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Robustness and Stability of Pure Impulsive Synchronization with Parametric Uncertainties and Mismatch

Ziqi Zhu
Institute for Pattern Recognition and Artificial Intelligence
Huazhong University of Science and Technology
Wuhan, China
ziqi_zhu@163.com

Hanping Hu
Institute for Pattern Recognition and Artificial Intelligence
Huazhong University of Science and Technology
Wuhan, China
hphu@mail.hust.edu.cn

***Abstract*—Impulsive control of a chaotic system has a great potential for applications in various fields. Therefore, the robustness and stability of impulsive synchronization is very important. In this paper, we introduce a new method for analyzing the robustness and stability of impulsive synchronization with parametric uncertainties and mismatch. By analyzing the oscillation process of the error between two chaotic systems, we establish a quantitive relationship between the prescribed synchronization threshold, the length of impulse interval, the bounds of uncertainties and mismatch and the impulse intensity. Numerical simulation results based on Chen system and Chua system are provided.**

***Index Terms*—Impulsive synchronization of chaotic systems; Uncertainty; Robustness**

I. INTRODUCTION

Since the pioneering work of Pecora and Carroll in 1990[1], synchronization of chaotic systems is an active research area due to great potential for applications in engineering (such as design of oscillators, vibrating wave generation, mechanical resonance, spatiotemporal pattern formation, and communications), as well as in biological systems. Numerous methods have been developed for chaos synchronization, including drive-response control [1][2], coupling control [3][4], adaptive control [5][6], fuzzy control [7], observer-based control [8][9], feedback control [10][11], and impulsive control [12][13], etc.

Impulsive control scheme is first proposed by L. Kocarev et al. in 1996 [12]. Impulsive control is attractive because it allows the stabilization of a chaotic system using only small control impulses, and it offers a direct method for modulating digital information onto a chaotic carrier signal for spread spectrum applications. In [13], T. Yang et al. present a theory of impulsive control of chaotic dynamical systems and provide an estimation of upper bound of the impulse interval. In Yang's theory, the parameters of the chaotic systems are known exactly and time invariant. Furthermore, the parameters should be identical for both chaotic systems. These conditions are impractical in real-world application. Since the chaotic system is sensitive to parameters, the robust synchronization theory has become quite attractive.

While analyzing the stability of impulse synchronization, the Lyapunov stability theory is the most using theory [14]-[17]. However, in this paper, what we use is different. Due to the parametrical mismatch and uncertain, the error between two chaotic systems will not approach to zero, but oscillate between certain bound. If the error can be ensured in the pre-specified synchronization threshold, we consider that the two chaotic systems are synchronized. Based on this definition of synchronization, we analyze the oscillation process of the error between the driving and driven systems, and derive a quantitative condition. The condition revealed the relation between the prescribed bound, the length of impulse interval, the bounds of uncertainties and mismatch and the impulse intensity which is also called impulse synchronization coefficient in this paper. It is shown that to achieve synchronization, the impulsive interval is not only determined by the pre-specified bound of the error and the bound of the parametric uncertainties and the mismatch, but also the impulse synchronization coefficient. The bigger the impulse synchronization coefficient is, the longer the impulsive interval can be allowed. Furthermore, by using this method, the robustness and stability of the impulsive synchronization scheme with both internal noise (noise-affected parameters) and external noise (noise-affected driving signal) can be analyzed. The further results will be published soon.

This remaining of the paper is organized as follows. In section II, the modeling of pure chaotic impulsive synchronization with parametric uncertainties and mismatch, illustrated by Chen system, is given. The condition which provides quantitative relation between the prescribed synchronization threshold, the length of impulse interval, the bounds of uncertainties and mismatch and impulse synchronization coefficient, is derived in section III. In section IV, some examples are illustrated to support the results deduced in section III. Finally, the paper is concluded in Section V.

II. SYNCHRONIZATION MODEL

In this section, we present the class of 3-dimension chaotic system to be considered with the Chen system as an illustrative example.

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The driving system is in the presence of parametric uncertainty is described by:

$$\dot{\mathbf{X}}_1 = (\mathbf{A} + \Delta\mathbf{A}_1(t))\mathbf{X}_1 + \varphi(\mathbf{X}_1) \tag{1}$$

where $\mathbf{X}_1 \in \mathbb{R}^3$ is the state vector and $\mathbf{A} \in \mathbb{R}^3$ is the nominal constant matrix, and $\varphi(\cdot)$ is the nonlinear part of Chen system, where

$$\mathbf{X}_1 = (x_1, y_1, z_1)^T, \varphi(\mathbf{X}_1) = \begin{pmatrix} 0 \\ -x_1 z_1 \\ x_1 y_1 \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$\Delta\mathbf{A}_1(t)$ denotes time-varying parametric uncertainties in the driving system where:

$$\Delta\mathbf{A}_1 = \begin{pmatrix} -\Delta a_1(t) & \Delta a_1(t) & 0 \\ \Delta c_1(t) - \Delta a_1(t) & \Delta c_1(t) & 0 \\ 0 & 0 & -\Delta b_1(t) \end{pmatrix}$$

