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A Decomposition Method for Optimal Firing Sequence Problems for First-order Hybrid Petri Nets

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***Abstract*—In this paper, we propose a general decomposition method for transition firing sequence problems for first order hybrid Petri Nets. The optimal transition firing sequence problem for first-order hybrid Petri Nets is formulated as a mixed integer programming problem. We propose a Lagrangian relaxation method for solving optimal transition firing sequence problems. The hybrid Petri Net is decomposed into several subnets in which the optimal firing sequence for each subnet is easily solved. The optimality of solution can be evaluated by duality gap derived by Lagrangian relaxation method. The proposed method is applied to a small-scale example. Computational experiments demonstrate the validity of the proposed formulation.**

***Index Terms*—first-order hybrid Petri Nets, optimization, Lagrangian relaxation, firing sequence problem, decomposition**

I. INTRODUCTION

Petri Nets have been used as a mathematical modeling tool for optimization problems with a wide variety of applications. It is possible to set up state equations, algebraic equations and other mathematical models governing the behavior of the systems. In order to use Petri Nets as a general optimization tool, it is required to develop the optimization algorithms for general purpose. Recently, hybrid system modeling techniques are promising for simulation, control, and optimization[6][7][8]. First-order hybrid Petri Net (FOHPN) is the formalism with discrete and continuous dynamics described by Alla and David[1][9] with the addition of algebraic analysis techniques that was first presented by Balduzzi et al.[2][3][4]. FOHPN is widely applied to distributed manufacturing systems[10], urban traffic control[11][12] etc. The objective of this paper is to propose a decomposition method for optimal firing sequence problems for FOHPN to avoid state explosion. A decomposition method for optimal firing sequence problems for timed Petri Nets have been studied[15][17][18][19]. For the simulation of FOHPN, linear programing technique is used to find admissible firing speed vector for each macro-period[4][5]. However, the optimization of firing vector and firing speeds for entire time horizon has not been considered in previous works. In this paper, we study the optimization method for firing sequence problem for FOHPN to determine a feasible firing vector and firing speed for total planning horizon to minimize cost function. The Petri Net decomposition approach with Lagrangian relaxation is used to derive a near-optimal solution effectively. The effectiveness of the proposed method is demonstrated for example problem. The paper consists of the following sections. Section II introduces FOHPN. Section III defines optimal firing sequence problem for FOHPN and Petri Net decomposition procedure is explained. Section V introduces Lagrangian relaxation to solve the problem efficiently. Section VI demonstrates the effectiveness of the proposed method. Section VII concludes the paper.

II. FIRST-ORDER HYBRID PETRI NETS

A. Definition of FOHPN

$\mathbb{N}$ is set of natural numbers, $\mathbb{R}$ is set of real numbers, $\mathbb{R}^+$ is non-negative real numbers, $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, $\mathbb{R}_\infty^+ = \mathbb{R}^+ \cup \{\infty\}$. For a set of A, B, and C. A^B represents a set of vectors of A with its dimension is $|B|$. $A^{B\times C}$ represents a set of matrices of A whose dimension is $|B| \times |C|$. A^{T} is transpose of A. The first-order hybrid Petri Net (FOHPN) is defined as $N = (P, T, \mathrm{Pre}, \mathrm{Post}, \mathcal{D}, \mathcal{C})$. $P = P_d \cup P_c$ is a finite set of places, P_d is a finite set of discrete places, P_c is a finite set of continuous places. $T = T_d \cup T_c$ is a finite set of transitions, T_d is a finite set of discrete transitions, and T_c is a finite set of continuous transitions. T_d is further partitioned into a set of immediate transitions T_I, a set of deterministic timed transitions T_D and a set of stochastic timed transitions T_E. We denote ${}^\bullet t$ as a set of input places for transition t, and $t^\bullet$ as a set of output places for transition t. A set of discrete places of ${}^\bullet t$ is denoted as ${}^{(d)}t =^\bullet t \cap P_d$, a set of continuous places is represented as ${}^{(c)}t =^\bullet t \cap P_c$. The pre- and post-incidence functions that specify the arcs are:

$$\mathrm{Pre}: \begin{cases} P_d \times T \to \mathbb{N} \\ P_c \times T \to \mathbb{R}_0^+ \end{cases} \tag{1}$$

$$\mathrm{Post}: \begin{cases} P_d \times T \to \mathbb{N} \\ P_c \times T \to \mathbb{R}_0^+. \end{cases} \tag{2}$$

$\mathcal{D} : T_t \to \mathbb{R}^+$ is associated with firing duration time for discrete transitions. For $t_i \in T_D$, $\delta_i = \mathcal{D}(t_i)$ is firing duration time. For $t_i \in T_E$, $\lambda_i = \mathcal{D}(t_i)$ is the average firing rate, i.e. the average firing delay is $1/\lambda_i$, where λ_i is the parameter of the corresponding exponential distribution. $\mathcal{C} : T_c \to \mathbb{R}_0^+ \times \mathbb{R}_\infty^+$ specifies the firing speeds associated with continuous transitions. $\mathcal{C}(t_i) = (V_i', V_i)(V_i' < V_i)$ for any continuous transition $t_j \in T_c$. V_i' represents the minimum firing speed (mfs) and V_j represents the maximum firing speed (MFS). In the following, unless explicitly specified, the mfs of a continuous transition t_i is $V_i' = 0$. The incidence matrices for FOHPN is defined

