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Necessary and Sufficient Conditions for General SISO Mamdani Fuzzy Systems on a Given Accuracy

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***Abstract*—Given a prescribed accuracy, necessary and sufficient conditions are investigated for general single input/single output (SISO) Mamdani fuzzy systems as approximators of continuous functions defined on compact domain. Since general SISO Mamdani fuzzy systems are monotonic on subintervals, necessary conditions for fuzzy systems on a given accuracy have been established firstly with the extreme of the desired continuous function. Simultaneously, a dynamically constructive method is proposed to show the conditions are sufficient. Furthermore, it is shown that existing results concerning necessary conditions are only special cases of ours. The necessary and sufficient conditions can not only be used practically to determine input/output fuzzy sets, and fuzzy rules for fuzzy systems, but also provide guidance on the membership function design. Finally, simulation examples are given to illustrate the conclusions and analyze the strength as well as the limitation of the fuzzy systems as function approximators: the number of fuzzy rules increases with the number of extreme whose swings are larger than the accuracy, not associated with the function's formulation.**

***Index Terms*—Fuzzy systems, Necessary and sufficient conditions, Fuzzy rules, Approximation accuracy.**

I. Introduction

In the past decade, fuzzy systems are widely used and extensively studied for a variety of applications, especially in control applications, mainly because of their excellent performance in representing nonlinear functions in an intuitive and informative manner. In most applications, fuzzy systems involved can be viewed as rule-based function approximators. Many scholars have proved that various fuzzy systems are universal approximators, that is, they can uniformly approximate any continuous functions [1-5]. Analyzing and comparing these literatures, it is found that approximation quality of a fuzzy system does not only depend on its system configuration, such as fuzzy sets, fuzzy rules and inference methods, but also depend on the character of the desired function. Given a continuous function, along with the improvement of accuracy, more and more fuzzy rules are needed, especially in high dimensions. Large numbers of rules are not desirable both in theory and in practice for they render difficulty in designing fuzzy systems, such as high storage consumption, increased computational complexity, etc.

One always wants to construct a fuzzy system using as less fuzzy rules as possible to approximate the given continuous function within accuracy. Therefore, many efforts have been made to reduce fuzzy rules, which could be categorized as two main lines of work. The first is through subset selection method, which is mainly based on orthogonal transformation or singular value decomposition [6-8]. The other mainly employs heuristic methods, which contain genetic algorithms [9-10], clustering methods [11, 12], and tree search [13], etc. Although above efforts have progressed much in reducing fuzzy rules, yet how far they are from the optimal configuration is still obscure. How many input fuzzy sets and rules are needed in order to guarantee the approximation accuracy? That are sufficient conditions for fuzzy systems, which have been investigated by many scholars based on the uniform approximate polynomial [14-16] and based on the derivative of the object function [17-18, 23]. The necessary conditions are more important for they establish the optimal configuration for fuzzy systems to achieve approximation accuracy under limited information. Despite its importance, only a few papers [19-22] have dealt with this problem. Generally, the number of the fuzzy rules needed increases with the number of extreme of the desired function. However, these conclusions were based on uniform approximation and will no longer be applicable under fixed approximation accuracy. More importantly, in most practical applications, designing system is to construct a fuzzy system, such that it approximates a prospective function within accuracy, so necessary and sufficient conditions for fuzzy systems on a given accuracy cry out for deep study.

In this paper, we establish necessary and sufficient conditions for general SISO Mamdani fuzzy systems on a prescribed accuracy and also show that the result in [19] is a special case of our conclusions. Simultaneously, according to these conditions, illustrative examples are used to explore and demonstrate the strength and limitation of the general Mamdani fuzzy system as function approximators. The rest of the paper is organized as follows. Configuration of general SISO Mamdani fuzzy systems and problem formulation are given in section 2. In section 3, we investigate necessary and sufficient conditions for general SISO Mamdani fuzzy systems on given accuracy. Example is illustrated to explore and analyze the character of general SISO Mamdani fuzzy systems in section 4. Finally, we end this paper with conclusions in section 5.

II. Configuration of General SISO Mamdani Fuzzy Systems and Problem Formulation

A. Configuration of general SISO Mamdani fuzzy systems

The interval $[a, b]$, on which the object function $f(x)$ is defined, is divided into N subintervals (S_i is endpoint of subintervals):

$$a = S_0 < S_1 < S_2 < \cdots < S_N = b$$

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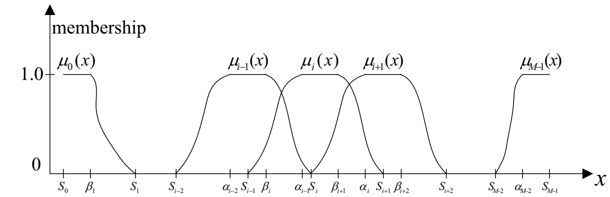

Fig. 1. Illustrative definition of input fuzzy sets.