Similarly, the driven system is described by:

$$\begin{cases} \dot{\mathbf{X}}_2 = (\mathbf{A} + \Delta\mathbf{A}_2(t))\mathbf{X}_2 + \varphi(\mathbf{X}_2) \\ \Delta\mathbf{X}|_{t=\tau_i} = -\mathbf{Be} \qquad , i = 1, 2, \ldots \end{cases} \tag{2}$$

where $\Delta\mathbf{A}_2(t)$ denotes time-varying parametric uncertainties in the driven system and the impulse synchronization coefficient $\mathbf{B}$, which denotes the pulse intensity, is a 3×3 diagonal matrix and $\mathbf{e}$ is the synchronization error, where:

$$\Delta\mathbf{A}_2 = \begin{pmatrix} -\Delta a_2(t) & \Delta a_2(t) & 0 \\ \Delta c_2(t) - \Delta a_2(t) & \Delta c_2(t) & 0 \\ 0 & 0 & -\Delta b_2(t) \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} b_1 & & \\ & b_2 & \\ & & b_3 \end{pmatrix},$$

$$\mathbf{e} = \begin{pmatrix} e_x(t) \\ e_y(t) \\ e_z(t) \end{pmatrix} = \begin{pmatrix} x_1(t) - x_2(t) \\ y_1(t) - y_2(t) \\ z_1(t) - z_2(t) \end{pmatrix}$$

$b_1 \in (0,1), b_2 \in (0,1), b_3 \in (0,1)$. $\tau_i, i = 1, 2, \ldots$ denotes the time instants as which impulses are sent to the driven system from the driving system. And we define the impulsive interval as:

$$\tau_{i+1} - \tau_i \equiv \Delta_1, i = 1, 2, \ldots$$

From (1) and (2), we can derive the error system

$$\begin{cases} \dot{\mathbf{e}} = ((\mathbf{A} + \Delta\mathbf{A}_1(t))\mathbf{X}_1 - (\mathbf{A} + \Delta\mathbf{A}_2(t))\mathbf{X}_2) \\ \qquad +(\varphi(\mathbf{X}_1) - \varphi(\mathbf{X}_2)) \\ \Delta\mathbf{X}|_{t=\tau_i} = -\mathbf{Be}, \qquad i = 1, 2, \ldots \end{cases} \tag{3}$$

In order to facilitate the analysis of stability of the error system (3), we re-write (3) as

$$\begin{cases} \dot{\mathbf{e}} = \mathbf{A}(t)\mathbf{e} + \Delta\mathbf{A}(t)\mathbf{X}_2 + \Psi(\mathbf{X}_1, \mathbf{X}_2) \\ \Delta\mathbf{X}|_{t=\tau_i} = -\mathbf{Be}, \qquad i = 1, 2, \ldots \end{cases} \tag{4}$$

where $a(t) = a + \Delta a_1(t)$, $b(t) = a + \Delta b_1(t)$, $c(t) = c + \Delta c_1(t)$ and $\Delta a(t) = \Delta a_1(t) - \Delta a_2(t)$, $\Delta b(t) = \Delta b_1(t) - \Delta b_2(t)$, $\Delta c(t) = \Delta c_1(t) - \Delta c_2(t)$ and

$$\mathbf{A}(t) = \begin{pmatrix} -a(t) & a(t) & 0 \\ c(t) - a(t) & c(t) & 0 \\ 0 & 0 & -b(t) \end{pmatrix},$$

$$\Psi(\mathbf{X}_1 - \mathbf{X}_2) = \begin{pmatrix} 0 \\ x_1 z_1 - x_2 z_2 \\ x_1 y_1 - x_2 y_2 \end{pmatrix},$$

$$\Delta\mathbf{A}(t) = \begin{pmatrix} -\Delta a(t) & \Delta a(t) & 0 \\ \Delta c(t) - \Delta a(t) & \Delta c(t) & 0 \\ 0 & 0 & \Delta b(t) \end{pmatrix}$$

While the parameters are identically for both chaotic systems, driven by the impulse signals, the error between two systems is asymptotically stable, which indicates that

$$\lim_{t \to \infty} e'_x(t) = 0, \lim_{t \to \infty} e'_y(t) = 0, \lim_{t \to \infty} e'_z(t) = 0$$

However, due to the parametric uncertainty of the chaotic systems, for any t_k, $\exists t > t_k$, such that

$$e_x(t) \neq 0, e_y(t) \neq 0, e_z(t) \neq 0$$

Thus we have some priori knowledge as given below.

