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ELLIPTIC DISCRETE FOURIER TRANSFORMS OF TYPE II

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ABSTRACT

This paper presents a novel concept of the N-point elliptic DFT of type II (EDFT-II), by considering and generalizing the N-point DFT in the real space R^{2N}. In the definition of such Fourier transformation, the block-wise representation of the matrix of the DFT is reserved and the Givens transformations for multiplication by the twiddle coefficients are substituted by other basic transformations. The elliptic transformations are defined by different Nth roots of the identity matrix 2×2, whose groups of motion move the point $(1, 0)$ around ellipses. The elliptic DFTs of type II are parameterized by two vector-parameters, exist for any order N, and differ from the class of elliptic DFT of type I whose basic transformations are defined by *the elliptic matrix* $\cos(\varphi)I + \sin(\varphi)R$, where R is such a matrix that $R^2 = -I$ and I is the identity matrix 2×2. Examples of application of the proposed N-block EDFT-II in signal and image processing are given.

1. INTRODUCTION

Many fast unitary transforms such as the Fourier, Hadamard, and cosine transforms, are widely used in different areas including data compression, pattern recognition and image reconstruction, interpolation, linear filtering, and spectral analysis. Since the introduction of the fast algorithm of the discrete Fourier transform (DFT), the Fourier analysis has become one of the most frequently used tools in signal and image processing and communication systems, and different fast algorithms have been introduced [1]-[5]. We mention the fast method of graph simplification of the DFT by the paired transforms [6], which splits calculations in such a way that the real and imaginary parts of the transform on each stage of the algorithm are calculated in parallel and separately. This method was generalized in [7], by using the block-wise representation of the matrix of the N-point DFT in the real space R^{2N}. In the traditional decomposition of the signal by the DFT, N roots of the unit are used, which represent themselves the Givens rotations, or multiplications by the twiddle factors. These basic rotations T are substituted by different Nth roots of the identity matrix 2×2, which describe the movement of the point $(1, 0)$ around ellipses, when considering the group of motion generated by T.

In this paper, we present another class of elliptic matrices and N-point block elliptic discrete Fourier transforms (EDFT) and describe their properties and examples of application in signal and image processing. The proposed Fourier transforms are parameterized by any pair of vectors. These transforms can be used in filtering, spectral analysis of signals and images, image enhancement, encryption, and other applications.

2. DFT IN THE REAL SPACE

The N-point discrete Fourier transform is defined as the decomposition of the signal by N roots of the unit

$$W^k = W_N^k = e^{-\frac{2\pi j}{N}k} = c_k - js_k = \cos(\frac{2\pi}{N}k) - j\sin(\frac{2\pi}{N}k),$$

where $k = 0 : (N-1)$, which are located on the unit circle. The discrete Fourier transform of the vector $\mathbf{f} = (f_0, f_1, f_2, ..., f_{N-1})'$,

$$F_p = \sum_{n=0}^{N-1} W^{np} f_n, \quad p = 0 : (N-1),$$

in the complex space C^N has the following matrix:

$$[\mathcal{F}_N] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & & & & & \\ 1 & W^{N-1} & W^{N-2} & W^{N-3} & & W^1 \end{bmatrix}.$$

During the calculation of the transform, data on different stages of calculation are multiplied by the twiddle factors and in the fast DFT the number of these operations is reduced to a minimum.

We describe, in matrix form, the multiplication of the complex number $\mathbf{x}$, which is considered as the vector $(x_1, x_2)'$, or row-vector (x_1, x_2) by the twiddle coefficients W^k, $k = 0 : (N-1)$,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow W^k\mathbf{x} = \begin{pmatrix} c_kx_1 + s_kx_2 \\ c_kx_2 - s_kx_1 \end{pmatrix},$$

where $W^k = (c_k, -s_k) = (\cos\varphi_k, -\sin\varphi_k) = \cos\varphi_k - j\sin\varphi_k$, and the angles $\varphi_k = 2\pi k/N$. In matrix form, this multiplication can be written as

$$T^k\mathbf{x} = \begin{pmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The matrix of rotation by the angle $\varphi_k = k\varphi_1$ is denoted by T^k.