Then there are $N+1$ completeness, normal and consistency input fuzzy sets defined on $[a,b]$ to fuzzify the input variable (Fig. 1). Each input fuzzy set denoted by $A_i(0 \le i \le N)$ has a membership function $\mu_i(x)$ defined as follows:

$$\mu_i(x) = \begin{cases} I_i(x) & x \in [S_{i-1}, \alpha_{i-1}) \\ 1 & x \in [\alpha_{i-1}, \beta_{i+1}] \\ D_i(x) & x \in (\beta_{i+1}, S_{i+1}] \\ 0 & x \in [a,b]/[S_{i-1}, S_{i+1}] \end{cases} \tag{1}$$

where both $I_i(x)$ and $D_i(x)$ are continuous, $I_i(x)$ is zero at $x = S_{i-1}$ and increases monotonically to one at $x = \alpha_{i-1}$, and $D_i(x)$ is one at $x = \beta_{i+1}$ and decreases monotonically to zero at $x = S_{i+1}$. $S_{-1} = S_0$, $S_{N+1} = S_N$, $S_{i-1} < \alpha_{i-1} \le S_i$, $S_i \le \beta_{i+1} < S_{i+1}$.

The $N+1$ fuzzy rules used are in the following form:

$$R_i: \ IF \ x \ is \ A_i, \ THEN \ y \ is \ B_i, \ i = 0, 1, \cdots, N \tag{2}$$

where A_i is a input fuzzy set with its membership function $\mu_i(x)$; B_i is a singleton output fuzzy set, whose membership function is one only at $y = y_i$ (an arbitrary constant) and is zero elsewhere. Using product inference engine and center-average defuzzifier, the output of the SISO Mamdani fuzzy system is

$$F(x) = \frac{\sum\limits_i \mu_i(x) y_i}{\sum\limits_i \mu_i(x)} \tag{3}$$

B. Statement of the approximation problem for fuzzy systems

For a SISO fuzzy system, the approximation problem can be stated as follows. Let $f(x)$ denote the continuous function defined on the compact domain Θ and suppose $\varepsilon \ge 0$ is an arbitrarily given approximation error bound. What are necessary and sufficient conditions under which there always exists a fuzzy system as defined in Section 2.1, whose output $F(x)$ satisfies

$$\|F(x) - f(x)\|_\infty \le \varepsilon \tag{4}$$

III. Necessary and Sufficient conditions for General SISO Mamdani Fuzzy Systems

In order to further derivation, the following lemmas are given firstly.

Lemma 1: The function $F(x)$ defined in (3) is continuous, and monotonic on $[S_{i-1}, S_i]$ for all $i = 1, 2, \cdots, N$ [19].

Lemma 2: Assume fuzzy system $F(x)$ defined in (3) can approximate $f(x)$ within approximation accuracy ε on $[a,b]$, for arbitrary $a \le x_1 < x_2 \le b$, if $f(x_1) - f(x_2) > 2\varepsilon$, then there exists $[x_{t1}, x_{t2}] \subseteq [x_1, x_2]$, such that $F(x)$ strictly decreases on $[x_{t1}, x_{t2}]$; for arbitrary $a \le x_3 < x_4 \le b$, if $f(x_3) - f(x_4) < -2\varepsilon$, then there exists $[x_{t3}, x_{t4}] \subseteq [x_3, x_4]$, such that $F(x)$ strictly increases on $[x_{t3}, x_{t4}]$.

Proof: (i) For arbitrary $a \le x_1 < x_2 \le b$, if $f(x_1) - f(x_2) > 2\varepsilon$, we should prove $F(x)$ strictly decreases on $[x_{t1}, x_{t2}]$, where $[x_{t1}, x_{t2}] \subseteq [x_1, x_2]$.

Using reduction to absurdity, it is assumed that the interval $[x_{t1}, x_{t2}]$, on which $F(x)$ strictly decreases, does not exist. Then $F(x)$ increases monotonically on $[x_1, x_2]$ so that we can get

$$F(x_2) \ge F(x_1) \tag{5}$$

Since $F(x)$ can approximate $f(x)$ within ε, we have

$$f(x_1) - F(x_1) \le \varepsilon \tag{6}$$

$$f(x_2) - F(x_2) \ge -\varepsilon \tag{7}$$

Subtracting (7) from (6) yields

$$f(x_1) - f(x_2) \le 2\varepsilon + F(x_1) - F(x_2) \tag{8}$$

Subtracting (5) from (8) yields

$$f(x_1) - f(x_2) \le 2\varepsilon \tag{9}$$

Formula (9) is contradictory with $f(x_1) - f(x_2) > 2\varepsilon$. This contradiction means that there exists $[x_{t1}, x_{t2}] \subseteq [x_1, x_2]$, such that $F(x)$ strictly decreases on $[x_{t1}, x_{t2}]$.