Definition 1: The driving system and the driven system are synchronized if $\exists t_0$, such that $\forall t > t_0$

$$|\ e_x(t)\ | < \varepsilon, |\ e_y(t)\ | < \varepsilon, |\ e_z(t)\ | < \varepsilon$$

where ε is the synchronization threshold.

Assumption 1: The parametric uncertainties and mismatch $\Delta\mathbf{A}_1(t)$ and $\Delta\mathbf{A}_2(t)$ are bounded as follows:

$$\begin{aligned} |\Delta a_1(t)| \leq \xi_a \cdot a, |\Delta a_2(t)| \leq \xi_a \cdot a \\ |\Delta b_1(t)| \leq \xi_b \cdot b, |\Delta b_2(t)| \leq \xi_b \cdot b \\ |\Delta c_1(t)| \leq \xi_c \cdot c, |\Delta c_2(t)| \leq \xi_c \cdot c \end{aligned}$$

where ξ is the error coefficient.

Thus, we have

$$\begin{aligned} |a(t)| \leq (1 + \xi_a) \cdot a, |\Delta a(t)| \leq 2\xi_a \cdot a \\ |b(t)| \leq (1 + \xi_b) \cdot b, |\Delta b(t)| \leq 2\xi_b \cdot b \\ |c(t)| \leq (1 + \xi_c) \cdot c, |\Delta c(t)| \leq 2\xi_c \cdot c \end{aligned}$$

III. Impulsive Synchronization Scheme

While analyzing the stability of impulse synchronization, the Lyapunov stability theory is the most using theory. However, in this paper, what we used is different. In this section, we analyze the oscillation process of the error between the driving system and the driven system. Furthermore a quantitative relationship between the length of the impulse interval and the impulse synchronization coefficient b_1, b_2, b_3, synchronization threshold ε, the error coefficient ξ is established.

Lemma 1: If $\exists n_0 \in \mathbb{Z}^+$, such that $\forall n > n_0$

$$\|e(t_n)\| \triangleq |e_x(t_n)| + |e_y(t_n)| + |e_z(t_n)| < \varepsilon \tag{5}$$

where $t_n = n \times \Delta_1$ and $e_x(t_n)$, $e_y(t_n)$, $e_x(t_n)$ denote the error at the time instants directly before the impulse signal sent to the driven system, then $\exists t_0$, such that for $\forall t > t_0$

$$|e_x(t)| < \varepsilon, |e_y(t)| < \varepsilon, |e_z(t)| < \varepsilon$$

Proof

The process of impulsive synchronization can be divided into many periods which contain the impulse and the period between two neighboring impulses as shown in Fig.1.

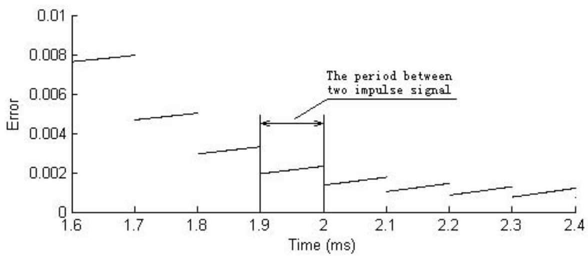


Fig.1 The error two chaotic systems which are controlled by impulsive synchronization scheme

Since the chaotic systems are sensitive to initial condition and parameters, the tiny error between two chaotic systems will increase with time. This phenomenon can be demonstrated by the results shown in Fig.1. During the period between two neighboring impulses, the error between two chaotic systems is increased with time. Thus the error is largest at the time instants $n \cdot \Delta_1$. If $\exists n_0 \in \mathbb{Z}^+$, such that $\forall n > n_0$

$$\|e(t_n)\| < \varepsilon$$

Therefore $\forall t > t_{n_0}$, $|e_x(t)| < \varepsilon$, $|e_y(t)| < \varepsilon$, $|e_z(t)| < \varepsilon$.

Lemma 2: If the following two conditions are satisfied,

1) If $\|e(t_n)\| \geq \varepsilon$

$$\|e(t_{n+1})\| < \|e(t_n)\| \tag{6}$$

2) If $\|e(t_n)\| < \varepsilon$

$$\|e(t_{n+1})\| < \varepsilon \tag{7}$$

Then $\exists n_0 \in \mathbb{Z}^+$, such that $\forall t > t_{n_0}$, $|e_x(t)| < \varepsilon$, $|e_y(t)| < \varepsilon$, $|e_z(t)| < \varepsilon$.

Proof

$\forall |e_x(t_1)|$, $|e_x(t_1)|$, $|e_z(t_1)|$, to show the result, we consider two cases: $\|e(t_1)\| \geq \varepsilon$ and $\|e(t_1)\| < \varepsilon$.

