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An Algebraic Approach to Calculating Stabilities in the Graph Model with Strength of Preference

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***Abstract*— An algebraic approach is developed to calculate stabilities in two decision maker graph models with strength of preference. The original graph model uses "simple preference" to represent a decision maker's relative preference between two states. This preference structure includes only a relative preference relation and an indifference relation. Basic stability definitions, and algorithms to calculate them, assume simple preference. But difficulties in coding the algorithms, mainly because of their logical formulation, led to the development of matrix representations of preference and explicit matrix algorithms to calculate stability. Here, the algebraic approach is extended to representation of strength-of-preference graph models, which feature multiple levels of preference, and stability analysis for such models. Matrix representation of stability definitions facilitates the development of new stability concepts and algorithms to calculate them. The method is illustrated using a simple model of a conflict over sustainable development.**

I. INTRODUCTION

A strategic conflict is a situation in which two or more decision-makers (DMs) have to make independent choices in face of differing preferences about possible outcomes for the DMs. Among the formal methodologies that handle strategic conflict, the Graph Model for Conflict Resolution (GMCR) [1], [9] provides a remarkable combination of simplicity and flexibility. A graph model is a structure $G\langle N, S, \{\succeq_i, A_i, i \in N\}\rangle$, where

- N is a non-empty finite set, called the set of DMs.
- S is a non-empty finite set, called the set of states.
- For each DM $i \in N$, $\succeq_i$ is a reflexive and complete binary relation on S, called i's weak preference.
- For each DM i, $A_i \subseteq S \times S$ is DM i's oriented arcs, representing *unilateral moves* by DM i, and $G_i = (S, A_i)$ is i's directed graph.

A state is stable for a DM if the DM has no incentive to move away from it unilaterally. Many stability definitions (solution concepts) have been proposed for graph models, reflecting the many possible patterns of reasoning of a DM [1]. Four basic definitions, consisting of Nash stability [10], [11], general metarationality (GMR) [8], symmetric metarationality (SMR) [8], and sequential stability (SEQ) [4] have been widely employed in practice. A possible resolution or equilibrium of a graph model is a state that all DMs find stable under an appropriate stability definition.

Preference information plays an important role in stability analysis. In the original graph model, a preference framework called "simple preference" [1] provides decision makers with a strict preference relation $\succ$ and an indifference (or equal) relation $\sim$. The triplet of relations on S, $\{\gg_i, >_i, \sim_i\}$, that takes strength of preference (strong or mild) into account, was developed by Hamouda et al. [5], [6], and is used in this paper. The preference structure is referred to as "strength of preference". For $s, q \in S$, $s \gg_i q$ denotes DM i strongly prefers state s to state q, $s >_i q$ means DM i mildly prefers s to q, and $s \sim_i q$ indicates that DM i is indifferent between states s and q. If for any states k, s, and q, $k \succ s$ and $s \succ q$ imply $k \succ q$, then the preference $\succ$ is transitive. Otherwise, preference is intransitive. Similarly, one can define transitive relation for mild preference $>$ and strong preference $\gg$. In this paper, transitivity of preferences is not required, so that all of the results hold when the preferences are transitive or intransitive.

Previously, solution concepts were defined logically, in terms of the underlying graphs. However, procedures to identify stable states based on these definitions are difficult to code because of the nature of the logical representations. GMCR II, a decision support system [2], [3], is available for basic stability analysis within the simple preference framework, but is difficult to modify or adapt to preference with strength. Matrix representation of solution concepts (MRSC) has shown to be feasible for four basic graph model stability definitions in simple preference structure [12], [13]. The explicit algebraic formulations allow algorithms to rapidly assess the stabilities of states, and to be applied to large and complicated conflict models. In this paper, the algebraic approach is expanded to strength of preference for capturing preference with strength and calculating potential resolutions in two DM conflicts. Because of the nature of explicit expressions, the algebraic approach is more flexible and easier to modify for new solution concepts and contexts.

The rest of the paper is organized as follows. Logical representation of the four basic solution concepts for strength of preference is shown in Section II. Next, matrix representation of stability definitions is introduced in Section III. In Section IV, the proposed algebraic approach is used to analyze a conflict between environmentalists and developers

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with strength of preference. The paper concludes with some comments in Section V.