We transfer the complex line into the 2-D real space, $C \rightarrow R^2$, and consider each operation of multiplication by the twiddle coefficient as the elementary rotation, or the Givens transformation, $W^k \rightarrow T^k$, $k = 0 : (N-1)$. We next consider the transform of the complex space C^N into the real space R^{2N} by

$$\mathbf{f} = (f_0, f_1, ..., f_{N-1})' \rightarrow \bar{\mathbf{f}} = (r_0, i_0, r_1, i_1, ..., r_{N-1}, i_{N-1})',$$

where we denote $r_k = \mathrm{Re} f_k$ and $i_k = \mathrm{Im} f_k$ for $k = 0 : (N-1)$. The vector $\bar{\mathbf{f}}$ is composed from the original vector $\mathbf{f}$, and its vector-components are denoted by $\bar{\mathbf{f}}_k = (\bar{f}_{2k}, \bar{f}_{2k+1})' = (r_k, i_k)'$. The

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N-point DFT of $\mathbf{f}$ is thus represented in the real space R^{2N} as the $2N$-point transform

$$\bar{\mathbf{F}}_p = \begin{bmatrix} R_p \\ I_p \end{bmatrix} = \sum_{k=0}^{N-1} T^{kp}\bar{\mathbf{f}}_k = \sum_{k=0}^{N-1} T^{kp} \begin{bmatrix} r_k \\ i_k \end{bmatrix}, \qquad (1)$$

where $p = 0 : (N-1)$. In matrix form, the Fourier transformation in the space R^{2N} is described by the following matrix $2N \times 2N$:

$$X = \begin{bmatrix} I & I & I & I & I & I \\ I & T^1 & T^2 & T^3 & \cdots & T^{N-1} \\ I & T^2 & T^4 & T^6 & \cdots & T^{N-2} \\ I & \cdots & \cdots & \cdots & \cdots & \cdots \\ I & T^{N-1} & T^{N-2} & T^{N-3} & & T^1 \end{bmatrix}. \qquad (2)$$

The (n,p)-th blocks 2×2 of this matrix are defined as $X_{(n,p)} = T^{np}$, where $n, p = 0 : (N-1)$. In the real space R^{2N}, the DFT is linear as in the complex plane C^N. According to the definition, the rotation matrices T^k compose the one-parametric group with period N. In other words

$$T^{k_1+k_2} = T^{k_1}T^{k_2}, \quad (T^0 = T^N = I),$$

for any $k_1, k_2 = 0 : (N-1)$.

In the real space R^{2N}, we define the $2N$-point transform over the vector $\bar{\mathbf{f}} = (f_0, f_1, f_2, f_3, ..., f_{2N-2}, f_{2N-1})'$, as a transform with the matrix defined by (2), where T is a matrix 2×2 with determinant one. It is assumed also that this matrix defines a one-parametric group with period N. We call the transformation $X : \mathbf{f} \to \mathbf{F}$ *the T-generated N-block discrete transformation*, or *N-block T-GDT*. In the case when each pair (f_{2n}, f_{2n+1}) represents the complex component of the vector $\mathbf{f}$ and the matrix T is the matrix of the Givens rotation by angle $2\pi/N$, this definition leads to the N-point DFT extended into the real space R^{2N}. We call this Fourier transform *the N-block discrete Fourier transform.*

2.0.1. *Elliptic DFT of Type I*

The class of parameterized transformations which are called *elliptic-type I Fourier transformations* are defined in the following way [7]. Given an integer $N > 1$ and angle $\varphi = \varphi_N = 2\pi/N$, the following matrix is considered:

$$T = T(\varphi) = \begin{bmatrix} \cos\varphi & \cos\varphi - 1 \\ \cos\varphi + 1 & \cos\varphi \end{bmatrix} = \cos\varphi \cdot I + \sin\varphi \cdot R$$

where

$$R = R(\varphi) = \begin{bmatrix} 0 & -\tan(\varphi/2) \\ \cot(\varphi/2) & 0 \end{bmatrix}, \quad (\det R = 1).$$

This definition leads to the equality $T^N(\varphi) = I$ for any integer N. The matrix R satisfies the condition $R^2(\phi) = -I$. In general $R^2 = -I$ case, we call the matrix $T = T(\varphi) = \cos\varphi \cdot I + \sin\varphi \cdot R$ *the generalized elliptic matrix.* For example, for the case when $N = 7$ and $R = R(2\pi/7)$, we have the following:

$$T = T_{\frac{2\pi}{7}} = \begin{bmatrix} 0.6235 & -0.3765 \\ 1.6235 & 0.6235 \end{bmatrix}, \quad \det T = 1.$$

We now discuss another class of elliptic discrete Fourier transformations, or *the elliptic-type II discrete Fourier transformations*.

3. ELLIPTIC DFT OF TYPE II

For the new class of elliptic DFTs, the construction of Nth roots of the identity matrix is based on the specific projection operators. We first discuss such transforms for the $N = 2, 3, 4$, and 5 cases, and then give an analytical formula for the general $N > 2$ case.

