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Multiobjective Random Fuzzy Portfolio Selection Problems based on CAPM

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***Abstract*—Many researchers have proposed portfolio models based on the stochastic and fuzzy approaches until now, and there are some models considering both random and ambiguous conditions, particularly using fuzzy random or random fuzzy variables. However, few studies with multiobjective random fuzzy models for the portfolio selection problems have been performed. Therefore, a multiobjective random fuzzy portfolio selection problem based on CAPM, which is one of the most standard factor models, is proposed. In the sense of mathematical programming, our proposed problems are not well-defined problems due to randomness and fuzziness. Therefore, setting some criterions and introducing chance constrains, main problems are transformed into deterministic programming problems. Finally, we construct a solution method to obtain a global optimal solution of the problem.**

***Keywords*—portfolio selection, random fuzzy programming, CAPM, deterministic equivalent transformation**

I. Introduction

In recent rapid expansions of investment and financial instability, the role of investment theory becomes more and more important, and so it is time to review the investment theory. Of course, it is easy to decide the most suitable financial assets allocation if investors can receive reliable information with respect to future returns a priori. However, there exist many cases that uncertainty from social conditions has a great influence on the future returns. In the real market, there are random factors derived from statistical analysis of historical data and ambiguous factors such as the psychological aspect of investors and lack of received efficient information. Under such uncertainty situations, they need to consider how to reduce a risk, and it becomes important whether they receive the greatest future profit.

Such a finance assets selection problem is generally called a portfolio selection problem, and various studies have been done until now. Markowitz [21] first proposed the mean-variance model in the sense of mathematical programming. Then, it has been the centre of research activities in the real financial field and numerous researchers have contributed to the development of the modern portfolio theory (for instance, Luenberger [20], Steinbach [26]). On the other hand, many researchers have proposed models of portfolio selection problems which extended Markowitz model; mean-absolute-deviation model (Konno [15], Konno, et al. [16]), semi-variance model (Bawa and Lindenberg [1]), safety-first model (Elton and Gruber [5]), Value at Risk and conditional Value at Risk model (Rockafellar and Uryasev [24]), etc.. Particularly, Capital Asset Pricing Model (CAPM) which is a single factor model proposed by Sharpe [25], Lintner [17] and Mossin [23], has been one of the most useful tool in the investment fields and also used in the performance measure of future returns for portfolios and the asset pricing theory.

In such previous researches, expected future return and variance of each asset are assumed to be known. Then, in previous many studies in the sense of mathematical programming for the investment, future returns are assumed to be continuous random variables according to normal distributions. However, from recent experimental studies of investment markets, it is often shown that future returns do not occur according to normal distribution, but fat or heavy-tail distributions. Therefore, we need to consider portfolio selection problems with more general random distributions in the sense of mathematical programming.

Furthermore, considering efficient or inefficient received information, the institution of expert decision maker and the existence of marginal random distribution, we need to consider that statistical distribution under these conditions includes some ambiguity and some flexibility. In this paper, we propose more extensional portfolio selection models including not only randomness but also fuzzy factors. Until now, there are some basic researches under various uncertainty conditions with respect to portfolio selection problems (Bilbao-Terol et al. [2], Carlsson et al. [3], Guo and Tanaka [6], Huang [9, 10],

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Inuiguchi et al. [11, 12], Katagiri et al. [13, 14], Tanaka et al. [27], Watada [28]). We also proposed some portfolio models with both randomness and fuzziness [7, 8]. However, there are few models considering both general random distributions and fuzziness, simultaneously. Furthermore, there are no researches which are analytically extended and solved these types of portfolio selection problems.

Our proposal portfolio model is not a well-defined problem due to including randomness and fuzziness in the sense of deterministic mathematical programming. Therefore, in this paper, we introduce chance constraints and transform the main problem into deterministic equivalent problems. Consequently, we construct the analytical solution method of proposed portfolio selection problems to apply those of previous portfolio models.

This paper is organized as follows. In Section 2, we introduce the definition of random fuzzy programming. Then, in Section 3, we introduce a single factor model; CAPM, and formulate the multiobjective random fuzzy portfolio selection problem maximizing the total future return. In Section 4, in order to solve our proposed model analytically, we introduce the degree of possibility, perform deterministic equivalent transformations and develop the solution method. In Section 5, we provide the simple example and compare our proposed model with the standard model. Finally in Section 6, we conclude this paper and discuss future research problems.

II. RANDOM FUZZY PROGRAMMING

A. *Random fuzzy variable*

A random fuzzy variable was defined by Liu [18, 19]. He was established the mathematical basis. Since this article utilizes a simple one, we define random fuzzy variables based on the study of Liu [18, 19] as follows.