II. Logical Representation of Stabilities in the Two Decision Maker Graph Model with Strength of Preference

A. Subsets of State Set S and Reachable Lists

Let $s, q \in S$ and $N = \{i, j\}$. The state set S is divided into a set of subsets based on strength of preference. For $i \in N$, the description of the subsets of S is presented in Table I.

TABLE I
Subsets of S for DM i for strength of preference [5]

Subsets of S	Description
$\Phi_i^{++}(s) = \{q : q \gg_i s\}$	States that DM i strongly prefers to s
$\Phi_i^{+}(s) = \{q : q >_i s\}$	States that DM i mildly prefers to s
$\Phi_i^{=}(s) = \{q : q \sim_i s\}$	States that are indifferent to s for DM i
$\Phi_i^{-}(s) = \{q : s >_i q\}$	States that DM i mildly less prefers than s
$\Phi_i^{--}(s) = \{q : s \gg_i q\}$	States that DM i strongly less prefers than s

Based on different preference structures, DM i can identify different subsets of S. As listed in Table I, for preference with strength, DM i can identify five subsets of S: $\Phi_i^{++}(s)$, $\Phi_i^{+}(s)$, $\Phi_i^{=}(s)$, $\Phi_i^{-}(s)$, and $\Phi_i^{--}(s)$.

The set $R_i(s)$ denotes DM $i's$ reachable list from state s, containing all states to which DM i can move from state s in one step and represents DM $i's$ unilateral moves (UMs). Let $\cup$ denote the union operation and $\cap$ indicate the intersection operation. For different preferences, DM i can control different unilateral movements that result in a set of reachable lists from state s by DM i. The relations among the subsets of S and the reachable lists from state s for DM i are presented in Table II.

For ease of use, some additional notation with respect to strength of preference is presented as follows:

TABLE II
Unilateral movements for DM i within appropriate preference structures [5]

Type of movements	Description
$R_i^{++}(s) = R_i(s) \cap \Phi_i^{++}(s)$	All strong unilateral improvements from state s for DM i
$R_i^{+}(s) = R_i(s) \cap \Phi_i^{+}(s)$	All mild unilateral improvements from state s for DM i
$R_i^{=}(s) = R_i(s) \cap \Phi_i^{=}(s)$	All equally preferred states reachable from state s by DM i
$R_i^{-}(s) = R_i(s) \cap \Phi_i^{-}(s)$	All mild unilateral disimprovements from state s for DM i
$R_i^{--}(s) = R_i(s) \cap \Phi_i^{--}(s)$	All strong unilateral disimprovements from state s for DM i

- $R_i^{+,++}(s) = R_i^{+}(s) \cup R_i^{++}(s)$ depicts DM i's mild improvements or strong improvements called *weak improvements* (WIs) from state s; and
- $\Phi_i^{--,-,=}(s) = \Phi_i^{--}(s) \cup \Phi_i^{-}(s) \cup \Phi_i^{=}(s)$.

B. Stabilities in the Graph Model with Strength of Preference

After strength of preference is introduced into GMCR for two DM conflicts, solution concepts can be defined for standard stabilities, strong stabilities, and weak stabilities based on the level of sanctioning [5]. The stabilities using logical representation are shown as follows [5].

*1) **Standard Stabilities in the Graph Model for Strength of Preference**:*

Definition 2.1: State s is Nash stable (Nash) for DM i, denoted by $s \in S_i^{Nash}$, iff $R_i^{+,++}(s) = \emptyset$.

Definition 2.2: State s is general metarational (GMR) for DM i, denoted by $s \in S_i^{GMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \Phi_i^{--,-,=}(s)$.

Definition 2.3: State s is symmetric metarational (SMR) for DM i, denoted by $s \in S_i^{SMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j(s_1)$, such that $s_2 \in \Phi_i^{--,-,=}(s)$ and $s_3 \in \Phi_i^{--,-,=}(s)$ for any $s_3 \in R_i(s_2)$.