A. Let $a = (\alpha_1, \alpha_2)$ and $b = (\beta_1, \beta_2)$ be two-dimensional vectors both with the norm one, $||a|| = ||b|| = 1$. We are looking for such a matrix H that satisfies the following equations:

$$Ha' = b', \qquad Hb' = a', \qquad (3)$$

which lead to the equation $H^2 = I$. The matrix H is considered in form $H = H(a,b) = v_1'a + v_2'b$, where v_1 and v_2 are two unknown vectors. It is not difficult to obtain from the system of equations (3) the following solution:

$$v_1' = \frac{1}{1-s^2}(b' - sa'), \quad v_2' = \frac{1}{1-s^2}(a' - sb'),$$

where s denotes the inner product $(a,b) = (b,a) = ab' = ba'$. For the $s = 1$ case, the matrix $H = \pm b'b$. Considering the $s \neq 1$ case, we obtain the following matrix:

$$H = \frac{1}{1-s^2}\left[(b'a + ab') - s(a'a + b'b)\right].$$

For instance, for the vectors $a = (1, -2)$ and $b = (3, -4)$, we obtain the matrix

$$H = H\left(\frac{a}{\sqrt{5}}, \frac{b}{5}\right) = \begin{bmatrix} -0.4472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix}, \ \det H = -1.$$

In general, when the vectors a and b are normalized, such a matrix has the form

$$H = \begin{bmatrix} h_1 & h_2 \\ h_2 & -h_1 \end{bmatrix} = \begin{bmatrix} \cos(w+u) & \sin(w+u) \\ \sin(w+u) & -\cos(w+u) \end{bmatrix},$$

considering that $a = (\cos w, \sin w)$ and $b = (\cos u, \sin u)$.

The matrix H can also be defined from equations in (3) in a different way. By adding and subtracting these equations, we obtain

$$H = \frac{1}{||y_1||}y_1'y_1 - \frac{1}{||y_2||}y_2'y_2 \qquad (4)$$

where $y_1 = a + b$ and $y_2 = a - b$ are two orthogonal vectors.

B. We now consider the $N = 3$ case. Let a_1, a_2, and a_3 be two-dimensional vectors each with the unit norm, $||a_k|| = 1$, $k = 1, 2, 3$. Let H be the matrix that satisfies the following conditions:

$$Ha_1' = a_2', \quad Ha_2' = a_3', \quad Ha_3' = a_1'. \qquad (5)$$

The matrix H is a solution of the equation $H^3 = I$. If the matrix H is considered in the form $H = v_1'a_1 + v_2'a_2 + v_3'a_3$, then we obtain the following matrix equation with unknown vectors v_1', v_2', and v_3' :

$$V \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 1 & s_{1,2} & s_{1,3} \\ s_{1,2} & 1 & s_{2,3} \\ s_{1,3} & s_{2,3} & 1 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} a_2' \\ a_3' \\ a_1' \end{bmatrix}$$

where $s_{m,n} = a_n a_m'$, for $n, m = 1, 2$, and 3. The determinant of the matrix V equals zero and we can use two vectors to define H. In other words, the matrix H can be found as a linear combination of any two independent vectors, and we will show one of such constructions. By adding and subtracting the first two equations in (5), we obtain the system of two equations

$$Hy_1' = y_2', \quad Hy_2' = -y_1' - y_2', \tag{6}$$

where $y_1 = a_1 - a_2$ and $y_2 = a_2 - a_3$. We therefore take the matrix H in the form $H = v_1'y_1 + v_2'y_2$ and consider $y_k = y_k/||y_k||$ for $k = 1, 2$. The required vectors are equal to

$$v_1' = \frac{1}{1-s^2}(y_2' + s(y_1' + y_2')),$$
$$v_2' = \frac{1}{1-s^2}(-sy_2' + (y_1' + y_2')),$$

where $s = s_{1,2} = (a_1, a_2) = a_1a_2'$. The matrix H thus equals

$$H = \frac{1}{1-s^2}\left[(y_2' + s(y_1' + y_2'))y_1 - (sy_2' - (y_1' + y_2'))y_2\right].$$