(ii) For arbitrary $a \le x_3 < x_4 \le b$, if $f(x_3) - f(x_4) < -2\varepsilon$, we can prove similarly that there exists $[x_{t3}, x_{t4}] \subseteq [x_3, x_4]$, such that $F(x)$ strictly increases on $[x_{t3}, x_{t4}]$. ■

From lemma 2, if the value of the desired function is greater two times accuracy than the other's at arbitrary two points, the monotonicity of the approximate fuzzy systems exists on its subintervals. In lemma 2, if both cases exist in an interval, the following lemma could be obtained.

Lemma 3: Suppose fuzzy system $F(x)$ defined in (3) can approximate $f(x)$ within approximation accuracy ε on $[a,b]$, for arbitrary $a \le x_1 < x_2 \le b$ and $a \le x_3 < x_4 \le b$, if $f(x_1) - f(x_2) > 2\varepsilon$, $f(x_3) - f(x_4) < -2\varepsilon$ then there exists division point in $[\min(x_1, x_3), \max(x_2, x_4)]$ for the fuzzy system.

Proof: For arbitrary $a \le x_1 < x_2 \le b$ and $a \le x_3 < x_4 \le b$, if $f(x_1) - f(x_2) > 2\varepsilon$ and $f(x_3) - f(x_4) < -2\varepsilon$, then according to lemma 2, there exists $[x_{t1}, x_{t2}] \subseteq [x_1, x_2]$, such that $F(x)$ strictly decreases on $[x_{t1}, x_{t2}]$ and exists $[x_{t3}, x_{t4}] \subseteq [x_3, x_4]$, such that $F(x)$ strictly increases on $[x_{t3}, x_{t4}]$.

Therefore, there exists two subintervals in $[\min(x_1, x_3), \max(x_2, x_4)]$, on which the approximate fuzzy system strictly increases and decreases respectively. According to lemma 1, $F(x)$ is monotonic on each subinterval such that there exists division point in $[\min(x_1, x_3), \max(x_2, x_4)]$. ■

From the above lemma, the relationship between whether the fuzzy systems need division points and the internal relationship among the arbitrary four points of the desired function is established. However, this relationship is not applicable because of its high computation requirement. Fortunately, following equivalent relation could be obtained as Lemma 4.

Lemma 4 (Equivalence 1): For arbitrary $a \leq x_1 < x_2 \leq b$ and $a \leq x_3 < x_4 \leq b$, if $f(x_1) - f(x_2) > 2\varepsilon$, $f(x_3) - f(x_4) < -2\varepsilon$, there exists $x_{i1} < x_{i2} < x_{i3} \in [\min(x_1, x_3), \max(x_2, x_4)]$, such that $f(x_{i1}) - f(x_{i2}) > 2\varepsilon$, $f(x_{i2}) - f(x_{i3}) < -2\varepsilon$ or $f(x_{i1}) - f(x_{i2}) < -2\varepsilon$, $f(x_{i2}) - f(x_{i3}) > 2\varepsilon$.

Proof: Without losing generality, it is assumed $x_1 \leq x_3$, and we will prove the result in the following two cases.

(a) When $x_2 \leq x_4$

(i) if $f(x_2) \geq f(x_3)$, with $f(x_1) - f(x_2) > 2\varepsilon$, we can get

$$f(x_1) - f(x_3) > 2\varepsilon \tag{10}$$

With $f(x_3) - f(x_4) < -2\varepsilon$, we can choose

$$x_{i1} = x_1,\ x_{i2} = x_3,\ x_{i3} = x_4 \tag{11}$$

(ii) if $f(x_2) < f(x_3)$, with $f(x_3) - f(x_4) < -2\varepsilon$, we can get

$$f(x_2) - f(x_4) < -2\varepsilon \tag{12}$$

With $f(x_1) - f(x_2) > 2\varepsilon$, we can choose

$$x_{i1} = x_1,\ x_{i2} = x_2,\ x_{i3} = x_4 \tag{13}$$

(b) When $x_2 > x_4$

(i) if $f(x_2) \geq f(x_3)$, with $f(x_1) - f(x_2) > 2\varepsilon$, we can get

$$f(x_1) - f(x_3) > 2\varepsilon \tag{14}$$

With $f(x_3) - f(x_4) < -2\varepsilon$, we can choose

$$x_{i1} = x_1,\ x_{i2} = x_3,\ x_{i3} = x_4 \tag{15}$$

(ii) if $f(x_2) < f(x_3)$, with $f(x_3) - f(x_4) < -2\varepsilon$, we can get

$$f(x_4) - f(x_2) > 2\varepsilon \tag{16}$$

With $f(x_3) - f(x_4) < -2\varepsilon$, we can choose

$$x_{i1} = x_3,\ x_{i2} = x_4,\ x_{i3} = x_2 \tag{17}$$

■

From lemma 4, it is shown that the relationship among arbitrary three points is sufficient, which would decrease computation complexity remarkably as compared with lemma 3. Furthermore, it serves as the basis of the algorithm and theorem below. Then rewrite lemma 3 as follows:

Lemma 3*: Suppose fuzzy system $F(x)$ defined in (3) can approximate $f(x)$ within accuracy ε, for each subinterval $[S_j, S_{j+1}]$ there does not exist $S_j \leq x_1 < x_2 < x_3 \leq S_{j+1}$, such that

$$\begin{gathered} f(x_1) - f(x_2) > 2\varepsilon,\ f(x_2) - f(x_3) < -2\varepsilon \\ \text{or } f(x_1) - f(x_2) < -2\varepsilon,\ f(x_2) - f(x_3) > 2\varepsilon \end{gathered}$$

However, the above assumptions imposed on the continuous function to be approximated require too much information, which may be not available and cause that the fuzzy systems perform worse than the other well-developed classical functional approximators. The following Equivalence Lemma guarantees that less information suffice to characterize the continuous function.

Lemma 5 (Equivalence 2): Following two results are equivalent for $f(x)$ on $[\tau_1, \tau_2] \subseteq [a, b]$:

(1) There exists $\tau_1 \leq x_1 < x_2 < x_3 \leq \tau_2$, such that $f(x_1) - f(x_2) > 2\varepsilon$, $f(x_2) - f(x_3) < -2\varepsilon$ or $f(x_1) - f(x_2) < -2\varepsilon$, $f(x_2) - f(x_3) > 2\varepsilon$.

(2) There exists $\tau_1 \leq \xi_1 < \xi_2 < \xi_3 \leq \tau_2(\xi_1,\ \xi_2$ and ξ_3 are extreme of $f(x)$ on $[\tau_1, \tau_2])$, such that $f(\xi_1) - f(\xi_2) > 2\varepsilon$, $f(\xi_2) - f(\xi_3) < -2\varepsilon$ or $f(\xi_1) - f(\xi_2) < -2\varepsilon$, $f(\xi_2) - f(\xi_3) > 2\varepsilon$.

Proof: (2)⇒(1) is trivial. We will prove (1)⇒(2)

Take $f(x_1) - f(x_2) > 2\varepsilon$, $f(x_2) - f(x_3) < -2\varepsilon$ as an example. Find the monotonic interval $[\beta_2, \beta_5]$, which includes x_2. Without losing generality, assume $f(x)$ increases monotonically on $[\beta_2, \beta_5]$. If $\beta_2 \leq x_1$, we will have $f(x_1) \leq f(x_2)$. This is contradictory with $f(x_1) - f(x_2) > 2\varepsilon$. So we get $x_1 < \beta_2 \leq x_2 < x_3$. With the assumption, we can know that $f(\beta_2) \leq f(x_2)$ and

$$f(x_1) - f(\beta_2) > 2\varepsilon,\ f(\beta_2) - f(x_3) < -2\varepsilon \tag{18}$$

Let $\xi_2 = \beta_2$. Similarly, ξ_1 and ξ_3 could be found. ■

We now establish, using lemma 3* and lemma 5, the necessary conditions for general SISO Mamdani fuzzy systems as function approximators on given accuracy.

Theorem 1 (Necessary Conditions for Fuzzy Systems): In order to achieve the approximation (4), one must choose at least such N, for the fuzzy system defined in (3), that divide $[a, b]$ in such a way that

(a) In each subinterval $[S_j, S_{j+1}]$, there does not exist $S_j \leq \xi_1 < \xi_2 < \xi_3 \leq S_{j+1}(\xi_1,\ \xi_2$ and ξ_3 are extreme of $f(x)$ on $[S_j, S_{j+1}])$, such that

$$\begin{gathered} f(\xi_1) - f(\xi_2) > 2\varepsilon,\ f(\xi_2) - f(\xi_3) < -2\varepsilon \\ \textit{or } f(\xi_1) - f(\xi_2) < -2\varepsilon,\ f(\xi_2) - f(\xi_3) > 2\varepsilon \end{gathered} \tag{19}$$

(b) Simultaneously, for arbitrary two adjacent subintervals $[S_{j-1}, S_j]$ and $[S_j, S_{j+1}]$, there does not exist $S_{j-1} \leq \xi_1 < \xi_2 \leq S_j \leq \xi_3 < \xi_4 \leq S_{j+1}(\xi_i(1 \leq i \leq 4)$ are extreme of $f(x)$ on $[S_{j-1}, S_{j+1}])$, such that

$$\begin{gathered} f(\xi_1) - f(\xi_2) > 2\varepsilon,\ f(\xi_3) - f(\xi_4) > 2\varepsilon \\ \textit{or } f(\xi_1) - f(\xi_2) < -2\varepsilon,\ f(\xi_3) - f(\xi_4) < -2\varepsilon \end{gathered} \tag{20}$$

Proof: With lemma 3* and lemma 5, the necessity of condition (a) is obvious.