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as $A = A^{+} - A^{-}$ where $(A^{+})_{ij} = \mathrm{Post}(p_i, t_j)$, $A^{-}_{ij} = \mathrm{Pre}(p_i, t_j)$. Especially, $A_{XY} = \mathrm{Pre}(p_i, t_j) - \mathrm{Post}(p_i, t_j)$ for $p_i \in P_Y, t_j \in T_X, X, Y \in \{c, d\}$). It is assumed that $A_{dc} = 0$. A marking

$$\boldsymbol{m} : \begin{cases} P_d \to \mathbb{N} \\ P_c \to \mathbb{R}_0^{+} \end{cases} \tag{3}$$

is a function that assigns to each discrete place a non-negative integer number of tokens, represented by black dots, and assigns to each continuous place a fluid volume; m_p denotes the marking of place p. The value of the marking at time τ is denoted as $\boldsymbol{m}(\tau)$. The restriction of m to P_d, and P_c is denoted as $\boldsymbol{m}^d, \boldsymbol{m}^c$, respectively.

B. Marking and enabling

The enabling of a discrete transition depends on the marking of all its input places both discrete and continuous. A discrete transition t is enabled for all $p \in^{\bullet} t,\;\; m_p \geq \mathrm{Pre}(p, t)$. The firing of a discrete transition t yields the marking $\boldsymbol{m'} = \boldsymbol{m} + A(p, t)$ $(\forall p \in P)$. A continuous transition is enabled only by the marking of input discrete places. A continuous transition t is enabled for all $p \in^{(d)} t$, $m_p \geq \mathrm{Pre}(p, t)$, however, it is used to distinguish between strongly and weakly enabling. We say that an enabled transition $t \in T_c$ is:

- strongly enabled for all places $p \in^{(c)} t$, $m_p \geq 0$;
- weakly enabled for some $\bar{p} \in^{(c)} t$, $m_{\bar{p}} = 0$.

The instantaneous firing speed (IFS) of a continuous transition $t_i \in T_c$ be denoted by v_i. The enabling state of a continuous transition t_i defines its admissible IFS v_j.

- If t_i is not enabled then $v_i = 0$.
- If t_i is strongly enabled, then it may fire with any firing speed $v_i \in [V_i', V_i]$.
- If t_i is weakly enabled, then it may fire with any firing speed $v_i \in [V_i', \bar{V}_i]$, where the upper bound $\overline{V}_j$ on the firing speed $\bar{V}_i \leq V_i$.

Let the IFS of a continuous transition $t_i \in T_c$ at time τ is $v_i(\tau)$. The marking evolution in time of a place $p \in P_c$ can be written as

$$\frac{dm_p(\tau)}{d\tau} = \sum_{t_i \in T_c} A(p, t_i) \cdot v_i(\tau) \tag{4}$$

Equation (4) holds assuming that at time τ no discrete transition is fired and that all speeds $v_j(\tau)$ are continuous in τ.

C. Marking and transition

A macro-event occurs (a) when a discrete transition fires, thus changing the discrete marking and enabling (or disabling) a continuous transition, or (b) a continuous place becomes empty, thus changing the enabling state of a continuous transition from strong to weak. Let τ_k and τ_{k+1} be the occurrence times of two consecutive macro-events as defined above. It is assumed that within the interval of time $[\tau_k, \tau_{k+1})$, we call macro-period k, $\Delta_k = \tau_{k+1} - \tau_k$ is constant and the IFS vector is constant during macro-period k. τ_0 is an initial time, $\tau_k (k > 0)$ is the occurrence time of macro-event, and $\boldsymbol{v}(\tau_k)$ is the IFS vector during Δ_k periods of macro-period k. The firing of a discrete transition yields the marking $\boldsymbol{m}(\tau_k)$ for each place k. Thus the behavior of $FOHPN$ can be written as

$$\begin{cases} \boldsymbol{m}^c(\tau_k) = \boldsymbol{m}^c(\tau_k^-) + A_{cd}^{\mathrm{T}} \cdot \boldsymbol{\sigma}(\tau_k) \\ \boldsymbol{m}^d(\tau_k) = \boldsymbol{m}^d(\tau_k^-) + A_{dd}^{\mathrm{T}} \cdot \boldsymbol{\sigma}(\tau_k) \end{cases} \tag{5}$$

where $\sigma(\tau_k)$ is the firing vector associated with the firing of time τ_k.

The continuous behavior of an $FOHPN$ for $\tau \in [\tau_k, \tau_{k+1})$ is described by

$$\begin{cases} \boldsymbol{m}^c(\tau) = \boldsymbol{m}^c(\tau_k) + A_{cc}^{\mathrm{T}} \cdot \boldsymbol{v}(\tau_k) \cdot (\tau - \tau_k) \\ \boldsymbol{m}^d(\tau) = \boldsymbol{m}^d(\tau_k). \end{cases} \tag{6}$$

In order to define the optimal transition firing problem for $FOHPN$, we consider that the time can be discretized into several time periods where the duration of unit time for continuous transition is ϵ.