For example, when vectors $a_1 = (1, 2)$, $a_2 = (6, 5)$, and $a_3 = (-1, -2)$, we obtain the matrix

$$H = H\left(\frac{a_1}{\sqrt{5}}, \frac{a_2}{\sqrt{89}}, \frac{a_3}{\sqrt{5}}\right) = \begin{bmatrix} -2.3806 & 2.5932 \\ -1.6530 & 1.3806 \end{bmatrix},$$

such that $\det H = 1$, $H^3 = I$ and $I + H + H^2 = 0$. In the real space R^6, the block transform is described by the matrix 6×6,

$$X = \begin{bmatrix} I & I & I \\ I & H^1 & H^2 \\ I & H^2 & H^1 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -2.3806 & 2.5932 & 1.3806 & -2.5932 \\ 0 & 1 & -1.6530 & 1.3806 & 1.6530 & -2.3806 \\ 1 & 0 & 1.3806 & -2.5932 & -2.3806 & 2.5932 \\ 0 & 1 & 1.6530 & -2.3806 & -1.6530 & 1.3806 \end{bmatrix},$$

such that $\det(X) = 27$ and $X^4 = 9I$, as for the matrix of the three-point DFT.

We now consider the representation of the matrix H by projection operators. This method will be generalized later for the $N \geq 2$ case. Let $y_1 = a_1/||a_1||$ and $y_2 = a_2/||a_2||$, and let $P_{n,m}$, where $n, m = 1, 2$, be the following matrices 2×2:

$$P_{1,1} = y_1'y_1,\ P_{1,2} = y_1'y_2,\ P_{2,1} = y_2'y_1,\ P_{2,2} = y_2'y_2.$$

Simple properties hold for these matrices:

$$P_{n,m}' = P_{m,n},\ \ P_{n,m}^2 = P_{n,m},\ \ P_{n,m}P_{p,k} = s^{|m-p|}P_{n,k}.$$

By using these properties, we can write the matrix H as

$$H = \frac{1}{1-s^2}\left(sP_{1,1} - P_{1,2} + (1+s)P_{2,1} - (s+1)P_{2,2}\right). \tag{7}$$

This matrix in the second equals

$$H^2 = \frac{1}{1-s^2}\left(-(s+1)P_{1,1} + (s+1)P_{1,2} - P_{2,1} + sP_{2,2}\right)$$

and $H^3 = I$. The identity matrix can also be represented by matrices $P_{n,m}$,

$$I = \frac{1}{1-s^2}\left(P_{1,1} + P_{2,2} - s(P_{1,2} + P_{2,1})\right).$$

In terms of the projection matrices, the matrix H for the $N = 2$ case can be written as

$$H = \frac{1}{1-s^2}\left(P_{2,1} + P_{1,2} - s(P_{1,1} + P_{2,2})\right). \tag{8}$$

C. The 4th root of the identity matrix can be found similarly,

$$H(a,b) = \frac{1}{1-s^2}\left(sP_{1,1} - P_{1,2} + P_{2,1} - sP_{2,2}\right).$$

D. The 5th root of the identity matrix equals

$$H(a,b) = \frac{1}{1-s^2}\left(h_{1,1}P_{1,1} + h_{1,2}P_{1,2} + h_{2,1}P_{2,1} + h_{2,3}P_{2,2}\right),$$

$$h_{1,1} = -sx,\ \ h_{1,2} = x,\ \ h_{2,1} = -sy + 1,\ \ h_{2,2} = y - s,$$

and the numbers x and y are calculated as

$$x = ky^2 = \frac{-3 \pm \sqrt{5}}{2}y^2, \quad y = (k(2k+1))^{\frac{1}{5}}.$$

For example, when vectors $a_1 = (1, 2)$ and $a_2 = (3, 4)$, we obtain the matrix

$$H = H\left(\frac{a_1}{\sqrt{5}}, \frac{a_2}{5}\right) = \begin{bmatrix} 0.4603 & -0.4407 \\ 2.1044 & 0.1578 \end{bmatrix},$$

such that $H^5 = I$ and $I + H + H^2 + H^3 + H^4 = 0$.

We now consider the general $N > 2$ case and denote $s = s_{1,2}$. The Nth root of the identity matrix is defined as

$$\begin{aligned} H &= H(a_1, a_2) = H(y_1, y_2) \\ &= \frac{1}{1-s^2}\Big[s(P_{1,1} - P_{2,2}) - P_{1,2} + P_{2,1} + \\ &\qquad +2\cos(\frac{2\pi}{N})(P_{2,2} - sP_{2,1})\Big]. \end{aligned} \tag{9}$$

This matrix also can be written in the form of

$$H = H(a_1, a_2) = S + 2\cos(\frac{2\pi}{N})Q, \tag{10}$$

where the matrices S and Q are defined as

$$\begin{aligned} S &= \frac{1}{1-s^2}[s(P_{1,1} - P_{2,2}) - P_{1,2} + P_{2,1}], \\ Q &= \frac{1}{1-s^2}[P_{2,2} - sP_{2,1}]. \end{aligned} \tag{11}$$