Now let us use a contradiction argument to analyze the condition (b) is necessary. Without losing generality, suppose there exist $S_{j-1} \leq \xi_1 < \xi_2 \leq S_j \leq \xi_3 < \xi_4 \leq S_{j+1}$ in two adjacent subintervals $[S_{j-1}, S_j]$ and $[S_j, S_{j+1}]$, such that

$$f(\xi_1) - f(\xi_2) > 2\varepsilon,\ f(\xi_3) - f(\xi_4) > 2\varepsilon \tag{21}$$

Since there exists a fuzzy system that satisfies the approximation, there does not exist $S_{j-1} \leq \xi_{j1} < \xi_{j2} \leq S_j$ and $S_j \leq \xi_{j3} < \xi_{j4} \leq S_{j+1}$ such that

$$f(\xi_{j1}) - f(\xi_{j2}) < -2\varepsilon,\ f(\xi_{j3}) - f(\xi_{j4}) < -2\varepsilon \tag{22}$$

(i) If there exists $S_{j-1} \le \xi_{j5} < S_j < \xi_{j6} \le S_{j+1}$, such that

$$f(\xi_{j5}) - f(\xi_{j6}) < -2\varepsilon \tag{23}$$

Therefore, based on lemma 2, there exists three subintervals $[x_{t1}, x_{t2}] \subseteq [\xi_1, \xi_2]$, $[x_{t3}, x_{t4}] \subseteq [\xi_2, \xi_3]$ and $[x_{t5}, x_{t6}] \subseteq [\xi_3, \xi_4]$, such that $F(x)$ strictly increases on $[x_{t1}, x_{t2}]$ and $[x_{t5}, x_{t6}]$, and strictly decreases on $[x_{t3}, x_{t4}]$. According to lemma 3, two division points are necessary in $[S_{j-1}, S_{j+1}]$. This is contradicted with there are only a division point S_j in $[S_{j-1}, S_{j+1}]$.

(ii) If there does not exist $S_{j-1} \le \xi_{j5} < S_j < \xi_{j6} \le S_{j+1}$, such that

$$f(\xi_{j5}) - f(\xi_{j6}) < -2\varepsilon \tag{24}$$

With (22) and (23), for arbitrary $S_{j-1} \le \xi_1 < \xi_2 \le S_{j+1}$, $f(\xi_1) - f(\xi_2) > 2\varepsilon$ is confirmed, which means the division point S_j is not necessary.

These contradictions show that the condition (b) is necessary. ∎

Next, we will emphasize that the conditions established in theorem 1 are also sufficient ones. For each subinterval $[S_j, S_{j+1}]$ if there does not exist $S_j \le x_1 < x_2 < x_3 \le S_{j+1}$, such that

$$f(x_1) - f(x_2) > 2\varepsilon,\ f(x_2) - f(x_3) < -2\varepsilon$$
$$\text{or } f(x_1) - f(x_2) < -2\varepsilon,\ f(x_2) - f(x_3) > 2\varepsilon.$$

This means that for arbitrary $x_1 < x_2 \in [S_j, S_{j+1}]$, the inequality $f(x_1) - f(x_2) \ge -2\varepsilon$ holds or $f(x_1) - f(x_2) \le 2\varepsilon$ holds. Then the converse theorem of lemma 3* is given and proved constructively in lemma 6.

Lemma 6: If $f(x_1) - f(x_2) \ge -2\varepsilon$ for all $x_1 < x_2 \in [S_j, S_{j+1}]$ or $f(x_1) - f(x_2) \le 2\varepsilon$ for all $x_1 < x_2 \in [S_j, S_{j+1}]$, then there exists an fuzzy system $F(x)$ to achieve the approximation (4) on subinterval $[S_j, S_{j+1}]$.

Proof: A constructive method will be used here to prove justification of this lemma.

(i) If $f(x_1) - f(x_2) \ge -2\varepsilon$ for arbitrary $x_1 < x_2 \in [S_j, S_{j+1}]$, then the output $F(x)$ of an fuzzy system that can approximate $f(x)$ within accuracy on $[S_j, S_{j+1}]$ can be constructed as follows (Fig. 2).