A firing vector $\sigma_\tau : T \to \{0, 1\}$ represents whether transition at time $\tau \in \mathbb{R}^+$ is fired or not. A firing vector $\sigma'_\tau : T \to \{0, 1\}$ represents whether the firing duration is completed or not at time τ whose elements takes 1 if the firing duration is completed and 0 otherwise. A marking $M_\tau : \begin{cases} P_d \to \mathbb{N} \\ P_c \to \mathbb{R}_0^{+} \end{cases}$ represents the vector comprising the number of tokens on each place at time τ. The firing vector $v_\tau : T_c \to \mathbb{R}_0^{+}$ is the IFS vector for continuous transition at time τ. These equations are denoted by column vectors as follows.

$$(M_{d,\tau})_i = M_{d,\tau}(p_{d,i}) \quad (\forall p_{d,i} \in P_d) \tag{7}$$
$$(M_{c,\tau})_i = M_{c,\tau}(p_{c,i}) \quad (\forall p_{c,i} \in P_c) \tag{8}$$
$$(\sigma_{d,\tau})_j = \sigma_{d,\tau}(t_{d,j}) \quad (\forall t_{d,j} \in T_d) \tag{9}$$
$$(\sigma_{c,\tau})_j = \sigma_{c,\tau}(t_{c,j}) \quad (\forall t_{c,j} \in T_c) \tag{10}$$
$$(v_\tau)_j = v_\tau(t_{c,j}) \quad (\forall t_{c,j} \in T_c) \tag{11}$$

For each continuous transition $t_j \in T_c$, the minimum and maximum firing speed is given as $v(t_j) \in [V_j', V_j]$.

III. Optimal firing sequence problem for FOHPN and Petri Net decomposition

In this section, we define the optimal firing sequence problem for $FOHPN$. A Petri Net decomposition approach using Lagrangian relaxation for $FOHPN$ is developed.

A. Definition of optimal firing sequence problem

For an $FOHPN$, we denote the problem to determine the firing sequence of transitions to minimize J as an optimal firing sequence problem defined as Given $N = (P, T, \mathrm{Pre}, \mathrm{Post}, \mathcal{D}, \mathcal{C})$, initial marking $M_0(p)$, final marking $M_f(p)$

$$M_0 : \begin{cases} P_d \to \mathbb{N} \\ P_c \to \mathbb{R}_0^{+} \end{cases}, \quad M_f : \begin{cases} P_d \to \mathbb{N} \\ P_c \to \mathbb{R}_0^{+} \end{cases} \tag{12}$$

, time horizon $N_T \in \mathbb{N} - \{0\}$, and the cost function to be minimized $J : (\{0,1\}^T)^{N_T} \to \mathbb{R}$, the problem is to find an feasible set of firing vectors $(\sigma_{d,0}, \sigma_{d,1}, \ldots, \sigma_{d,N_T-1}, \sigma_{c,0}, \sigma_{c,1}, \ldots, \sigma_{c,N_T-1}) \in$

$(\{0,1\}^{T_d \cup T_c})^{N_T}$ and the set of firing speed vectors $(v_0, v_1, \ldots, v_{N_T-1})$ satisfying $M_{N_T}(p) = M_f(p)(\forall p \in P_c \cup P_d)$ to minimize the cost function J. To formulate the problem, the total time horizon N_T is divided into N_t steps with an interval ϵ. The problem can be formulated as:

$$\min\ J = \sum_{k=0}^{N_T} \sum_{\tau \in \mathcal{T}_k} \epsilon \delta_\tau \tag{13}$$

$$\text{s. t.}\quad M_{d,\tau+1} = M_{d,\tau} + (A_{dd}^+)^{\mathrm{T}} \sigma'_{d,n} - (A_{dd}^-)^{\mathrm{T}} \sigma_{d,\tau} \tag{14}$$

$$M_{c,\tau+1} = M_{c,\tau} + (A_{cd}^+)^{\mathrm{T}} \sigma'_{d,\tau} - (A_{cd}^-)^{\mathrm{T}} \sigma_{d,\tau} + (A_{cc})^{\mathrm{T}} v_\tau \epsilon \tag{15}$$

$$M_{d,\tau} - (A_{dd}^-)^{\mathrm{T}} \sigma_{d,\tau} - (A_{cd}^-)^{\mathrm{T}} \sigma_{c,\tau} \geq 0 \tag{16}$$

$$M_{c,\tau} - (A_{dd}^-)^{\mathrm{T}} \sigma_{d,\tau} \geq 0 \tag{17}$$

$$V'(t)\sigma_{c,\tau}(t) \leq v_\tau(t) \leq V(t)\sigma_{c,\tau}(t) \tag{18}$$

$$v_{\tau'} = v_\tau \quad (\forall \tau', \tau \in \mathcal{T}_k, \forall k = 0, \ldots, N_T) \tag{19}$$

$$\delta_\tau = \begin{cases} 1 & \{(M_{d,\tau} \neq M_{d,f}) \vee (M_{c,\tau} \neq M_{c,f})\} \\ & \wedge(\delta_{\tau_{k-1}} = 1) \\ 0 & \{(M_{d,\tau} = M_{d,f}) \wedge (M_{c,\tau} = M_{c,f})\} \\ & \vee(\delta_{\tau-1} = 0) \end{cases} \tag{20}$$

$$(\forall \tau \in \mathcal{T}_k, \forall k = 0, \ldots, N_T).$$

(13) is the objective function of total transition time where the time counting variable δ_τ defined as (20) indicates whether the marking is the final marking or not at time τ. δ_τ takes 0 if the marking is the final marking, and 1 otherwise, then the total sum of δ_τ is the total transition time from initial marking to final marking. (14) is the marking evolution constraints for discrete places and (15) is the marking evolution constraints for continuous places. (16) and (17) ensure the enabling condition for discrete transitions, continuous transitions, respectively. (18) and (19) defines the IFS firing speed vector. $\mathcal{T}_k$ is the set of discretized time periods in macro-period k.