It is not difficult to show, that the matrices S and Q possess the following properties:

$$S^2 = -I, \quad Q^2 = Q, \quad QS + SQ = S.$$

As an example, consider two vectors $a_1 = (-1, 2)$ and $a_2 = (3, 4)$. For the $N = 15$ case, the matrix H is equal to

$$H = H\left(\frac{a_1}{\sqrt{5}}, \frac{a_2}{5}\right) = \begin{bmatrix} 1.0068 & 1.1742 \\ -0.1483 & 0.8203 \end{bmatrix},$$

and $\det H = 1$, $H^{15} = I$, and $I + H + H^2 + ... + H^{14} = 0$. In this case, $s = 0.4472$ and the matrices S and Q are equal to

$$S = \begin{bmatrix} -0.0894 & 0.6261 \\ -1.6100 & 0.0894 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.6000 & 0.3000 \\ 0.8000 & 0.4000 \end{bmatrix}.$$

Figure 1 shows the successive movement of the unit circle O_1 by the groups of motion H^n, when $N = 0 : 15$. The unit circle is

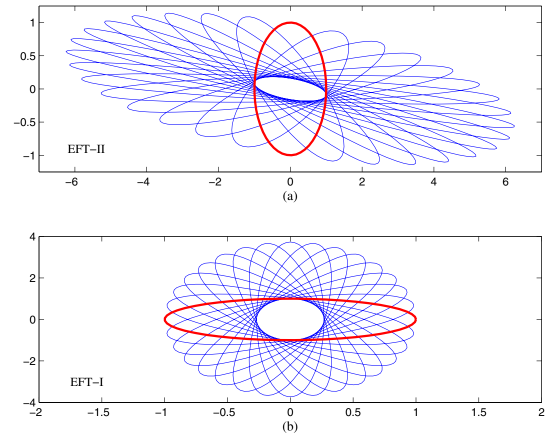

Fig. 1. The successive movement of the unit circle when applying the basic transformations of (a) the EDFT-II and (b) EDFT-I.

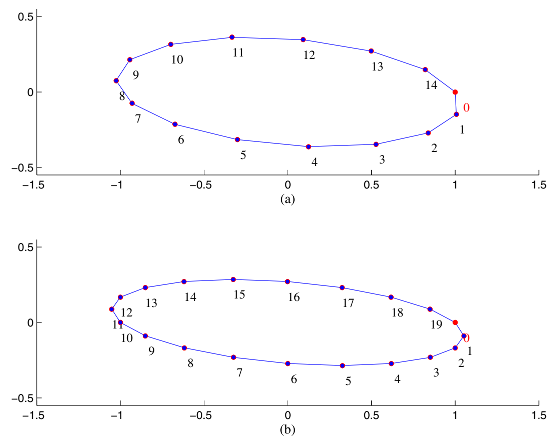

Fig. 2. The movement of the point $(1, 0)$ for the cases (a) $N = 15$ and (b) $N = 20$.

rotated and changes its form on each stage of movement. For comparison, the movement of the same circle when applying the group of motion of the EDFT of type I is shown in b. One can notice that all points of the unit circle are rotated around the corresponding ellipses. Figure 2 shows the movement of the point $(1, 0)$ by the groups of motion H^k, when $k = 0 : N$, for the $N = 15$ case in part a, along with the point movement for the $N = 20$ case in b.

The next figures show the movement of the cosine and sine waves when applying the elementary rotation and the elliptic transformation of type II. Figure 3 shows the full movement of the discrete-time sine wave $\sin(2\pi t)$ in the time interval $[0, 2\pi]$, when applying the group of rotation $\{W_{16}^k;\ k = 0 : 15\}$ in part a, along with the movement of the discrete-time cosine wave $\cos(2\pi t)$ in b. This is the case of the 16-point DFT when the rotation of waves is performed anti-clock wise. The movement of the same sine and cosine waves when applying sequentially the half of the group of rotation $\{W_{32}^k;\ k = 0 : 15\}$ are shown in c and d, respectively,

We now consider the movement of the cosine and sine waves when applying the EDFT of type II. The matrix H is the 16th