Definition 2.4: State s is sequentially stable (SEQ) for DM i, denoted by $s \in S_i^{SEQ}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j^{+,++}(s_1)$ with $s_2 \in \Phi_i^{--,-,=}(s)$.

According to the strength of the possible sanction, states can be strongly or weakly stable. Strong and weak stabilities are only given for GMR, SMR, and SEQ because Nash stability does not involve sanctions.

*2) **Strong Stabilities in the Graph Model for Strength of Preference**:*

Definition 2.5: State s is strongly general metarational (SGMR) for DM i, denoted by $s \in S_i^{SGMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.

Definition 2.6: State s is strongly symmetric metarational (SSMR) for DM i, denoted by $s \in S_i^{SSMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j(s_1)$, such that $s_2 \in \Phi_i^{--}(s)$ and $s_3 \in \Phi_i^{--}(s)$ for all $s_3 \in R_i(s_2)$.

Definition 2.7: State s is strongly sequentially stable (SSEQ) for DM i, denoted by $s \in S_i^{SSEQ}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j^{+,++}(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.

*3) **Weak Stabilities in the Graph Model for Strength of Preference**:* A state is weakly stable iff it is standard stable but not strongly stable.

III. Matrix Representation of Stabilities in the Two Decision Maker Graph Model for Strength of Preference

A. Matrix Representation of Strength of Preference

The set $R_i(s)$ denotes DM $i's$ reachable list from a state s and represents DM $i's$ unilateral moves (UMs). Let $m = |S|$ be the number of the states in S. J_i is an $m \times m$ 0-1 matrix defined by

$$J_i(s,q) = \begin{cases} 1 & if\ q \in R_i(s), \\ 0 & otherwise. \end{cases}$$

Then J_i is called a unilateral movement (UM) matrix or reachable matrix. If $R_i(s)$ is written as a 0-1 row vector, then

$$R_i(s) = e_s^T \cdot J_i,$$

where e_s^T denotes the transpose of the s^{th} standard basis vector of the m-dimensional Euclidean space. For DM i, a WI matrix $J_i^{+,++}$ is defined by

$$J_i^{+,++}(s,q) = \begin{cases} 1 & if\ q \in R_i^{+,++}(s), \\ 0 & otherwise. \end{cases}$$

Obviously, $R_i^{+,++}(s) = e_s^T \cdot J_i^{+,++}$, if $R_i^{+}(s)$ is written as a 0-1 row vector.

The UM and WI matrices depict DM i's movements in one step. To carry out a stability analysis, a set of matrices corresponding to strength of preference is constructed next.

Let E represent an $m \times m$ matrix with each entry being 1. For two $m \times m$ matrices, M and G, let $M(s,q)$ and $G(s,q)$ denote the (s,q) entries of matrices M and G, respectively. Then, $W = M \circ G$ is defined as the $m \times m$ matrix with (s,q) entry $W(s,q) = M(s,q) \cdot G(s,q)$. ("$\circ$" denotes the Hadamard product.) $H = M \vee G$ is defined as the $m \times m$ matrix with (s,q) entry

$$H(s,q) = \begin{cases} 1 & if\ M(s,q) + G(s,q) \neq 0, \\ 0 & otherwise. \end{cases}$$

The sign function, $sign(M)$, is defined as the $m \times m$ matrix with (s,q) entry

$$sign[M(s,q)] = \begin{cases} 1 & M(s,q) > 0, \\ 0 & M(s,q) = 0, \\ -1 & M(s,q) < 0. \end{cases}$$

Below, several matrices representing strength of preference for DM i are defined.

$$P_i^{++}(s,q) = \begin{cases} 1 & if\ q \gg_i s, \\ 0 & otherwise, \end{cases}$$

and

$$P_i^{--}(s,q) = \begin{cases} 1 & if\ s \gg_i q, \\ 0 & otherwise. \end{cases}$$

Therefore, $(P_i^{++})^T = P_i^{--}$, where T denotes the transpose of a matrix.

$$P_i^{--,-,=}(s,q) = \begin{cases} 1 & if\ q \in \Phi_i^{--,-,=}(s), \\ 0 & otherwise, \end{cases}$$

and

$$P_i^{+,++}(s,q) = \begin{cases} 1 & if\ q \gg_i s\ or\ q >_i s, \\ 0 & otherwise. \end{cases}$$

For strength of preference, $P_i^{--,-,=}(s,q) = 1 - P_i^{+,++}(s,q)$ for $s, q \in S$ and $s \neq q$.