IV. Decomposition of FOHPN and Lagrangian relaxation

It is extremely difficult to solve firing sequence problem for $FOHPN$ optimally when the number of transitions are increased. In order to solve the problem efficiently, we propose a decomposition method for $FOHPN$ by using Lagrangian relaxation. In the proposed method, Petri Net is decomposed into several independent subnets by the idea of place duplication[18][19]. The shared places are removed and the entire Petri Net is decomposed into independent subnets. The subproblem is defined as the subnets and shared places with appropriate connection. The subproblem for each decomposed subnets are solved repeatedly until the firing condition for the entire subnets is satisfied.

A. Decomposition procedure

In order to use Petri Net decomposition approach, the following condition must be satisfied.

- The objective function J is represented as $J = \sum_{i=1}^{m} J_i$, the sum of the objective function J_i for the decomposed subnets.
- The final marking is not specified for the duplicated places for decomposition.

The decomposition procedure is explained here. The set of transitions are decomposed into the set of transitions for each subnet i. The set of transitions T_i is defined as (21).

$$T = \bigcup_{i=1}^{m} T_i \quad (T_i \cap T_j = 0,\ i \neq j) \tag{21}$$

Then the set of places for subnet i is defined as:

$$P_i = \{p \mid p^\bullet \subseteq T_i,\ {}^\bullet p \subseteq T_i\} \tag{22}$$

P_R is the set of places that satisfies (23).

$$P_R = P \backslash \bigcup_{i=1}^{m} P_i = \emptyset \quad (i \neq j) \tag{23}$$

The column vector with the elements of $\sigma,\ v,\ \mathcal{D},\ \mathcal{C}$ for $t \in T_i$ is denoted as $\sigma_i,\ v_i,\ \mathcal{D}_i,\ \mathcal{C}_i$, respectively. The column vector with the elements of $p \in P_i, p \in P_R$ is denoted as M_i, M_R, respectively. The incidence matrices for $p \in P_i$ are $A_i^+ \in \mathbb{N}^{T_i \times P_i}$, and $A_i^- \in \mathbb{N}^{T_i \times P_i}$, and the incidence matrices for $p \in P_R$ are $B^+ \in \mathbb{N}^{T \times P_R}$, and $B^- \in \mathbb{N}^{T \times P_R}$ where B^+, B^- is defined as:

$$\begin{aligned} B^+ &= [(B_1^+)^{\mathrm{T}}, \ldots, (B_m^+)^{\mathrm{T}}]^{\mathrm{T}} \quad (B_i^+ \in \mathbb{N}^{T_i \times P_R}) \\ B^- &= [(B_1^-)^{\mathrm{T}}, \ldots, (B_m^-)^{\mathrm{T}}]^{\mathrm{T}} \quad (B_i^- \in \mathbb{N}^{T_i \times P_R}). \end{aligned}$$

By using the decomposition procedure, the equations for marking evolution can be given by (24) $\sim$ (27), the enabling conditions are represented by (28) $\sim$ (31).

$$M_{i,d,\tau+1} = M_{i,d,\tau} + (A_{i,dd}^+)^{\mathrm{T}} \sigma'_{i,d,\tau} - (A_{i,dd}^-)^{\mathrm{T}} \sigma_{i,d,\tau} \tag{24}$$

$$M_{i,c,\tau+1} = M_{i,c,\tau} + (A_{i,dc}^+)^{\mathrm{T}} \sigma'_{i,d,\tau} - (A_{i,dc}^-)^{\mathrm{T}} \sigma_{i,d,\tau} + (A_{i,cc})^{\mathrm{T}} v_{i,\tau} \cdot \epsilon \tag{25}$$

$$M_{R,d,\tau+1} = M_{R,d,\tau} + (B_{dd}^+)^{\mathrm{T}} \sigma'_{d,\tau} - (B_{dd}^-)^{\mathrm{T}} \sigma_{d,\tau} \tag{26}$$

$$M_{R,c,\tau+1} = M_{R,c,\tau} + (B_{dc}^+)^{\mathrm{T}} \sigma'_{d,\tau} - (B_{dc}^-)^{\mathrm{T}} \sigma_{d,\tau} + (B_{cc})^{\mathrm{T}} v_\tau \cdot \epsilon \tag{27}$$

$$M_{i,d,\tau} - (A_{i,dd}^-)^{\mathrm{T}} \sigma_{i,d,\tau} - (A_{i,cd}^-)^{\mathrm{T}} \sigma_{i,c,\tau} \geq 0 \tag{28}$$

$$M_{i,c,\tau} - (A_{i,dd}^-)^{\mathrm{T}} \sigma_{i,d,\tau} \geq 0 \tag{29}$$

$$M_{R,d,\tau} - (B_{dd}^-)^{\mathrm{T}} \sigma_{d,\tau} - (B_{cd}^-)^{\mathrm{T}} \sigma_{c,\tau} \geq 0 \tag{30}$$

$$M_{R,c,\tau} - (B_{dd}^-)^{\mathrm{T}} \sigma_{d,\tau} \geq 0 \tag{31}$$

The optimization problem can be rewritten as:

$$\begin{aligned} &\min J = \sum_{i=1}^{m} J_i \\ &\text{s. t.} \quad (24) \sim (31). \end{aligned}$$

Based on the decomposition for the set of transitions and the set of places, the subproblem for each subnet i can be decomposed into $N_i = (P_i \cup P_{Ri}, T_i, \mathrm{Pre}_i, \mathrm{Post}_i, \mathcal{C}_i, \mathcal{D}_i)$. Here, P_{Ri} is the set of duplicated places which has one-to-one relationship between P_R. The weight on arc can be defined as follows by using bijection $\zeta : P_{Ri} \rightarrow P_R$.