Step 1. Calculate the extreme of the object function $f(x)$ on $[S_j, S_{j+1}]$ $\xi_0 = S_j, \xi_1, \cdots, \xi_{n+1} = S_{j+1}$. let $i = 0$, $e = \xi_0$

Step 2. WHILE $i < n$ DO

Let $F(x) = \min(f(x) + \varepsilon, f(e) + \varepsilon)$ on $x \in [\xi_i, \xi_{i+1}]$.

If $f(e) - f(\xi_{i+1}) \ge 0$, then $e = \xi_{i+1}$.

$i = i + 1$.

Step 3. If $f(\xi_n) - f(\xi_{n+1}) \ge 0$, let

$F(x) = f(x) - \varepsilon + \frac{\xi_{n+1} - x}{\xi_{n+1} - \xi_n}(f(e) - f(\xi_n) + 2\varepsilon)$, else let

$F(x) = f(e) + \varepsilon + \frac{x - \xi_n}{\xi_{n+1} - \xi_n}(f(\xi_{n+1}) - f(e) - 2\varepsilon)$ on $[\xi_n, \xi_{n+1}]$.

(ii) If $f(x_1) - f(x_2) \le 2\varepsilon$ for arbitrary $x_1 < x_2 \in [S_j, S_{j+1}]$. The corresponding fuzzy system output $F(x)$ can be constructed in the similar way:

Step 1. Calculate the extreme of the object function $f(x)$ on $[S_j, S_{j+1}]$ $\xi_0 = S_j, \xi_1, \cdots, \xi_{n+1} = S_{j+1}$. Let $i = 0$, $e = \xi_0$

Step 2. WHILE $i < n$ DO

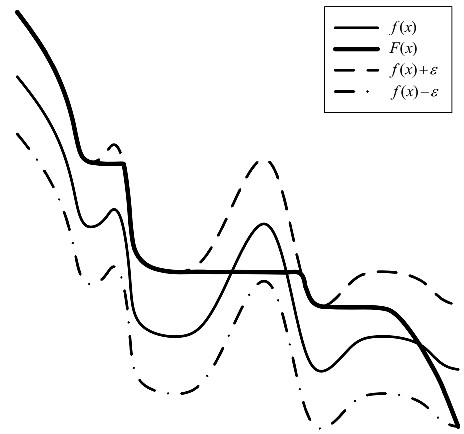

Fig. 2. Illustration of the constructing method.

Let $F(x) = \max(f(x) - \varepsilon, f(e) - \varepsilon)$ on $x \in [\xi_i, \xi_{i+1}]$

If $f(e) - f(\xi_{i+1}) \le 0$, then $e = \xi_{i+1}$.

$i = i + 1$.

Step 3. If $f(\xi_n) - f(\xi_{n+1}) \le 0$, let

$F(x) = f(x) + \varepsilon + \frac{\xi_{n+1} - x}{\xi_{n+1} - \xi_n}(f(e) - f(\xi_n) - 2\varepsilon)$, else let

$F(x) = f(e) - \varepsilon + \frac{x - \xi_n}{\xi_{n+1} - \xi_n}(f(\xi_{n+1}) - f(e) + 2\varepsilon)$ on $[\xi_n, \xi_{n+1}]$.

Then, the input-output fuzzy sets can be constructed as follows. $\mu_1(x) = (\sup F - F(x))/(\sup F - \inf F)$, $\mu_2(x) = 1 - \mu_1(x)$, $y_1 = F(S_j)$ and $y_2 = F(S_{j+1})$. This concludes the proof. ∎

Remark 1: The constructive methods in Lemma 6 just show that there exist fuzzy systems to achieve the approximation (4). The constructed input-output fuzzy sets may have complicated memberships, while fuzzy sets with simple membership are more realistic in practical applications.

With theorem 1 and lemma 6, we will establish some necessary and sufficient conditions for general SISO Mamdani fuzzy system on prescribed approximation accuracy.