Based on the aforementioned definitions, for DM i, UM matrix J_i, WI matrix $J_i^{+,++}$, and preference matrix $P_i^{+,++}$ have the following relationship among them:

$$J_i^{+,++} = J_i \circ P_i^{+,++}.$$

B. Matrix Representation of Stabilities in a Two DM Graph Model for Strength of Preference

Equivalent matrix representations of the aforementioned logical definitions for Nash stability, GMR, SMR, and SEQ in a two-DM graph model can be determined directly by using the relationship that has been established between matrix elements and the state set of a graph model, and by using preference relation matrices among the states.

Let $i \in N$, $|N| = 2$, and $m = |S|$.

Theorem 3.1: State s is Nash stable for DM i iff $e_s \cdot J_i^{+,++} \cdot e = 0$, where e denotes the m-dimensional column vector with each element being set to 1.

Define the $m \times m$ matrix M_i^{GMR} as

$$M_i^{GMR} = J_i^{+,++} \cdot [E - sign\left(J_j \cdot (P_i^{--,-,=})^T\right)].$$

Theorem 3.2: State s is GMR for DM i iff $M_i^{GMR}(s,s) = 0$.

Proof: Since $M_i^{GMR}(s,s) =$

$$(e_s^T \cdot J_i^{+,++}) \cdot [\left(E - sign\left(J_j \cdot (P_i^{--,-,=})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot J_j) \cdot (e_s^T \cdot P_i^{--,-,=})^T\right)],$$

then $M_i^{GMR}(s,s) = 0$ iff

$$J_i^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot J_j) \cdot (e_s^T \cdot P_i^{--,-,=})^T\right)] = 0,$$

$\forall s_1 \in S$. This implies that $M_i^{GMR}(s,s) = 0$ iff

$$(e_{s_1}^T \cdot J_j) \cdot (e_s^T \cdot P_i^{--,-,=})^T \neq 0, \forall s_1 \in R_i^{+,++}(s). \quad (1)$$

By (1), for any $s_1 \in R_i^{+,++}(s)$, there exists $s_2 \in S$ such that the m-dimensional row vector $e_{s_1}^T \cdot J_j$ has s_2^{th} element 1 and the m-dimensional column vector $(P_i^{--,-,=})^T \cdot e_s$ has s_2^{th} element 1.

Therefore, $M_i^{GMR}(s,s) = 0$ iff for any $s_1 \in R_i^{+,++}(s)$, there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \Phi_i^{--,-,=}(s)$.

Define the $m \times m$ matrix M_i^{SMR} as

$$M_i^{SMR} = J_i^{+,++} \cdot [E - sign(H)],$$

with

$$H = J_j \cdot [(P_i^{--,-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^{+,++})^T\right)\right)].$$

Theorem 3.3: State s is SMR for DM i iff $M_i^{SMR}(s,s) = 0$.

Proof: Let

$$G = (P_i^{--,-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^{+,++})^T\right)\right).$$

Since

$$M_i^{SMR}(s,s) = (e_s^T \cdot J_i^{+,++}) \cdot [(E - sign(H)) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s,s_1)[1 - sign\left(H(s_1,s)\right)]$$

with

$$H(s_1,s) = \sum_{s_2=1}^{m} J_j(s_1,s_2) \cdot G(s_2,s),$$

and $G(s_2, s) =$

$$P_i^{--,-,=}(s, s_2)[1 - sign\left(\sum_{s_3=1}^{m} \left(J_i(s_2, s_3)P_i^{+,+}(s, s_3)\right)\right)],$$

thus, $M_i^{SMR}(s, s) = 0$ holds iff $H(s_1, s) \neq 0$, $\forall s_1 \in R_i^{+,++}(s)$, which is equivalent to the statement that, $\forall s_1 \in R_i^{+,++}(s), \exists s_2 \in R_j(s_1)$ such that