$$\text{Post}_i(p,t) = \text{Post}(p,t) \quad (\forall p \in P_i;\ \forall t \in T_i) \tag{32}$$
$$\text{Pre}_i(p,t) = \text{Pre}(p,t) \quad (\forall p \in P_i;\ \forall t \in T_i) \tag{33}$$
$$\text{Post}_i(p,t) = \text{Post}(\zeta(p),t) \quad (\forall p \in P_{Ri};\ \forall t \in T_i) \tag{34}$$
$$\text{Pre}_i(p,t) = \text{Pre}(\zeta(p),t) \quad (\forall p \in P_{Ri};\ \forall t \in T_i) \tag{35}$$

The marking evolution for $M_{Ri,k}$ and the enabling condition can be represented as (36) $\sim$ (39).

$$M_{Ri,d,\tau+1} = M_{Ri,d,\tau} + (B^+_{i,dd})^{\mathrm{T}}\sigma'_{i,d,\tau} - (B^-_{i,dd})^{\mathrm{T}}\sigma_{i,d,\tau} \tag{36}$$
$$M_{Ri,c,\tau+1} = M_{Ri,c,\tau} + (B^+_{i,dc})^{\mathrm{T}}\sigma'_{i,d,\tau} - (B^-_{i,dc})^{\mathrm{T}}\sigma_{i,d,\tau} + (B^+_{i,cc})^{\mathrm{T}}v_{i,\tau}\cdot\epsilon \tag{37}$$
$$M_{Ri,d,\tau} - (B^-_{i,dd})^{\mathrm{T}}\sigma_{i,d,\tau} - (B^-_{i,cd})^{\mathrm{T}}\sigma_{i,c,\tau} \geq 0 \tag{38}$$
$$M_{Ri,c,\tau} - (B^-_{i,dd})^{\mathrm{T}}\sigma_{i,d,\tau} \geq 0 \tag{39}$$

B. Example of decomposition

Consider an optimal firing sequence problem for $FOHPN$ shown in Fig. 1. The initial marking is $M_{d,0} = [1\ 0\ 0\ 0]^{\mathrm{T}}$, $M_{c,0} = [0, c_0, 0]^{\mathrm{T}}$, the final marking is $M_{d,f} = [0\ 0\ 1\ 0]^{\mathrm{T}}$, $M_{c,f} = [0, 0, c_0]^{\mathrm{T}}$. The optimal firing sequence problem to minimize the total transition time is treated as an example. In this example, if the place p_4 is removed and the duplicated places $p_4^{'}$ and $p_4^{''}$ for p_4 are connected with the $FOHPN$ can be decomposed into independent subnets as shown in Fig. 2.

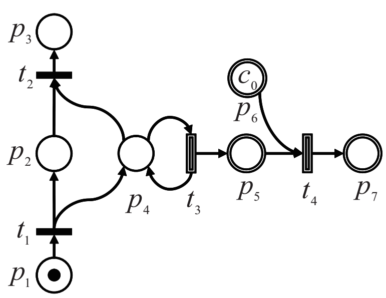

Fig. 1. An example of original FOHPN

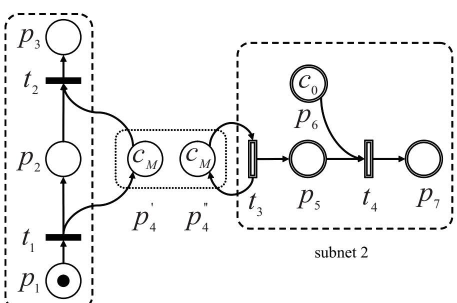

Fig. 2. Decomposed subnets for FOHPN

V. Lagrangian relaxation

Lagrangian relaxation is used for obtaining lower bound of the combinatorial optimization problem. Recently, the method is widely applied to solve scheduling problems. The optimality of solution can be evaluated by upper bound and lower bound of the problem.

A. Lagrange function

The constraints for enabling condition for the set of places P_R represented by (30) and (31) are relaxed by Lagrange multipliers λ_d, λ_c, respectively and added to the objective function multiplied by Lagrange multipliers. The relaxed probledm can be rewritten as follows where $N_t = N_T/\epsilon$ is the total time horizon.

$$\min \quad L \tag{40}$$
$$L = \sum_{i=1}^{m} J_i + \sum_{\tau=0}^{N_t} \lambda^{\mathrm{T}}_{d,\tau}\left\{(B^-_{dd})^{\mathrm{T}}\sigma_{d,\tau} + (B^-_{cc})^{\mathrm{T}}\sigma_{c,\tau} - M_{R,d,\tau}\right\} + \sum_{\tau=0}^{N_t} \lambda_{c,\tau}\{(B^-_{cc})^{\mathrm{T}}\sigma_{d,\tau} - M_{R,c,\tau}\} \tag{41}$$
$$\text{s. t.} \quad (18), (19), (20), (24), (25), (28), (29), (36) \sim (39)$$

