf{x}$ and $\epsilon_1(\mathbf{x})$, $\epsilon_{2,s_\tau}(\mathbf{x}),\ \epsilon_{3ij}(\mathbf{x}),\ \epsilon_{4ij}(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$ are pre-defined polynomial scalars and the local feedback gains can be derived from (15).

**Remark 3:** If $\hat{\mathbf{x}} = \mathbf{x}$, $\mathbf{A}_i(\mathbf{x}) = \mathbf{A}_i$ and $\mathbf{B}_i(\mathbf{x}) = \mathbf{B}_i$, then the polynomial fuzzy control system (3) is reduced to the T-S FMB control system (7). Hence investigation of system stability for T-S FMB control systems is considered as a special case of the proposed approach.

## IV. Simulation Example

The effectiveness of the proposed stability analysis approach is illustrated by the following simulation example. Consider a polynomial fuzzy model with three rules as in the following,

$$\begin{aligned}
\mathbf{A}_1(\mathbf{x}) &= \begin{pmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -a & -6 \end{pmatrix}, \\
\mathbf{A}_2(\mathbf{x}) &= \begin{pmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 0 & -6 \end{pmatrix}, \\
\mathbf{A}_3(\mathbf{x}) &= \begin{pmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 0.2172a & -6 \end{pmatrix}, \\
\mathbf{B}_1(\mathbf{x}) &= \mathbf{B}_2(\mathbf{x}) = \mathbf{B}_3(\mathbf{x}) = \begin{pmatrix} x_1 \\ b \end{pmatrix}, \qquad (42)
\end{aligned}$$

where $a$ and $b$ are parameters in the proposed fuzzy model. A three-rule polynomial fuzzy controller in the form of (5) is employed to close the feedback loop. Then the polynomial FMB control system in the form of (6) is obtained.

The membership functions for the polynomial fuzzy model are chosen as

$$\mu_1(x_1) = \frac{1}{1 + e^{-\frac{(x_1-4)}{2}}}, \qquad \mu_3(x_1) = \frac{1}{1 + e^{\frac{(x_1+4)}{2}}}$$
$$\mu_2(x_1) = 1 - \mu_1(x_1) - \mu_3(x_1), \qquad (43)$$

and for the polynomial fuzzy controller are chosen as follows.

$$\lambda_1(x_1) = \begin{cases} 1 & -15 \leq x_1 \leq -10 \\ -\dfrac{-x_1+2}{12} & -10 \leq x_1 \leq 2 \\ 0 & 2 \leq x_1 \leq 15 \end{cases},$$
$$\lambda_3(x_1) = \begin{cases} 0 & -15 \leq x_1 \leq -2 \\ \dfrac{x_1+2}{12} & -2 \leq x_1 \leq 10 \\ 1 & 10 \leq x_1 \leq 15 \end{cases}, \qquad (44)$$
$$\lambda_2(x_1) = 1 - \lambda_1(x_1) - \lambda_3(x_1).$$

The plots of membership functions of (43) and (44) are shown in Fig. 1. Then each product term $\mu_i(x_1)\lambda_j(x_1)$ is approximated by a polynomial. In this way, the operating region of fuzzy membership functions is divided to 11 sub-regions, $s_1, ...s_{11}$. Then corresponding to each sub-region, a polynomial is derived as an approximated term. To release conservativeness resulting from consideration of whole operating domain for all regionally approximated polynomials via SOSTOOLS, the conditions in Theorem 2, containing some slack expressions, are employed for which corresponding to each sub-region $s_1$ to $s_{11}$, slack expressions $L_{s_1}(x_1)$ to $L_{s_{11}}(x_1)$ are introduced. Due to lack of space, the approximated membership functions, boundaries of the error terms and slack expressions have been omitted. For comparison purposes the stability conditions proposed in Theorem 2 [9], is employed to investigate stability analysis for $2 \leq a \leq 9$, $7.4 \leq b \leq 9$. Due to using PDC design techniques reported in [9], the membership functions of both fuzzy model and fuzzy controller have been chosen as in (43). Consequently, this design technique causes much more complicated implementation. The stability region is demonstrated in Fig. 2. It is shown that compared to the stability conditions in [9], the proposed SOS-based stability conditions in Theorem 1 and Theorem 2 are able to offer larger stable regions. In this example the following assumptions are considered $\epsilon_1(\mathbf{x}),\ \epsilon_{2,s_\tau}(\mathbf{x}),\ \epsilon_{3,ij}(\mathbf{x}) = \epsilon_{4,ij}(\mathbf{x}) = 0.001$ for all $i,\ j,\ \tau$.

## V. Conclusion

This paper has presented relaxed SOS-based stability conditions for the polynomial FMB control systems. Continuous form of membership functions have been taken to the stability analysis leading to release conservatism. Furthermore, the proposed approach can be applied to the polynomial FMB control systems in which fuzzy model and fuzzy controller do not share the same membership functions under non-PDC design technique. With the free third-party MATLAB toolbox, SOSTOOLS, the solution for the SOS-based stability conditions could be found numerically. Simulation example has been given to illustrate the effectiveness of the proposed approach.