$$P_i^{--,-,=}(s, s_2) \neq 0, \quad (2)$$

and

$$\sum_{s_3=1}^{m} \left(J_i(s_2, s_3)P_i^{+,++}(s, s_3)\right) = 0. \quad (3)$$

Obviously, for $\forall s_1 \in R_i^{+,++}(s), \exists s_2 \in R_j(s_1)$, statement (2) holds iff $s_2 \in \Phi_i^{--,-,=}(s)$. For $\forall s_1 \in R_i^{+,++}(s), \exists s_2 \in R_j(s_1)$, equation (3) holds iff for all $s_3 \in R_i(s_2)$, $P_i^{+,++}(s, s_3) = 0$ which implies $s_3 \in \Phi_i^{--,-,=}(s)$.

Therefore, $M_i^{SMR} = 0$ iff for every $s_1 \in R_i^{+,++}(s)$ there exists $s_2 \in R_j(s_1)$ such that $s_2 \in \Phi_i^{--,-,=}(s)$ and $s_3 \in \Phi_i^{--,-,=}(s)$ for all $s_3 \in R_i(s_2)$.

Define the $m \times m$ matrix M_i^{SEQ} as

$$M_i^{SEQ} = J_i^{+,++} \cdot [E - sign\left(J_j^{+,++} \cdot (P_i^{--,-,=})^T\right)].$$

Theorem 3.4: State s is SEQ for DM i iff $M_i^{SEQ}(s, s) = 0$.

Proof: Since $M_i^{SEQ}(s, s) =$

$$(e_s^T \cdot J_i^{+,++}) \cdot [\left(E - sign\left(J_j^{+,++} \cdot (P_i^{--,-,=})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s, s_1)[1 - sign\left((e_{s_1}^T \cdot J_j^{+,++}) \cdot (e_s^T \cdot P_i^{--,-,=})^T\right)],$$

then $M_i^{SEQ}(s, s) = 0$ iff for any $s_1 \in S$,

$$J_i^{+,++}(s, s_1)[1 - sign\left((e_{s_1}^T \cdot J_j^{+,++}) \cdot (e_s^T \cdot P_i^{--,-,=})^T\right)] = 0.$$

This implies that $M_i^{SEQ}(s, s) = 0$ iff

$$(e_{s_1}^T \cdot J_j^{+,++}) \cdot (e_s^T \cdot P_i^{--,-,=})^T \neq 0, \forall s_1 \in R_i^{+,++}(s). \quad (4)$$

By (4), for any $s_1 \in R_i^{+,++}(s)$, there exists $s_2 \in S$ such that the m-dimensional row vector $e_{s_1}^T \cdot J_j^{+,++}$ has s_2^{th} element 1 and the m-dimensional column vector $(P_i^{--,-,=})^T \cdot e_s$ has s_2^{th} element 1.

Therefore, $M_i^{SEQ}(s, s) = 0$ iff for any $s_1 \in R_i^{+,++}(s)$, there exists at least one $s_2 \in R_j^{+,++}(s_1)$ with $s_2 \in \Phi_i^{--,-,=}(s)$.

C. Matrix Representation of Strong Stabilities in the Graph Model with Strength of Preference

Let $i \in N$, $|N| = 2$, and $m = |S|$ in the subsection. Define the $m \times m$ matrix M_i^{SGMR}

$$M_i^{SGMR} = J_i^{+,++} \cdot [E - sign\left(J_j \cdot (P_i^{--})^T\right)].$$

Theorem 3.5: State $s \in S$ is strongly GMR ($SGMR$) for DM i iff

$$M_i^{GMR}(s, s) = 0.$$

Define the $m \times m$ matrix M_i^{SSMR}

$$M_i^{SSMR} = J_i^{+,++} \cdot [E - sign(J_j \cdot H)],$$

with

$$H = (P_i^{++}) \circ [E - sign\left(J_i \cdot (E - P_i^{++})\right)].$$

Theorem 3.6: State $s \in S$ is strongly symmetric metarational ($SSMR$) for DM i iff

$$M_i^{SSMR}(s, s) = 0.$$

Define the $m \times m$ matrix M_i^{SSEQ}

$$M_i^{SSEQ} = J_i^{+,++} \cdot [E - sign\left(J_j^{+,++} \cdot (P_i^{--})^T\right)].$$

Theorem 3.7: State $s \in S$ is strongly sequentially stable ($SSEQ$) for DM i iff

$$M_i^{SSEQ}(s, s) = 0.$$

The proofs of the aforementioned Theorems 3.5 to 3.7 are similar to those of the three standard stabilities.