Using the marking evolution equation, $M_{R,d,\tau} = M_{R,d,0} + \sum_l^{\tau-1}\left((B^+_{dd})^{\mathrm{T}}\sigma'_{d,l} - (B^-_{dd})^{\mathrm{T}}\sigma_{d,l}\right)$, $M_{R,c,\tau} = M_{R,c,0} + \sum_l^{\tau-1}\left((B^+_{dc})^{\mathrm{T}}\sigma'_{d,l} - (B^-_{dc})^{\mathrm{T}}\sigma_{d,l} + (B_{cc})^{\mathrm{T}}v_l\cdot\epsilon\right)$, the function L can be rewritten as:

$$L = \sum_{i=1}^{m}\Bigg[J_i + \sum_{\tau=0}^{N_t}\lambda^{\mathrm{T}}_{d,\tau}\Bigg\{(B^-_{i,dd})^{\mathrm{T}}\sigma'_{i,d,\tau} + (B^-_{i,cd})^{\mathrm{T}}\sigma_{i,c,\tau} - \sum_{l}^{k-1}\left((B^+_{i,dd})^{\mathrm{T}}\sigma'_{i,d,l} - (B^-_{i,dd})^{\mathrm{T}}\sigma_{i,d,l}\right)\Bigg\} + \sum_{\tau=0}^{N_t}\lambda^{\mathrm{T}}_{c,\tau}\Bigg\{(B^-_{i,dc})^{\mathrm{T}}\sigma_{i,d,\tau} - \sum_{l}^{\tau-1}((B^+_{i,dc})^{\mathrm{T}}\sigma_{i,d,l} - (B^-_{i,dc})^{\mathrm{T}}\sigma_{i,d,l} + (B_{i,cc})^{\mathrm{T}}v_{i,l}\cdot\epsilon\Bigg\}\Bigg] - \sum_{\tau=0}^{N_t}(\lambda^{\mathrm{T}}_{\tau}M_{R,d,0} + \lambda^{\mathrm{T}}_{c,\tau}M_{R,c,0}) \tag{42}$$

(42) can be rewritten as $L = \sum_{i=1}^{m} L_i - \sum_{\tau=0}^{N_t}(\lambda^{\mathrm{T}}_{d,\tau}M_{R,d,0} + \lambda^{\mathrm{T}}_{c,\tau}M_{R,c,0})$ when L_i is defined as $L_i = J_i + \sum_{\tau=0}^{N_t}\Big[\lambda^{\mathrm{T}}_{d,\tau}\Big\{(B^-_{i,dd})^{\mathrm{T}}\sigma'_{i,d,\tau} + (B^-_{i,cd})^{\mathrm{T}}\sigma_{i,c,\tau} - \sum_l^{\tau-1}\left((B^+_{i,dd})^{\mathrm{T}}\sigma'_{i,d,l} - (B^-_{i,dd})^{\mathrm{T}}\sigma_{i,d,l}\right)\Big\} + \lambda^{\mathrm{T}}_{c,\tau}\left\{(B^-_{i,dc})^{\mathrm{T}}\sigma_{i,d,\tau} - \sum_l^{\tau-1}((B^+_{i,dc})^{\mathrm{T}}\sigma_{i,d,\tau} - (B^-_{i,dc})^{\mathrm{T}}\sigma_{i,d,\tau} + (B^{\mathrm{T}}_{i,cc})v_{i,l}\cdot\epsilon\right\}\Big]$.

Therefore the relaxed problem with Larange multiplier λ_τ for the minimization of L is decomposable into subproblem for the minimization of L_i.

B. Optimization algorithm of Lagrangian relaxation

The optimization algorithm for Lagrangian relaxation is as follows.

Step 1 Initialization

The number of iterations $N := 1$. The Lagrange multipliers are set to an initial value ($\lambda_\tau^{(N)} = \{0\}$).

Step 2 Solving subproblems

Each subproblem $RP_i \quad (i = 1, \ldots, m)$ is solved sequentially. The derived solution provides a lower bound LB for the original problem.

$$(RP_i) \ \min L_i \tag{43}$$
$$\text{s. t. } (18), (19), (20), (24), (25), (28), (29), (36) \sim (39)$$

Step 3 Construction of feasible solution

The solution derived at Step 2 is not always feasible. Therefore a feasible solution is constructed by the algorithm explained in Section V-D. A feasible solution provides an upper bound UB for the original problem.

Step 4 Evaluation of convergence

If the lower bound is not updated for a pre-determined number of times, the algorithm is finished.

Step 5 Update of Lagrange multipliers

The value of Lagrange multipliers are updated by subgradient optimization method. The subgradient can be given by:

$$g_\tau = (B_{i,dd}^{-})^{\mathrm{T}} \tilde{\sigma}_{d,\tau} + (B_{cc}^{-})^{\mathrm{T}} \tilde{\sigma}_{c,\tau} - M_{R,d,\tau} \tag{44}$$

where $\tilde{\sigma}_{d,\tau}, \tilde{\sigma}_{c,\tau}$ is the firing vector derived by solving subproblem derived at Step 2. The Lagrange multipliers are updated by

$$\lambda_\tau^{(N+1)} = \max\{0, \ \lambda_\tau^{(n)} + \alpha \frac{UB - LB}{\sum_{\tau=0}^{N_t} |g_\tau|^2} g_\tau\} \tag{45}$$

UB is upper bound, LB is lower bound, and α is the parameter for the update of Lagrange multiplies. $N := N + 1$ and return to Step 2.

C. Solving subproblem

The subproblem in Section V-B is formulated as a mixed integer programming problem to determine firing vector and firing speed. The problem can be solved by a commercial solver. However, we propose an efficient algorithm to solve the problem by state space search algorithm with the enumeration of reachability graph[16]. The algorithm for solving subproblem is as follows.