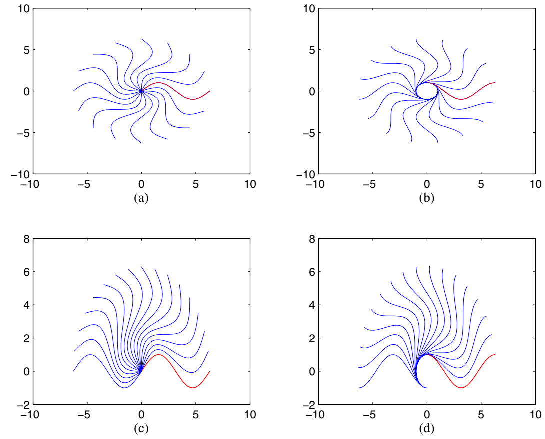

Fig. 3. The movement of the discrete-time (a) sine wave $\sin(2\pi t)$, (b) cosine wave $\cos(2\pi t)$ when rotating by matrices of W_{16}^k, $k = 0 : 16$, and (c) sine wave $\sin(2\pi t)$ and (d) cosine wave $\cos(2\pi t)$ when rotating by matrices of W_{32}^k, $k = 0 : 16$.

root of the identity matrix and is generated by two vectors $a_1 = (-1, 2)$ and $a_2 = (3, 4)$. Figure 4 shows the full movement of the discrete-time sine, $\sin(2\pi t)$, and cosine, $\cos(2\pi t)$, waves in parts a and b, respectively, when applying the half of the group of motion H^n, $n = 1 : 16$. This case corresponds to the 16-point EDFT. The movement of the same sine and cosine waves are shown in c and d, respectively, when applying the group of motion H_1^n, $n = 0 : 15$. The matrix H_1 is the 32th root of the identity matrix and is generated by the same vectors a_1 and a_2. These exam-

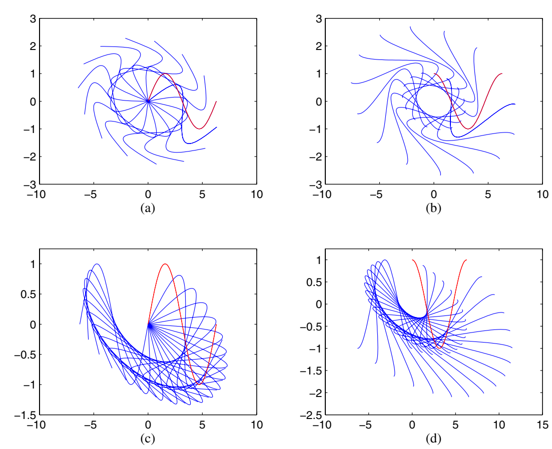

Fig. 4. The movement of the (a) sine wave $\sin(2\pi t)$ and (b) cosine wave $\cos(2\pi t)$ when applying sequentially the elliptic matrix of order 16, and (c) sine wave $\sin(2\pi t)$, (d) cosine wave $\cos(2\pi t)$ when applying sequentially the elliptic matrix of order 32.

ples demonstrate the complexity of movement of different waves when applying the groups of motion that correspond to the elliptic matrices. Such movement depends on the vector-generators a_1 and a_2.

4. EXPERIMENTAL RESULTS

It is interesting to know if the described elliptic discrete Fourier transforms of type II can distinguish the carrier frequencies of the cosine or sine waves as the DFT does, or better. As shown [7], the EDFTs of type I possess this property, and the imaginary part of these transforms is more sensitive to such frequencies than that for the DFT. The EDFTs of type II are also able to distinguish the carrier frequencies of the waves in a different degree which depends on the vector-generators a_1 and a_2. To show that, we consider the example of the discrete wave sampled from the signal

$$x(t) = \cos(\omega t) + 0.4\cos(4\omega t) - 0.2\cos(16t\omega - 0.4),$$

where the low frequency $\omega = \pi/32$ and the time t runs $N = 64$ points in the interval $[0, 2\pi]$. Figure 5 shows this signal in part a, along with the real part of the 64-block EDFT in b. The time- and frequency-points are shown in the interval $[0, 2\pi]$. The vectors a_1 and a_2 for this transform are $a_1 = (1, 2)$ and $a_2 = (3, 4)$. One

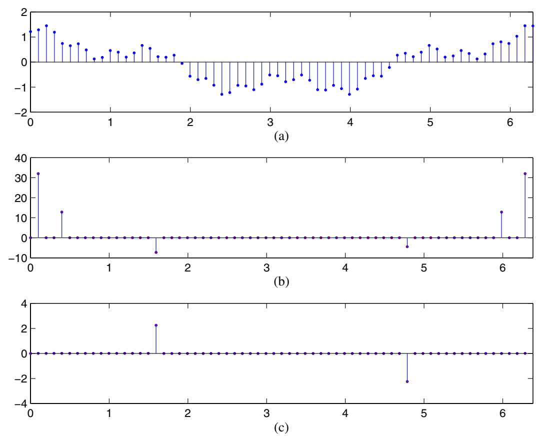

Fig. 5. (a) Signal $x(t)$ and (b) real and (c) imaginary parts of the 64-block EDFT.

can notice that the EDFT has maximums at the carrying frequency-points $p = 1, 4$, and 16, as well as at frequency-points $N - p$, since the signal is real. The imaginary part of the EDFT is given in c and has a maximum at frequency-points $p = 16$ and $N - 16 = 48$.