D. Weak stabilities

In the following definition, the symbol SC denote a solution concept, GMR, SMR, or SEQ. Then SSC refers to the strong solution concept of SC, and WSC refers to the weak solution concept of SC (defined below). The symbol $s \in S_i^{SC}$ denotes that $s \in S$ is stable for DM i according to stability SC. Similarly, $s \in S_i^{SSC}$ denotes that $s \in S$ is strongly stable for DM i according to strong stability SSC. A state is weakly stable iff it is stable, but not strongly stable. The formal weak stability concept is defined next.

Definition 3.1: State s is weakly stable WSC for DM i according to stability SC, denoted by $s \in S_i^{WSC}$, iff $s \in S_i^{SC}$ and $s \notin S_i^{SSC}$.

Theorem 3.8: State $s \in S$ is weakly GMR for DM i iff

$$M_i^{GMR}(s, s) = 0 \; and \; M_i^{SGMR}(s, s) \neq 0.$$

Theorem 3.9: State $s \in S$ is weakly SMR for DM i iff

$$M_i^{SMR}(s, s) = 0 \; and \; M_i^{SSMR}(s, s) \neq 0.$$

Theorem 3.10: State $s \in S$ is weakly SEQ for DM i iff

$$M_i^{SEQ}(s, s) = 0 \; and \; M_i^{SSEQ}(s, s) \neq 0.$$

IV. APPLICATION

In this section, Theorems 3.1 to 3.10 are used to analyze a two-DM conflict over sustainable development. Hipel [7] developed a model for a conflict regarding sustainable development that was also studied by Hamouda et al. [5]. Specifically, the conflict consists of two DMs: environmental agencies (DM 1: E) and developers (DM 2: D); and a total of four options: DM 1 controls the two options of being proactive (labeled P) and being reactive (labeled R) in monitoring developers' activities and their impacts on the environment, and DM 2 has the two options of practicing sustainable development (labeled S) and practicing unsustainable development (labeled U) for properly treating the environment. These options are combined to form four feasible states: s_1: PS, s_2: PU, s_3: RS, and s_4: RU. The four options together four feasible states are listed in

Table III, where a “Y” indicates that an option is selected by the DM controlling it and an “N” means that the option is not chosen.

TABLE III
OPTIONS AND FEASIBLE STATES FOR THE SUSTAINABLE DEVELOPMENT CONFLICT

E: environmentalists				
1. Proactive (labeled P)	Y	Y	N	Y
2. Reactive (labeled R)	N	N	Y	Y
D: developers				
3. Sustainable development (labeled S)	Y	N	Y	N
4. Unsustainable development (labeled U)	N	Y	N	Y
State number	s_1	s_2	s_3	s_4

The graph model for each DM in this conflict is depicted in Fig. 1, where vertices designate states and arcs represent movement between states. The letter on a given arc indicates which DM controls the movement while the arrowhead shows the direction of movement. The preference information for each DM is:

DM 1: $s_1 >_1 s_3 \gg_1 s_2 \sim_1 s_4$;

DM 2: $s_3 >_2 s_1 \gg_2 s_4 \sim_2 s_2$.

DM 1 and DM 2’s preference information includes strength.

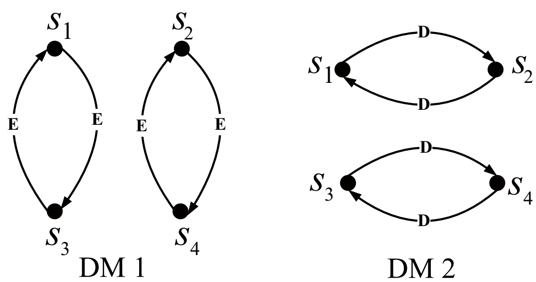

Fig. 1. Graph model for the sustainable development conflict.