Step 1 For the optimization of subnet i, initial state ξ_0 , the number of steps $n := 0$, the initial marking $M_{i,0}$ are defined. Let $\xi = \xi_0$.

Step 2 Generate the successor state ξ' from state ξ by the following steps.

1) Select discrete transitions from the enabled discrete transitions. M'_n is the marking during the transitions.
2) Select continuous transitions from the enabled continuous under the marking M'_n.
3) For the selected continuous transition, determine a firing speed that can maximize $1^{\mathrm{T}} v$ under the following conditions.

$$\begin{cases} V_j - v_j \geq 0 & \forall t_j \in T_{\mathcal{E}}(M'_n) \\ v_j - V'_j \geq 0 & \forall t_j \in T_{\mathcal{E}}(M'_n) \\ v_j = 0 \geq 0 & \forall t_j \in T_{\mathcal{N}}(M'_n) \\ \displaystyle\sum_{t_j \in T_{\mathcal{E}}} A(p, t_j) \cdot v_j \geq 0 & \forall p \in P_{\mathcal{E}}(M'_n) \end{cases}$$

where $T_{\mathcal{E}}(M'_n) \in T_c$ is the subset of continuous transitions enabled at marking M'_n, and $T_{\mathcal{N}}(M'_n) \in T_c$ is the subset of continuous transitions not enabled at marking M'_n. $P_{\mathcal{E}}(M'_n)$ is the subset of empty continuous places.
4) If at least one discrete transition is fired at 1), $n' := n + \Delta\tau$, otherwise $n' := n + 1$ and the time is determined for ξ'. The state ξ' is added to the list of states. If the same states already exist in the reachability graph, the cost function is compared. Then the state with larger value of objective function is deleted.

Step 3. Repeat the procedure of Step 2 until all of states are searched. The state with minimum cost is set to the next state ξ.

Step 4. If the marking of state ξ is final marking, the search is completed. Otherwise return to Step 2 and repeat the search again.

D. Construction of a feasible solution

The solution derived by solving subproblem is not always feasible for the entire $FOHPN$. In this section, the algorithm to construct a feasible solution is explained in the algorithm explained in Section V-B. Before the execution of the algorithm, the solution derived at each subnet is divided into multiple macro-periods. The algorithm for the setting of solution for each macro-period for subnet i is as follows.

Step 1. Time step $n := 0$. Set the period number $k_{(i)} = 0$, $\sigma_{i,0}^p = \sigma_{i,0}, v_{i,0}^p = v_{i,0}$.

Step 2. If $\sigma'_{i,d,n}(t_a) = 1$ at time step n for transition $t_a \in T_d$, that is to say, the firing of transition t_a is completed or $v_{i,n} \neq v_{i,n+1}$, complete the period $k_{(i)}$, and $\sigma_{i,d,k_{(i)}}^p(t_a) = 1, k_{(i)} := k_{(i)} + 1$.

Step 3. If the Step 2 is examined for all time steps, the algorithm is completed. Otherwise $n := n + 1, \sigma_{i,d,k_{(i)}}^p = \sigma_{i,d,n}, v_{i,k_{(i)}}^p = v_{i,n}$ and return to Step 2.

where $\sigma_{i,k_{(i)}}^p, v_{i,k_{(i)}}^p$ is the firing vector for period $k_{(i)}$, and firing speed for period $k_{(i)}$, respectively.

By dividing the solution of firing vector and firing speed into multiple macro-periods, the algorithm to construct a feasible solution is executed.

Step 1. Initialize $n = 0, k = 0, k_{(i)} = 0$.

Step 2. For each subnet, derive the minimum time step n_{min} when the macro-period is started. n_{min} is the occurrence time of the next macro-event at macro-period $k + 1$.

Step 3. Check the conflicts of solution of subnets at time step n_{min}. If there exists conflicts, they are resolved by delaying the starting time of occurrence time of an event.

Step 4. Determine the time step n_{min} when the next macro-event is occurred. For the subnet i which completes the macro-period at time step n_{min}, $k_{(i)} := k_{(i)} + 1$.
Step 5. If all of macro-events are completed, the algorithm is finished.

VI. Computational experiments

Computational experiments are demonstrated to show the effectiveness of the proposed method for the example problem in Fig. 1. The initial marking is $M_0(p_1) = 1$, $M_0(p_6) = 45$ and final marking is $M_f(p_3) = 1$, $M_f(p_6) = 45$. The condition for firing speed for continuous transition t_3, t_4 is $[0, 15]$, $[0, 10]$, respectively. The unit discretization time is $\epsilon = 0.1$. The results derived by solving the optimal firing sequence problem are shown in Fig. 3. Fig. 3 shows the firing speed for continuous places t_3 and t_4. A Pentium D 3.4GHz processor with 1GB memory is used for computation.

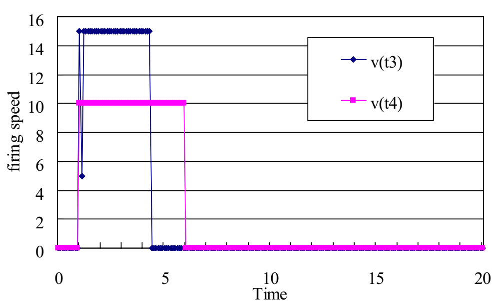

Fig. 3. Computational results for firing speeds of continuous places

Table I shows the comparison between the performance between the proposed method using Petri Net decomposition and CPLEX without decomposition using the formulation of (13)-(20). The upper bound and lower bound is almost the same. The computation time is reduced by the decomposition. The results demonstrate that the proposed method can derive an optimal solution with less computational effort for the problem.