Case 1 $\|e(t_1)\| \geq \varepsilon$

Based on (6), the error between two chaotic systems decreases gradually till $\exists n_0 \in \mathbb{Z}^+$, such that

$$\|e(t_{n_0})\| < \varepsilon$$

Once $\|e(t_{n_0})\| < \varepsilon$, from (6), $\forall n > n_0$, we have

$$\|e(t_n)\| < \varepsilon$$

According to *Lemma* 1, $\forall t > t_{n_0}$, $|e_x(t)| < \varepsilon$, $|e_y(t)| < \varepsilon$, $|e_z(t)| < \varepsilon$.

Case 2 $\|e(t_1)\| < \varepsilon$

Since $\|e(t_1)\| < \varepsilon$, from (7), $\forall n > 1$, we have

$$\|e(t_n)\| < \varepsilon$$

According to *Lemma* 1, $\forall t > t_1$, $|e_x(t)| < \varepsilon$, $|e_y(t)| < \varepsilon$, $|e_z(t)| < \varepsilon$.

Since the driven system is Chen system, it is obviously that $\sup(x_2(t)) > 0$, $\sup(y_2(t)) > 0$, $\sup(z_2(t)) > 0$, $\sup(e_x(t)) > 0$, $\sup(e_y(t)) > 0$, $\sup(e_z(t)) > 0$, and

$$\sup(x_2(t) - y_2(t)) > 0, \ \sup((c-a)x_2(t) - cy_2(t)) > 0$$

Theorem 1: While the driving and driven systems are both Chen system, if the length of the impulsive interval satisfied:

$$\Delta_1 < \frac{\min(b_1, b_2, b_3)}{\max(m(\xi_c), n(\xi_a, \xi_c), o(\xi_b)) + C(\xi_a, \xi_b, \xi_c)/\varepsilon} \tag{8}$$

where:

$$\begin{aligned}
m(\xi_c) &= c \cdot \xi_c + (c + \sup(y_2(t)) + \sup(z_2(t))) \\
n(\xi_a, \xi_c) &= a \cdot \xi_a + c \cdot \xi_c + a + c + sup(x_2(t)) + \\
&\quad sup(e_x(t)) \\
o(\xi_b) &= b \cdot \xi_b + b + \sup(z_2(t)) + \sup(e_x(t)) \\
C(\xi_a, \xi_b, \xi_c) &= 2(a \cdot \xi_a + c \cdot \xi_c) \cdot \sup(x_2(t) - y_2(t)) + \\
&\quad 2a \cdot \xi_a \cdot \sup(x_2(t)) + 2b \cdot \xi_b \cdot \sup(z_2(t))
\end{aligned}$$

then $\exists t_0$, such that $\forall t > t_0$

$$|e_x(t)| < \varepsilon, |e_y(t)| < \varepsilon, |e_z(t)| < \varepsilon$$

Proof:

Case 1: $\|e(t_1)\| \geq \varepsilon$

According to the result proposed in [16], we already know that the impulsive interval Δ_1 must be very short so that two chaotic systems can be synchronized with parametric uncertain and mismatch. Thus, we have

$$\begin{aligned}
|e_x(t_{n+1})| &\approx (1 - b_1)|e_x(t_n)| + \Delta_1 |\dot{e}_x(t_n)| \\
|e_y(t_{n+1})| &\approx (1 - b_2)|e_y(t_n)| + \Delta_1 |\dot{e}_y(t_n)| \\
|e_z(t_{n+1})| &\approx (1 - b_3)|e_z(t_n)| + \Delta_1 |\dot{e}_z(t_n)|
\end{aligned} \tag{9}$$

From (9), we have

$$\begin{aligned}
|e_x(t_{n+1})| - |e_x(t_n)| &\approx \Delta_1 |\dot{e}_x(t_n)| - b_1 |e_x(t_n)| \\
|e_y(t_{n+1})| - |e_y(t_n)| &\approx \Delta_1 |\dot{e}_y(t_n)| - b_1 |e_y(t_n)| \\
|e_z(t_{n+1})| - |e_z(t_n)| &\approx \Delta_1 |\dot{e}_z(t_n)| - b_1 |e_z(t_n)|
\end{aligned} \tag{10}$$

Therefore

$$\begin{aligned}
&(|e_x(t_{n+1})| - |e_x(t_n)|) + (|e_y(t_{n+1})| - |e_y(t_n)|) + \\
&(|e_z(t_{n+1})| - |e_z(t_n)|) \\
\approx &(\Delta_1 |\dot{e}_x(t_n)| - k|e_x(t_n)|) + (\Delta_1 |\dot{e}_y(t_n)| - \\
&k|e_y(t_n)|) + (\Delta_1 |\dot{e}_z(t_n)| - k|e_z(t_n)|) \\
= &\Delta_1 (|\dot{e}_x(t_n)| + |\dot{e}_y(t_n)| + |\dot{e}_z(t_n)|) - \\
&b_1 |e_x(t_n)| - b_2 |e_y(t_n)| - b_3 |e_z(t_n)|
\end{aligned} \tag{11}$$