Figure 6 shows the real and imaginary parts of another 64-block EDFT in parts a and b, respectively. The vectors a_1 and a_2 for this transform are $a_1 = (1, 2)$ and $a_2 = (12, 2)$. The basic elliptic matrix equals

$$H = H\left(\frac{a_1}{\sqrt{5}}, \frac{a_2}{\sqrt{148}}\right) = \begin{bmatrix} 1.4762 & 0.3647 \\ -0.6607 & 0.5142 \end{bmatrix}.$$

The amplitudes of the components of the EDFT at the frequency-points $p = 16$ and 48 become big. This transform distinguishes the carrier frequencies of the signal $x(t)$ better than the EDFT with vectors $a_1 = (1, 2)$ and $a_2 = (3, 4)$ and better than the DFT. Figure 7 shows the amplitude spectrum of the signal $x(t)$ in part a when using the 64-point EDFT by vectors $a_1 = (1, 2)$ and $a_2 = (12, 2)$. For comparison, the spectrum of the 64-point DFT is given b. The transforms have been shifted cyclicly to the center.

Figure 8 shows the real signal of length 256 in part a, along with the real parts of the 256-block DFT and EDFT of the signal in

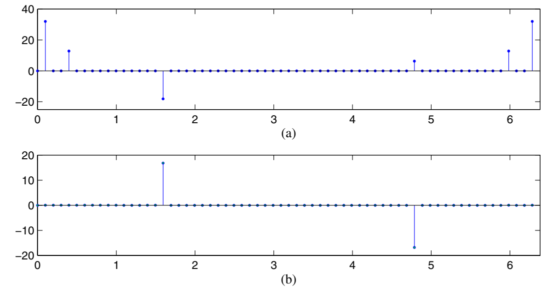

Fig. 6. (a) Real and (b) imaginary parts of the 64-block EDFT.

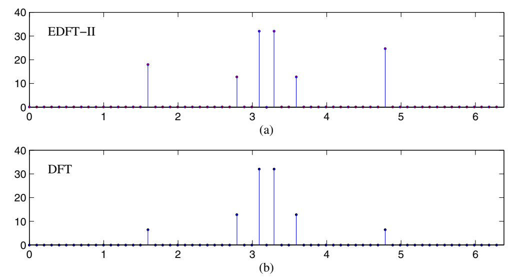

Fig. 7. The 64-block (a) EDFT and (b) DFT of the signal.

b and c, respectively. The transform is generated by vectors $a_1 = (3, 5)$ and $a_2 = (3, 4)$. These two vectors together are referred to

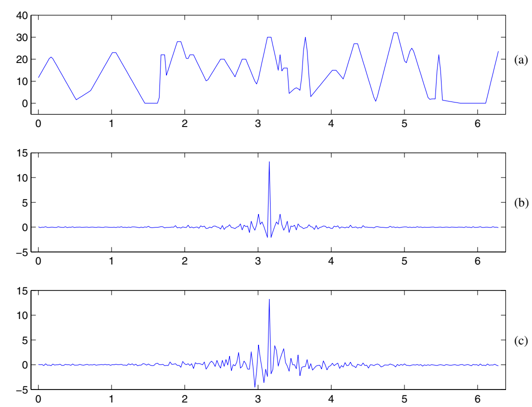

Fig. 8. (a) Signal of length 256 and the real parts of the (b) DFT and (c) EDFT of the signal.

as a key which allows for reconstructing the original signal from its spectrum represented by the elliptic DFTs. By changing this key we can vary the original form of the signal. As an example, Figure 9 shows the signals obtained from their elliptic DFTs by applying the traditional inverse DFT, instead of the inverse EDFTs. This is the case when assuming that the key-vectors a_1 and a_2 are unknown. The inverse DFT over the EDFT of Figure 8(c) is shown in a, and similar results are shown in b and c, when the 64-block EDFTs of the signals were generated by vectors $a_1 = (-1, 2)$ and $a_2 = (3, 4)$ and $a_1 = (1, 2)$ and $a_2 = (-12, 5)$, respectively.