TABLE I
Computational results for the example

	The proposed method	CPLEX
UB	10.5	10.5
LB	10.45	10.5
DGAP[%]	0.45	0.0
Time[s]	0.007	0.42

VII. Conclusion

In this paper, we have proposed a decomposition method for optimal firing sequence problem for first-order hybrid Petri Nets. The Petri Net decomposition with Lagrangian relaxation has been developed to solve the problem. In the proposed method, the original FOHPN is decomposed into several subnets by removing shared places and connecting subnets with the duplicated places with appropriate connection. The subproblem for determining firing vector and firing speed has been solved by Dijkstra algorithm with reachability graph. The algorithm to construct a feasible solution has also been proposed. Computational experiments demonstrate that the proposed method can derive near-optimal solution with less computational effort compared with commercial solver before decomposition. Further research is to investigate the effectiveness of the proposed method for large-sized problems.

References

[1] H. Alla and R. David, "Continuous and Hybrid Petri Nets," *Journal of Circuits, System and Computers*, vol. 8, no. 1, pp. 159–188, 1998.

[2] F. Bulduzzi, A. Giua and G. Menga, "Hybrid Stochastic Petri Nets: Firing Speed Computation and FMS Modeling," *Proceeding of the 4th Workshop on Discrete Event Systems*, pp. 432–438, 1998.

[3] F. Bulduzzi, A. Giua and G. Menga, "Optimal Speed Allocation and Sensitivity Analysis of Hybrid Stochastic Petri Nets," *Proceeding of the IEEE Conference on Systems, Man and Cybernetics*, pp. 656–662, 1998.

[4] F. Bulduzzi, A. Giua and G. Menga, "First-order Hybrid Petri Nets: A Model for Optimization and Control," *IEEE Transactions on Robotics and Automation*, vol. 16, no. 4, pp. 382–399, 2000.

[5] F. Bulduzzi, A. Giua and C. Seatzu, "Modelling and Simulation of Manufacturing Systems with First-order Hybrid Petri Nets," *International Journal of Production Research*, vol. 39, no. 2, pp. 255–282, 2001.

[6] A. Bemporad, and M. Morari, "Control of Systems Integrating Logic, Dynamics, and Constraints," *Automatica*, vol. 35, no. 3, pp. 407–427, 1999.

[7] A. Bemporad, and N. Giorgetti, "Logic-based Solution Methods for Optimal Control of Hybrid Systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 6, pp. 963–976, 2006.

[8] G. Ciardo, D. M. Nicol, and K. S. Trivedi, "Discrete-event Simulation of Fluid Stochastic Petri Nets," *IEEE Transactions on Software Engineering*, vol. 25, no. 2, pp. 207–217, 2005.

[9] R. David, and H. Alla, "On Hybrid Petri Nets," *Discrete Event Dynamic Systems: Theory and Applications*, vol. 11, pp. 9–40, 2001.

[10] M. Dotoli, M. P. Fanti, A. Giua, and C. Seatzu, "First-order Hybrid Petri Nets. An Application to Distributed Manufacturing Systems," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 2, pp. 408–430, 2008.

[11] A.D. Febbraro, D. Giglio and N. Sacco, "Urban Traffic Control Structure Based on Hybrid Petri Nets," *IEEE Transactions on Intelligent Transportation Systems*, vol. 5, no. 4, pp. 224–237, 2004.

[12] A.D. Febbraro and N. Sacco, "On Modelling Urban Transportation Networks via Hybrid Petri Nets," *Control Engineering Practice*, vol. 12, no. 10, pp. 1225–1239, 2004.

[13] M.L. Fisher, "The Lagrangian Relaxation Method for Solving Integer Programming Problems," *Management Science*, vol. 27, no. 1, pp. 1–18, 1981.

[14] L. Ghomri, and H. Alla, "Modeling and Analysis Using Hybrid Petri Nets," *Nonlinear Analysis: Hybrid Systems*, vol. 1, no. 2, pp. 141–153, 2007.

[15] D.W. He, B. Strege, H. Tolle and A. Kusiak, "Decomposition in Automatic Generation of Petri Nets for Manufacturing System Control and Engineering," *International Journal of Production Research*, vol. 38, no. 6, pp. 1437–1457, 2000.

[16] D.Y. Lee and F. DiCeare, "Scheduling Flexible Manufacturing Systems Using Petri Nets and Heuristic Search," *IEEE Transactions on Robotics and Automation*, vol. 10, no. 2, pp. 123–131, 1994.

[17] D.Y. Liao, M. D. Jeng, and M. C. Zhou, "Application of Petri Nets and Lagrangian Relaxation to Scheduling Automatic Material-Handling Vecles in 300-mm Semiconductor Manufacturing," *IEEE Transactions on Systems, Man, and Cybernetics. Part C*, vol. 37, no. 4, pp. 504–516, 2007.

[18] T. Nishi, K. Shimatani, M. Inuiguchi, "Decomposition of Petri nets and Lagrangian relaxation for solving routing problems for AGVs," *International Journal of Production Research*, vol. 47, no. 14, pp. 3957–3977, 2009.

[19] T. Nishi and R. Maeno, "Petri Net Modeling and Decomposition Method for Solving Production Scheduling Problems," *Journal of Advanced Mechanical Design, Systems, and Manufacturing*, vol. 1, no. 2, pp. 262–271, 2007.