From (4), we have

$$\begin{aligned}&|\dot{e}_x(t_n)|\\=&|(-a(t_n)e_x(t_n)+a(t_n)e_y(t_n))-(-\Delta a(t_n)x_2(t_n)+\\&\Delta a(t_n)y_2(t_n))|\\<&|a(t_n)|(|e_y(t_n)|+e_x(t_n))+|\Delta a(t_n)|\cdot|x_2(t_n)-y_2(t_n)|\end{aligned}\tag{12}$$

$$\begin{aligned}&|\dot{e}_y(t_n)|\\=&|(c(t_n)-a(t_n))\cdot e_x(t_n)+c(t_n)e_y(t_n)+x_1(t_n)z_1(t_n)-\\&(((\Delta c(t_n)-\Delta a(t_n))x_2(t_n)-\Delta c(t_n)y_2(t_n))+\\&x_2(t_n)z_2(t_n))|\\=&|(c(t_n)-a(t_n))e_x(t_n)+c(t_n)e_y(t_n)+\Delta c(t_n)(x_2(t_n)-\\&y_2(t_n))+\Delta a(t_n)x_2(t_n)+e_x(t_n)e_z(t_n)+x_2(t_n)e_x(t_n)+\\&z_2(t_n)e_x(t_n)|\\<&|(c(t_n)-a(t_n))e_x(t_n)|+|c(t_n)e_y(t_n)|+|\Delta c(t_n)\times\\&(x_2(t_n)-y_2(t_n))|+|\Delta a(t_n)x_2(t_n)|+|e_x(t_n)e_z(t_n)|+\\&|x_2(t_n)e_x(t_n)|+|z_2(t_n)e_z(t_n)|\end{aligned}\tag{13}$$

$$\begin{aligned}&|\dot{e}_z(t_n)|\\=&|b(t_n)e_z(t_n)+x_1(t_n)y_1(t_n)-(\Delta b(t_n)z_2(t_n)\\&+x_2(t_n)y_2(t_n))|\\=&|b(t_n)e_z(t_n)-\Delta b(t_n)z_2(t_n)+(x_2(t_n)+e_x(t_n))\times\\&(y_2(t_n)+e_y(t_n))-x_2(t_n)y_2(t_n)|\\<&|b(t_n)e_z(t_n)|+|\Delta b(t_n)z_2(t_n)|+|x_2(t_n)e_y(t_n)|+\\&|y_2(t_n)e_x(t_n)|+|e_x(t_n)e_y(t_n)|\end{aligned}\tag{14}$$

From (12), (13) and (14), we can derive the following inequation

$$\begin{aligned}&\Delta_1\cdot(|\dot{e}_x(t_n)|+|\dot{e}_y(t_n)|+|\dot{e}_z(t_n)|)-b_1|e_x(t_n)|-\\&b_2|e_y(t_n)|-b_3|e_z(t_n)|\\<&|e_x(t_n)|(\Delta_1((1+\xi_c)c+\sup(x_2(t_n))+\sup(y_2(t_n)))-b_1)\\&+|e_y(t_n)|(\Delta_1((1+\xi_a)a+(1+\xi_c)c+\sup(x_2(t_n))+\\&\sup(e_x(t_n)))-b_2)+|e_z(t_n)|(\Delta_1((1+\xi_b)b+\sup(e_x(t_n))\\&+\sup(x_2(t_n)))-b_3)+\Delta_1((2a\cdot\xi_a+2c\cdot\xi_c)\sup(x_2(t_n)\\&-y_2(t_n))+2a\cdot\xi_a\sup(x_2(t_n))+2b\cdot\xi_b\cdot\sup(z_2(t_n)))\\<&(|e_x(t_n)|+|e_y(t_n)|+|e_z(t_n)|)\times\\&(\Delta_1\cdot\max(m(\xi_c),n(\xi_a,\xi_c),o(\xi_b))-\min(b_1,b_2,b_3))+\\&\Delta\cdot C(\xi_a,\xi_b,\xi_c)\end{aligned}\tag{15}$$

Thus we have

$$\begin{aligned}&(|e_x(t_n)|+|e_x(t_n)|+|e_x(t_n)|)\times\\&(\Delta_1\cdot\max(m(\xi_c),n(\xi_a,\xi_c),o(\xi_b))-\min(b_1,b_2,b_3))+\\&\Delta\cdot C(\xi_a,\xi_b,\xi_c)<0\end{aligned}$$

only if

$$\Delta_1\cdot\max(m(\xi_c),n(\xi_a,\xi_c),o(\xi_b))-\min(b_1,b_2,b_3)<0$$

Hence

$$\begin{aligned}&(|e_x(t_n)|+|e_x(t_n)|+|e_x(t_n)|)\times\\&(\Delta_1\cdot\max(m(\xi_c),n(\xi_a,\xi_c),o(\xi_b))-\min(b_1,b_2,b_3))+\\&\Delta_1\cdot C(\xi_a,\xi_b,\xi_c)\\<&\varepsilon\cdot(\Delta_1\cdot\max(m(\xi_c),n(\xi_a,\xi_c))-\min(b_1,b_2,b_3))+\\&\Delta_1\cdot C(\xi_a,\xi_b,\xi_c)\end{aligned}\tag{16}$$