The reachable matrices for DM 1 and DM 2 are

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The preference relation matrices for DMs 1 and 2 are given as:

$$P_1^{++} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, P_1^{+,++} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$P_2^{++} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, P_2^{+,++} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$J_i^{+,++} = J_i \circ P_i^{+,++}$ $P_i^{--,-,=} = E - I - P_i^{+,++}$, $P_i^{--} = (P_i^{++})^T$ for $i = 1, 2$.

The matrices introduced by Theorems 3.1 to 3.7 are included in Table IV, which are used to calculate the extended standard stabilities of Nash, GMR, SMR, and SEQ, as well as the strong stabilities of GMR, SMR, and SEQ, respectively.

TABLE IV
THE CONSTRUCTION OF STABILITY MATRICES

Types of stabilities	Stability matrices
Standard stabilities	$M_i^{Nash} = J_i^{+,++}$
	$M_i^{GMR} = J_i^{+,++} \cdot [E - sign\left(J_j \cdot (P_i^{--,-,=})^T\right)]$
	$M_i^{SMR} = J_i^{+,++} \cdot [E - sign(J_j \cdot H)]$, with $H = (P_i^{--,-,=})^T \circ [E - sign\left(J_i \cdot (P_i^{+,++})^T\right)]$
	$M_i^{SEQ} = J_i^{+,++} \cdot [E - sign\left(J_j^{+,++} \cdot (P_i^{--,-,=})^T\right)]$
Strong stabilities	$M_i^{SGMR} = J_i^{+,++} \cdot [E - sign\left(J_j \cdot (P_i^{--})^T\right)]$
	$M_i^{SSMR} = J_i^{+,++} \cdot [E - sign(J_j \cdot H)]$, with $H = (P_i^{++}) \circ [E - sign\left(J_i \cdot (E - P_i^{++})\right)]$
	$M_i^{SSEQ} = J_i^{+,++} \cdot [E - sign\left(J_j^{+,++} \cdot (P_i^{--})^T\right)]$

The stable states and equilibria for the sustainable development conflict are summarized in Table V, in which “$\surd$” for a given state means that this state is stable for DM 1 or DM 2 and “Eq” is an equilibrium for a corresponding solution concept.

The results provided by Table V show that state s_1 is strong equilibrium for the four basic stabilities. State s_3 is strongly stable for GMR and SMR stabilities. Hence, s_1 and s_3 are better choices for decision makers.

V. Conclusions and Future Work

A. *Conclusions*

In this paper, the matrix representation of stability definitions for graph models using simple preference is extended to produce a matrix method that represents several basic solution concepts for two-DM conflicts with strength of preference. The nature of the logical representations of solution concepts in the presence of strength of preference [5] makes them difficult to code, which may explain why algorithms for these solution concepts have not yet been integrated into a decision support system. The algebraic approach defined here handles this problem efficiently, and therefore facilitates the development of improved algorithms to assess the stabilities of states in graph models with strength of preference. The extended method offers the advantages of easy calculation and easy coding, and can be expected to stimulate further theoretical and empirical research.

B. *Future Work*

To demonstrate the efficiency of the new representations, it would be worthwhile to extend the matrix method introduced here to multiple-decision-maker conflicts with strength of preference. Then the method may be further extended to tackle more complex problems, such as models with combining strength and uncertainty for preferences [14].

TABLE V

STABILITY RESULT OF THE SUSTAINABLE DEVELOPMENT GAME WITH STRENGTH OF PREFERENCE

State	Nash			GMR			SMR			SEQ			SGMR			SSMR			SSEQ			WGMR			WSMR			WSEQ		
	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq
s_1	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√									
s_2	√			√			√			√			√			√			√											
s_3		√		√	√	√	√	√	√		√		√	√	√	√	√	√		√										
s_4	√			√			√			√			√			√			√											

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