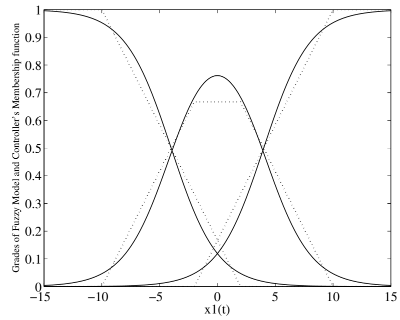


Fig. 1. Membership functions for Simulation Example.

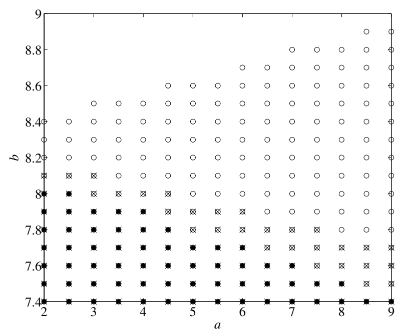


Fig. 2. Stability regions given by [9] (•), Theorem 1 (×) and Theorem 2 (○) for Simulation Example.

## References


[1] H. K. Khalil, *Nonlinear Systems*. Prentice Hall Upper Saddle River, NJ, 1996.

[2] S. P. Boyd, *Linear matrix inequalities in system and control theory*. Society for Industrial & Applied, 1994.

[3] K. Tanaka, T. Ikeda, and H. O. Wang, "Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs," *IEEE Trans. Fuzzy Systems*, vol. 6, no. 2, pp. 250–265, May 1998.

[4] H. K. Lam and F. H. F. Leung, "Stability analysis of fuzzy control systems subject to uncertain grades of membership," *IEEE Trans. Systems, Man, and Cybernetics, Part B*, vol. 35, no. 6, pp. 1322–1325, Dec. 2005.

[5] C. Ariño and A. Sala, "Extensions to "Stability analysis of fuzzy control systems subject to uncertain grades of membership"," *IEEE Trans. Systems, Man, and Cybernetics, Part B*, vol. 38, no. 2, pp. 558–563, Apr. 2008.

[6] H. K. Lam and M. Narimani, "Stability analysis and performance design for fuzzy-model-based control system under imperfect premise matching," *IEEE Trans. Fuzzy Systems*, Available on-line.

[7] A. Sala and C. Ariño, "Relaxed stability and performance conditions for Takagi-Sugeno fuzzy systems with knowledge on membership function overlap," *IEEE Trans. Systems, Man, and Cybernetics, Part B*, vol. 37, no. 3, pp. 727–732, Jun. 2007.

[8] ——, "Relaxed stability and performance LMI conditions for Takagi–Sugeno fuzzy systems with polynomial constraints on membership function shapes," *IEEE Trans. Fuzzy Systems*, vol. 16, no. 5, pp. 1328–1336, Oct. 2008.

[9] K. Tanaka, H. Yoshida, H. Ohtake, and H. Wang, "A sum of squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," *IEEE Trans. Fuzzy Systems*, (Avilable online).

[10] K. Tanaka, H. Ohtake, and H. Wang, "Guaranteed cost control of polynomial fuzzy systems via a sum of squares approach," *IEEE Trans. Systems, Man, and Cybernetics, Part B*, (Avilable online).

[11] A. Sala, "Introducing shape-dependent relaxed conditions in fuzzy control of nonlinear systems in Takagi-Sugeno form," in *IEEE World Congress on Computational Intelligence, FUZZ-IEEE 2008.*, June 2008, pp. 512–517.

[12] A. Sala and T. Guerra, "Stability Analysis of Fuzzy Systems: membership-shape and polynomial approaches," in *Proceedings of the 17th World Congress IFAC*, July 2008, pp. 5605–5610.

[13] M. Bernal, T. Guerra, and A. Kruszewski, "A membership-function-dependent stability analysis of Takagi-Sugeno models," in *Proceedings of the 17th World Congress the IFAC*, July 2008, pp. 5611–5616.

[14] S. Prajna, A. Papachristodoulou, and P. Parrilo, "Introducing SOSTOOLS: A general purpose sum of squares programming solver," in *proceeding of the* $41^{st}$ *IEEE Conference On Decision And Control*, vol. 1, Dec. 2002, pp. 741–746.

[15] K. Tanaka and H. O. Wang, *Fuzzy control systems design and analysis: A linear matrix inequality approach*. John Wiley & Sons, Inc. New York, NY, USA, 2001.

[16] K. Tanaka and M. Sugeno, "Stability analysis and design of fuzzy control systems," *Fuzzy Sets and Systems*, vol. 45, no. 2, pp. 135–156, 1992.

[17] K. Tanaka and M. Sano, "A robust stabilization problem of fuzzy control systems and its application to backing up control of a truck-trailer," *IEEE Trans. Fuzzy Systems*, vol. 2, no. 2, pp. 119–134, May 1994.

[18] S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear control synthesis by sum of squares optimization: a lyapunov-based approach," *Control Conference, 2004. 5th Asian*, vol. 1, pp. 157–165 Vol.1, July 2004.