Theorem 2 (Necessary and Sufficient Conditions): The fuzzy system defined in (3) can achieve the approximation (4), if and only if one must choose at least such N that divide $[a, b]$ in such a way that

(a) In each subinterval $[S_j, S_{j+1}]$, there does not exist $S_j \le \xi_1 < \xi_2 < \xi_3 \le S_{j+1}$($\xi_1$, ξ_2 and ξ_3 are extreme of $f(x)$ on $[S_j, S_{j+1}]$), such that

$$f(\xi_1) - f(\xi_2) > 2\varepsilon,\ f(\xi_2) - f(\xi_3) < -2\varepsilon$$
$$\textit{or } f(\xi_1) - f(\xi_2) < -2\varepsilon,\ f(\xi_2) - f(\xi_3) > 2\varepsilon \tag{25}$$

(b) Simultaneously, for arbitrary two adjacent subintervals $[S_{j-1}, S_j]$ and $[S_j, S_{j+1}]$, there does not exist $S_{j-1} \le \xi_1 < \xi_2 \le S_j \le \xi_3 < \xi_4 \le S_{j+1}$($\xi_i (1 \le i \le 4)$ are extreme of $f(x)$ on $[S_{j-1}, S_{j+1}]$), such that

$$f(\xi_1) - f(\xi_2) > 2\varepsilon,\ f(\xi_3) - f(\xi_4) > 2\varepsilon$$
$$\textit{or } f(\xi_1) - f(\xi_2) < -2\varepsilon,\ f(\xi_3) - f(\xi_4) < -2\varepsilon \tag{26}$$

Proof: The necessity has been shown in theorem 1. Now we prove these conditions are sufficient.

For arbitrary two adjacent subintervals $[S_{j-1}, S_j]$ and $[S_j, S_{j+1}]$, without losing generality, assume there exists $S_{j-1} \leq \xi_1 < \xi_2 \leq S_j$, such that

$$f(\xi_1) - f(\xi_2) > 2\varepsilon \tag{27}$$

With (25), there does not exist $S_{j-1} \leq \xi_{j1} < \xi_{j2} \leq S_j$, such that

$$f(\xi_{j1}) - f(\xi_{j2}) < -2\varepsilon \tag{28}$$

which means $f(x_1) - f(x_2) \geq -2\varepsilon$ holds for arbitrary $x_1 < x_2 \in [S_{j-1}, S_j]$. Then the output $F(x)$ of fuzzy system on $[S_{j-1}, S_j]$ can be constructed using the first way in lemma 6.

With (26) and (27), there does not exist $S_j \leq \xi_3 < \xi_4 \leq S_{j+1}$, such that

$$f(\xi_3) - f(\xi_4) > 2\varepsilon \tag{29}$$

which means $f(x_1) - f(x_2) \leq 2\varepsilon$ holds for arbitrary $x_1 < x_2 \in [S_j, S_{j+1}]$. Then the output $F(x)$ of fuzzy system on $[S_j, S_{j+1}]$ can be constructed using the second way in lemma 6.

Furthermore, the value of the end point in the first way in lemma 6 is $F(S_j) = f(S_j) - \varepsilon$, which equals to the value of the start point in the second way. Therefore, a continuous output for fuzzy systems can be constructed using the constructive methods in lemma 6 alternately, which means that there exists a fuzzy system to achieve the approximation (4). ■

Remark 2: According to theorem 2, the following assumption is required to found the necessary and sufficient conditions for general SISO Mamdani fuzzy systems on a given approximation accuracy:

(1) Approximation accuracy $\varepsilon \geq 0$;

(2) The values of the desired function $f(x) \in C[a, b]$ at extrema ξ_i: $v_i = f(\xi_i)$, $i = 1, 2, \cdots, m$;

(3) The values of $f(x)$ at endpoints: $v_0 = f(a)$, $v_{m+1} = f(b)$.

The prerequisite available data about the desired function is probably the minimum amount of information for characterizing a well-behaved continuous function. Furthermore, such information can be gained easily in practice, which means the results derived in the paper can be used in many applications.

Remark 3: When approximation accuracy $\varepsilon \to 0$, for arbitrary extreme ξ_{i-1}, ξ_i, ξ_{i+1} , $i = 1, 2, \cdots, m$, we have

$$\begin{aligned} f(\xi_{i-1}) - f(\xi_i) > 2\varepsilon,\ f(\xi_i) - f(\xi_{i+1}) < -2\varepsilon \\ or\ f(\xi_{i-1}) - f(\xi_i) < -2\varepsilon,\ f(\xi_i) - f(\xi_{i+1}) > 2\varepsilon \end{aligned} \tag{30}$$

According to theorem 2, the input interval $[a, b]$ must be divided into at least $m + 1$ subintervals, the number of the fuzzy rules is $m + 2$. This is just the necessary conditions for fuzzy systems as universal approximators found in [19]. As a result, the approach proposed in [19] is a special case of our necessary conditions when ε approaches zero.

IV. Illustrative Examples

Example: Design a SISO Mamdani fuzzy system to uniformly approximate the continuous function $f(x)$ shown in Fig. 3 with error bound $\varepsilon = 0.08$ and $\varepsilon = 0.18$.