If the length of the impulse interval satisfied

$$\Delta_1<\frac{\min(b_1,b_2,b_3)}{\max(m(\xi_c),n(\xi_a,\xi_c))+C(\xi_a,\xi_b,\xi_c)/\varepsilon}$$

Based on the inequation (16), we have

$$\begin{aligned}&(|e_x(t_{n+1})|-|e_x(t_n)|)+(|e_y(t_{n+1})|-|e_y(t_n)|)+\\&(|e_y(t_{n+1})|-|e_y(t_n)|)<0\end{aligned}$$

Thus condition 1 of Lemma 2 is satisfied.

Case 1: $\|e(t_1)\|<\varepsilon$

Based on (10) and (12), (13), (14), we have

$$\begin{aligned}&|e_x(t_{n+1})|+|e_y(t_{n+1})|+|e_z(t_{n+1})|\\=&(1-b_1)|e_x(t_n)|+(1-b_2)|e_y(t_n)|+(1-b_3)|e_z(t_n)|+\\&\Delta_1(|\dot{e}_x(t_n)|+|\dot{e}_y(t_n)|+|\dot{e}_z(t_n)|)\\<&(1-\min(b_1,b_2,b_3)+\Delta_1\cdot\max(m(\xi_c),n(\xi_a,\xi_c)))\cdot\varepsilon+\\&\Delta_1\cdot C(\xi_a,\xi_b,\xi_c)\end{aligned}\tag{17}$$

Hence, if the length of the impulse interval satisfied

$$\Delta_1<\frac{\min(b_1,b_2,b_3)}{\max(m(\xi_c),n(\xi_a,\xi_c))+C(\xi_a,\xi_b,\xi_c)/\varepsilon}$$

Based on the inequation (17), we have

$$|e_x(t_{n+1})|+|e_y(t_{n+1})|+|e_z(t_{n+1})|<\varepsilon$$

Thus condition 2 of Lemma 2 is satisfied.

In summary, if the length of the impulsive interval satisfied:

$$\Delta_1<\frac{\min(b_1,b_2,b_3)}{\max(m(\xi_c),n(\xi_a,\xi_c))+C(\xi_a,\xi_b,\xi_c)/\varepsilon}$$

then $\exists t_0$, such that $\forall t>t_0$, $|e_x(t)|<\varepsilon$, $|e_y(t)|<\varepsilon$, $|e_z(t)|<\varepsilon$.

Remark 1: *Theorem* 1 provides a guideline to choose the length of the impulsive interval while the two chaotic system are parametric uncertain and mismatch. According to *Theorem* 1, it is obvious that the more uncertain the system parameters and/or the more the mismatch is, the shorter the impulsive intervals should be designed and the larger the chosen bound, the longer the impulsive intervals can be allowed. Furthermore, we note that the upper bound of the length of the impulsive intervals is also affected by the impulse synchronization coefficient. The bigger $\min(b_1,b_2,b_3)$ is the longer the impulsive intervals can be allowed.

Remark 2: *Theorem* 1 only corresponds to Chen system. However, the method can also be used while the driving and driven systems are other chaotic systems. *Theorem* 2 provides a guideline to choose the length of the impulsive interval while the driving and driven systems are both Chua system as shown below:

$$\begin{cases} \dot{x} = \alpha \cdot (y - x - h(x)) \\ \dot{y} = x - y + z \\ \dot{z} = -\beta \cdot y - \gamma \cdot z \end{cases}$$

$$h(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x-1|)$$

Theorem 2: While the driving and driven systems are both Chua system, if the length of the impulsive interval satisfied:

$$\Delta_1 < min(b_1, b_2, b_3)/(\max(m'(\xi_\alpha, \xi_b), n'(\xi_\alpha, \xi_\beta), o'(\xi_\gamma))+ C'(\xi_\alpha, \xi_\beta, \xi_\gamma, \xi_a, \xi_b)/\varepsilon) \tag{18}$$