Definition 1

A random fuzzy variable is a function ξ from a possibility space $\left(\Theta, P(\Theta), \mathrm{Pos}\right)$ to collection of random variables R. An n-dimensional random fuzzy vector $\boldsymbol{\xi} = \left(\xi_1, \xi_2, ..., \xi_n\right)$ is an n-tuple of random fuzzy variables $\xi_1, \xi_2, ..., \xi_n$.

That is, a random fuzzy variable is a fuzzy set defined on a universal set of random variables.

B. *Random fuzzy programming problem*

Liu [18, 19] first considered a random fuzzy programming (RFP) problem. Generally speaking, since RFP problems are not well-defined problems, some decision making models can be considered to satisfy variety of preference of decision makers.

In order to construct a decision making model taking account of both random and fuzzy conditions, in this paper, we incorporate a possibilistic programming (PP) approach with the stochastic programming (SP) approach.

In SP, many researchers have proposed various models such as E-model (expectation optimization model), the V-model (variance minimization model), chance constrained model such P-model (probability maximization model) and F-model (probability fractile optimization model), etc.

Then, in possibilistic programming, the concepts of possibility and necessity measures (Zadeh [29]) were introduced to deal with the ambiguity included in the objective function and/or constraints. Dubois and Prade [4] proposed possibilistic programming and considered a degree of possibility or necessity that fuzzy goals for the objective function and/or constraints are satisfied. Inuiguchi and Ramik [11] viewed various types of PP models and developed modality constrained programming models.

On the other hand, random fuzzy variables are considered as the concepts of level 2 fuzzy sets. A level 2 fuzzy set is an extended version of fuzzy set in the respect that elements in the domain become fuzzy sets. Meanwhile, in the random fuzzy variable, the elements in the domain are random variables.

Most recently, some researchers have been introduced RFP models by incorporating PP models and SP models. In this paper, we focus on a multiobjective programming problem by incorporating model PP model and probability maximization model which is one of SP models.

III. MULTIOBJECTIVE RANDOM FUZZY PROGRAMMING PROBLEM

A. *Capital Asset Pricing Model*

First, in portfolio models, Capital Asset Pricing Model (CAPM) proposed by Sharpe [25], Lintner [17], and Mossin [23] has been used in many practical investment cases by not only researchers but also practical investors. The main advantage of CAPM is easily-handled since the relation between returns of each asset and market portfolio such as NASDAQ and TOPIX can be represented as the following linear formula;

$$r_j = d_j^1 + d_j^2 r_m$$

where r_m is the return of market portfolio. Then, d_j^1 and d_j^2 are inherent values derived from historical data in investment fields. However, market portfolio r_m is not entirely equal to NASDAQ and TOPIX, and so it is almost impossible to observe r_m exactly in the investment field. Furthermore, in the case that the decision maker predicts the future return using CAPM, it is obvious that market portfolio r_m also occurs according to a random distribution. Therefore, considering these situations, we propose a random fuzzy CAPM model which is assumed r_m to be a random fuzzy variable.

B. *Problem Formulation*

We mainly reconsider the following basic random fuzzy programming problem:

$$\text{Maximize } \sum_{j=1}^{n} \bar{\tilde{r}}_{ij} x_j,\ i = 1,2,...,k$$

$$\text{subject to } \boldsymbol{x} \in X \triangleq \left\{ \boldsymbol{x} \,\middle|\, \begin{array}{l} \sum_{j=1}^{n} x_j = 1, \\ 0 \le x_j \le b_j,\ j = 1,2,...,n \end{array} \right\} \tag{1}$$

where $\boldsymbol{x}$ is an n-dimensional decision variable column vector for the rate of asset allocations and b_j is an upper limited value for each decision variable. In this paper, we denote randomness and fuzziness of the coefficients by the "dash above" and "wave above", i.e., "－" and "~", respectively.