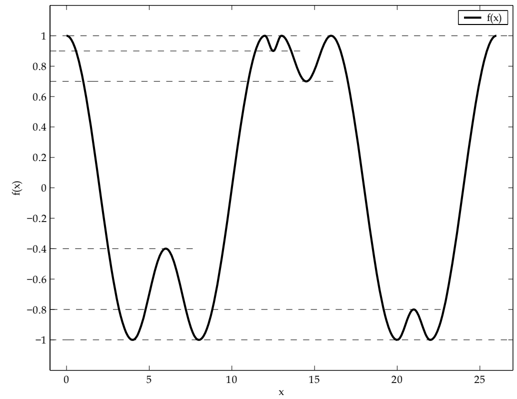

Fig. 3. The desired function$f(x)$ in the example.

(1) $\varepsilon = 0.08$

The upper and lower boundaries of $f(x)$ is obtained by moving it vertically by 0.08 upwards and downwards (the dashed lines in Fig. 4). A fuzzy system's output, whose monotonicity changes 9 times, is constructed in the tunnel encircled by the boundaries to approximate $f(x)$ (the dotted line in Fig. 4). Therefore, 11 fuzzy rules are necessary for achieving accuracy.

(2) $\varepsilon = 0.18$

The upper and lower boundaries of $f(x)$ is obtained by moving it vertically by 0.18 upwards and downwards (the dashed lines in Fig. 5). A fuzzy system's output, whose monotonicity changes 5 times, is constructed in the tunnel encircled by the boundaries to approximate $f(x)$ (the dotted line in Fig. 5). Therefore, 7 fuzzy rules are necessary for achieving accuracy.

With theorem 2, the strength as well as limitation of the general SISO Mamdani fuzzy systems as function approximators is showed in the example. We observe that the minimal number of needed fuzzy rules, M, increases with increasing number of extreme, whose swings are larger than

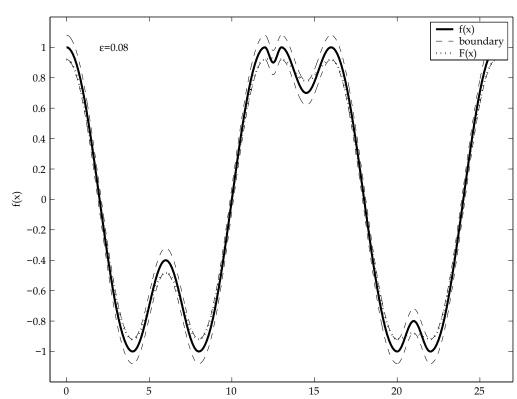

Fig. 4. An output $F(x)$ for fuzzy systems is constructed to approximate $f(x)$ with accuracy 0.08.

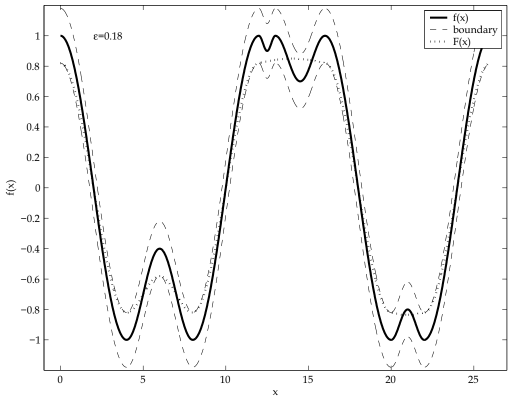

Fig. 5. An output $F(x)$ for fuzzy systems is constructed to approximate $f(x)$ with accuracy 0.18.

the approximate accuracy, of the desired continuous function, Ω. That is, although the desired function has a complicated formulation, if Ω is a small number, a small number of fuzzy rules suffice to achieve the approximation. This also means that fuzzy systems are not ideal function approximators for applications involving highly oscillatory continuous functions whose swings are larger than the approximate accuracy.

V. Conclusion

In the paper, necessary and sufficient conditions for general SISO Mamdani fuzzy systems under a given approximation accuracy have been studied. Based on the extreme of the object function, the necessary conditions are established firstly according to the monotonicity of the fuzzy system. Then these conditions are constructively proved to be sufficient. Because the assumption imposed on the object function is minimal and realistic, the necessary and sufficient conditions derived in the paper can be used in many applications. At the same time, the strength and the limitation of general SISO Mamdani fuzzy systems as function approximators are also revealed, which have significant theoretical and practical implications on fuzzy control and fuzzy modeling.

Yet, in this stage the conclusions are limited to the general SISO Mamdani case. Whether the necessary conditions obtained in this paper are applicable to multiple-input single-output (MISO) Mamdani fuzzy systems and Takagi-Sugeno (TS) fuzzy systems is an open question and need further research. Another side, the general SISO Mamdani fuzzy systems usually have complicated membership function, which is impossible in many realistic applications. So investigating the necessary conditions for fuzzy systems on a given membership function is another research area.

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