where

$$\begin{aligned} &m'(\xi_\alpha, \xi_b) = 1 + |(1+\xi_\alpha)\cdot\alpha| + |(1+\xi_\alpha)(1+\xi_b)\cdot a \cdot b| \\ &n'(\xi_\alpha, \xi_\beta) = 1 + |(1+\xi_\alpha)\cdot\alpha| + |(1+\xi_\beta)\cdot\beta| \\ &o'(\xi_\gamma) = 1 + |(1+\xi_\gamma)\cdot\gamma| \\ &C'(\xi_\alpha, \xi_\beta, \xi_\gamma, \xi_a, \xi_b) = |2a\cdot\xi_\alpha \cdot \sup(y_2(t) - x_2(t))| + \\ &|2(\xi_\alpha + \xi_b + \xi_\alpha\cdot\xi_b)\cdot\alpha\cdot b\cdot\sup(x_2(t))| + \\ &|2\alpha(1+\xi_\alpha)((1+\xi_a)\cdot a - (1-\xi_b)\cdot b)| + \\ &|2\xi_\beta\cdot\beta\cdot\sup(y_2(t))| + |2\xi_\gamma\cdot\gamma\cdot\sup(z_2(t))| \end{aligned} \tag{19}$$

then $\exists t_0$, such that $\forall t > t_0$

$$|e_x(t)| < \varepsilon, |e_y(t)| < \varepsilon, |e_z(t)| < \varepsilon$$

IV. Illustrative Examples

In this section, we conduct simulation studies on both Chen model and Chua model to illustrate the effectiveness of the scheme proposed.

Example 1: Suppose that the Chen model at the driving system and the driven system are perturbed with parametric uncertainties. The parameters are set as follows.

$$\begin{aligned} \Delta a_1(t) = \xi_a\cdot a\cdot\sin(t), \Delta a_2(t) = \xi_a\cdot a\cdot\cos(t) \\ \Delta b_1(t) = \xi_b\cdot b\cdot\sin(t), \Delta b_2(t) = \xi_b\cdot b\cdot\cos(t) \\ \Delta c_1(t) = \xi_c\cdot c\cdot\sin(t), \Delta c_2(t) = \xi_c\cdot c\cdot\cos(t) \end{aligned}$$

$$\mathbf{A} = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0.9 & & \\ & 0.8 & \\ & & 0.7 \end{pmatrix}$$

The synchronization threshold $\varepsilon = 0.01$ and the the error coefficient $\xi_a = \xi_b = \xi_c = 0.05$.

According to the result shown in [18], Chen system is bounded. There exist $0 < \eta < 1$ such that

$$\eta^4 + \frac{2(b+c)}{c}\eta^3 + \frac{2(b-c)}{c}\eta - 1 = 0$$

and we have

$$x^2 + y^2 + (z-c)^2 \le R^2 \tag{20}$$

where x, y, z are the state of Chen system and

$$R^2 = \frac{(a+c)(a-c)^2(1+\eta)^2}{16a\eta^2}(\frac{b}{c}(1-\eta)^2 + (1+\eta)^2) + \frac{4(a^2-c^2)(a^2+c^2-bc)+c^2(2a-b)^2}{4a(b+c)}$$

Thus

$$\begin{aligned} &min(b_1, b_2, b_3) = 0.7 \\ &max(m(\xi_c), n(\xi_a, \xi_c), o(\xi_b)) = 190.2426 \\ &C(\xi_a, \xi_b, \xi_c) = 531.8080 \end{aligned} \tag{21}$$

According to *Theorem* 1,

$$\Delta_1 < 1.3116 \times 10^{-5}$$

We choose $\Delta_1 = 1.2 \times 10^{-5}$ as the impulse interval in the simulation. The simulation results are illustrated in Fig.2. As it shown in Fig.2, the error between two systems is decreasing visibly while the error is bigger than the prescribed synchronization threshold. When the error is smaller than the synchronization threshold, the error keeps oscillating within the prescribed bound.

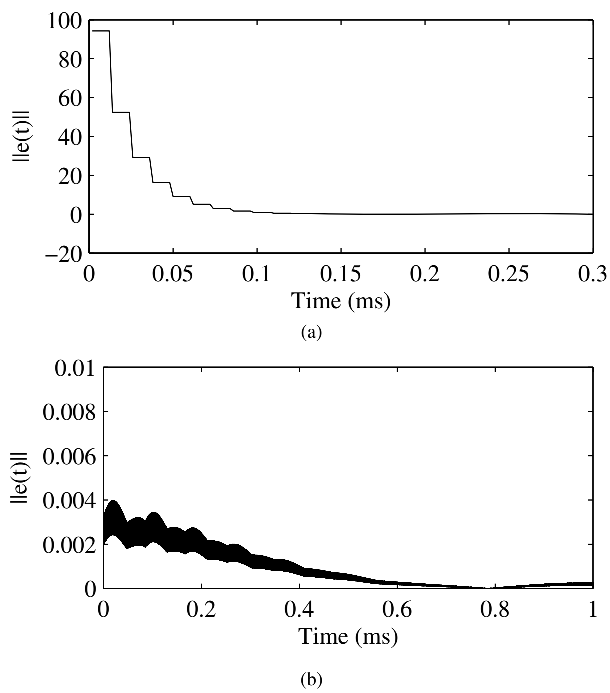


Fig.2 Simulation results for Chen system (a) The decreasing process of $\|e(t)\|$ when the systems are synchronized. (b) The oscillation process of $\|e(t)\|$ when the systems are synchronized.