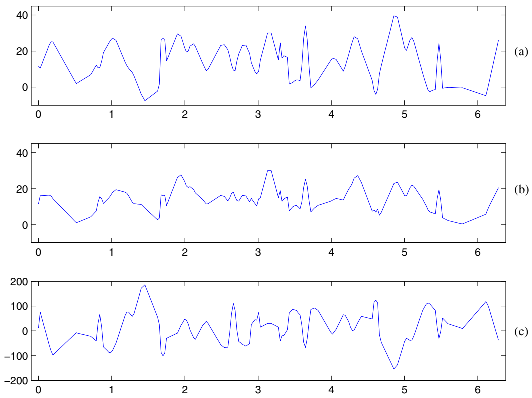

Fig. 9. The inverse DFT of the 64-block EDFTs generated by vectors (a) $(3, 5)$ and $(3, 4)$, (b) $(-1, 2)$ and $(3, 4)$, and (c) $(1, 2)$ and $(-12, 5)$.

The EDFT with the same or different vector-generators can be used for representing and processing images in frequency domain. As an example, Figure 10 shows the tree image in part a, along with the real and imaginary parts of 1-D EDFTs performed by rows in b and c, respectively. The amplitude spectrum of the 2-D EDFT calculated by 1-D EDFTs of the columns of the obtained data is shown in d. For all 1-D EDFT, in this example, the same

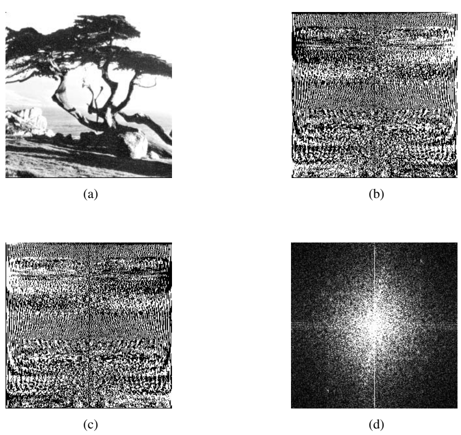

Fig. 10. (a) The tree image, and (b) real and (c) imaginary parts of 1-D EDFTs by rows, and (d) amplitude spectrum of the 2-D EDFT.

vectors $a_1 = (1, 2)$ and $a_2 = (4, 3)$ are used. The image in part a of Figure 11 shows the result of the inverse 2-D DFT of the 2-D EDFT, when assuming that these key-vectors are unknown. One can notice that the resulting image differs greatly from the original. It should be noted that, when calculating 1-D EDFT in the separable 2-D EDFT, different vector-generators a_1 and a_2 can be chosen for 1-D EDFTs. For instance, we consider the following key for processing the tree image:

$$\mathcal{K}_{256,(3)} = \{\underline{1, 2, 3, 4, 80}, \underline{3, 5, 3, 4, 160}, \underline{1, 7, 1, -3, 16}\}.$$

The first 80 rows will be processed by the 256-block EDFT with

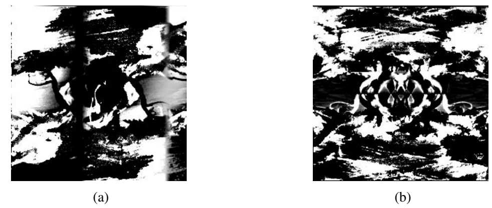

Fig. 11. The inverse 2-D DFTs of 2-D EDFTs of the tree image by (a) the vectors $(1, 2)$ and $(4, 3)$, and (b) key-vector $\mathcal{K}$.

vector-generators $(1, 2)$ and $(3, 4)$. The next 160 rows by the EDFT with vectors $(3, 5)$ and $(3, 4)$, and the remaining 16 rows by the EDFT with vectors $(1, 7)$ and $(1, -3)$. The same key can be used for processing all columns in the second stage of calculation of the 2-D EDFT. The result of processing the tree image is shown in b. It is clear that the probability of finding this key and obtaining the original image from the 2-D EDFT in b is almost zero, especially when the range of the vector-generators is large.

5. CONCLUSIONS

In this paper, we have generalized the concept of the N-point DFT, by introducing the block-type elliptic DFT of type II in the real space R^{2N}. The class of the 1-D EDFT is parameterized by two vectors. We believe that the proposed parameterized EDFT of type II can be used in different areas in signal and image processing, such as filtration, enhancement, and encryption.

6. REFERENCES

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