Example 2: Similarly, we use Chua model at the driving system and the driving system, which are perturbed with parametric uncertainties, to demonstrate *Theorem* 2. The systems are set as below.

$$\begin{aligned} \alpha = 15, \Delta\alpha_1(t) = \xi_\alpha\cdot\alpha\cdot R_1(t), \Delta\alpha_2(t) = \xi_\alpha\cdot\alpha\cdot R_2(t) \\ \beta = 20, \Delta\beta_1(t) = \xi_\beta\cdot\beta\cdot R_3(t), \Delta\beta_2(t) = \xi_\beta\cdot\beta\cdot R_4(t) \\ \gamma = 0.5, \Delta\gamma_1(t) = \xi_\gamma\cdot\gamma\cdot R_5(t), \Delta\gamma_2(t) = \xi_\gamma\cdot\gamma\cdot R_6(t) \\ a = -8/7, \Delta a_1(t) = \xi_a\cdot a\cdot R_7(t), \Delta a_2(t) = \xi_a\cdot a\cdot R_8(t) \\ b = -5/7, \Delta b_1(t) = \xi_b\cdot b\cdot R_9(t), \Delta b_2(t) = \xi_b\cdot b\cdot R_{10}(t) \end{aligned}$$

$$\mathbf{B} = \begin{pmatrix} 0.9 & & \\ & 0.8 & \\ & & 0.7 \end{pmatrix}$$

Where $R_i(t), i = 1, 2, \ldots, 10$ are different uniform random functions within the same bound of $[-1, 1]$. The synchronization threshold $\varepsilon = 0.01$ and the the error coefficient $\xi_\alpha = \xi_\beta = \xi_\gamma = \xi_a = \xi_b = 0.05$.

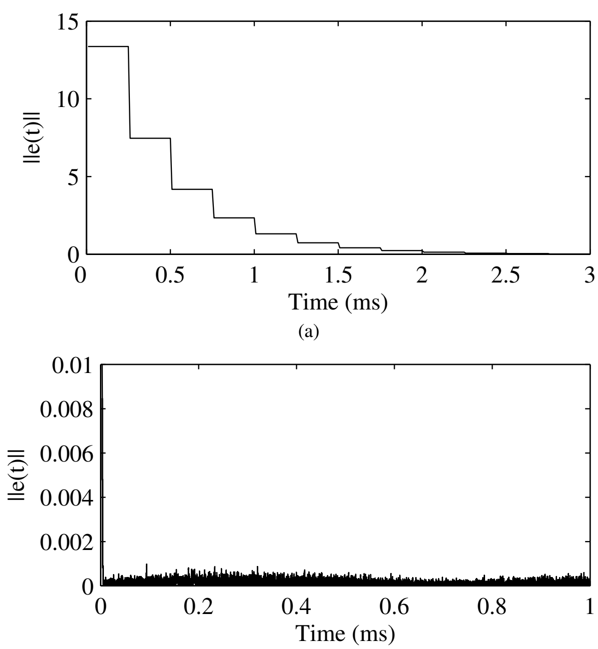
Thus we have

$$min(b_1, b_2, b_3) = 0.7$$
$$max(m'(\xi_\alpha, \xi_b), n'(\xi_\alpha, \xi_\beta), o'(\xi_\gamma)) = 37.75$$
$$C'(\xi_\alpha, \xi_\beta, \xi_\gamma, \xi_a, \xi_b) = 26.6411$$

Based on *Theorem* 2, the impulsive interval should satisfy

$$\Delta_1 < 2.5908 \times 10^{-4}$$

We choose $\Delta_1 = 2.5 \times 10^{-4}$ as the impulse interval in the simulation. The simulation results are illustrated in Fig.3.



(b)

Fig.3 Simulation results for Chua system (a) The decreasing process of $\|e(t)\|$ when the systems are synchronized. (b)The oscillation process of $\|e(t)\|$ when the systems are synchronized.

V. Conclusion

In this paper, we present a method to analyze the stability and robustness of pure impulsive control of two chaotic dynamical systems with parametric uncertainties and mismatch. By analyzing the oscillation process of the error between the driving system and the driven system, we derive a quantitative condition, which provides guideline in designing a chaotic impulsive synchronization scheme. The condition revealed the relation between the prescribed bound, the length of impulse interval, the bounds of uncertainties and mismatch and the pulse intensity. The more uncertain the system parameters and/or the more the mismatch, the shorter the impulsive intervals should be designed and the larger the chosen bound, the longer the impulsive intervals can be allowed. The upper bound of the length of the impulsive intervals is also determined by the pulse intensity. The more intensity of the impulse is, the longer the impulsive intervals can be allowed. Simulation results on Chen system and Chua system verify the effectiveness of this method.

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