Subsequently, using CAPM, let us assume that coefficients of objective functions $\bar{\tilde{r}}_j = d_j^1 + \bar{\tilde{r}}_m d_j^2$ where d_j^1 and d_j^2 are constants, and $\bar{\tilde{r}}_m$ is a random variable with variance σ^2 and mean $\tilde{m}$ characterized by the following membership functions:

$$\mu_{\tilde{m}}(\xi) = \begin{cases} L\left(\dfrac{\bar{m} - \xi}{\alpha}\right) & (\bar{m} \ge \xi) \\ R\left(\dfrac{\xi - \bar{m}}{\beta}\right) & (\bar{m} < \xi) \end{cases} \tag{2}$$

where $L(x)$ and $R(x)$ are nonincreasing reference functions to satisfy $L(0) = R(0) = 1$, $L(1) = R(1) = 0$ and the parameters α and β represent the spreads corresponding to the left and the right sides, respectively, and both parameters are positive values. Then, the return coefficient $\bar{\tilde{r}}_j$ is a random fuzzy variable characterized by the following membership function:

$$\mu_{\bar{\tilde{r}}_j}(\bar{\gamma}_j) = \sup_{s_j} \left\{ \mu_{\tilde{M}_j}(s_j) \,\middle|\, \begin{array}{l} \bar{\gamma}_j \sim T_j(s_j, \sigma_j^2), \\ s_j = d_j^1 + \tilde{m} d_j^2 \end{array} \right\},\ \forall \bar{\gamma}_j \in \Gamma \tag{3}$$

where Γ is a universal set of normal random variable and $T_j(m_j, \sigma_j^2)$ is a probability density function for the random variable with mean m_j and variance σ_j^2 . From this membership function (3), objective function $Z = \sum_{j=1}^{n} \bar{\tilde{r}}_j x_j$ is defined as a random fuzzy variable characterized by the following membership function:

$$\mu_{\bar{\tilde{Z}}}(\bar{u}) = \sup_{\bar{\gamma}_j} \left\{ \min_{1 \le j \le n} \mu_{\bar{\tilde{r}}_j}(\bar{\gamma}_j) \,\middle|\, \begin{array}{l} \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j, \\ \bar{\gamma}_j \sim T_j(s_j, \sigma_j^2),\ s_j = d_j^1 + \tilde{m} d_j^2 \end{array} \right\}, \tag{4}$$

$$\forall \bar{u} \in Y \triangleq \left\{ \sum_{j=1}^{n} \bar{\gamma}_j x_j \,\middle|\, \bar{\gamma}_j \in \Gamma,\ j = 1,...,n \right\}$$

where $\bar{\gamma} = (\gamma_1, ..., \gamma_n)$. Furthermore, we discuss probabilities $\Pr\left\{ \omega \,\middle|\, \sum_{j=1}^{n} \bar{\tilde{r}}_j x_j \ge f \right\}$ satisfying that the objective function value is greater than or equal to an aspiration level f . Since $\sum_{j=1}^{n} \bar{\tilde{r}}_j x_j$ is represented with a random fuzzy variable, we express each probability $\Pr\left\{ \omega \,\middle|\, \sum_{j=1}^{n} \bar{\tilde{r}}_j x_j \ge f \right\}$ as a fuzzy set $\tilde{P}$ and define the membership function of $\tilde{P}$ as follows:

$$\begin{aligned} &\mu_{\tilde{P}}(p) \\ &= \sup_{\bar{u}} \left\{ \mu_Z(\bar{u}) \,\middle|\, p = \Pr\left\{ \omega \,\middle|\, \bar{u}(\omega) \ge f \right\} \right\} \\ &= \sup_{s} \min_{1 \le j \le n} \left\{ \mu_{\tilde{M}_j}(s_j) \,\middle|\, \begin{array}{l} p = \Pr\left\{ \omega \,\middle|\, \bar{u}(\omega) \ge f \right\},\ \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j, \\ \bar{\gamma}_j \sim T_j(s_j, \sigma_j^2),\ s_j = d_j^1 + \tilde{m} d_j^2 \end{array} \right\} \end{aligned} \tag{5}$$

Subsequently, since problem (1) is not a well-defined problem due to including random fuzzy variable returns, we need to set a criterion with respect to probability and possibility of future returns for the deterministic optimization. In general decision cases with respect to investment, an investor usually focused on maximizing either the goal of the total profit or that of achieve probability. Therefore, in this subsection, we propose a possibility maximization model for probability maximization model in random fuzzy programming problems.

IV. POSSIBILITY MAXIMIZATION MODEL FOR PROBABILITY MAXIMIZATION MODEL

In this subsection, we mainly consider the case that a decision maker sets the target value f and she or he considers a multi-criteria programming problem maximizing the probability such as the objective function is more than the target value f . First of all, we introduce a basic probability maximization model introducing the probability chance constraint as follows:

$$\begin{aligned} &\text{Maximize} \quad \tilde{P} = \Pr\left\{ \sum_{j=1}^{n} \bar{\tilde{r}}_j x_j \ge f \right\} \\ &\text{subject to} \quad \boldsymbol{x} \in X \end{aligned} \tag{6}$$

However, a decision maker usually has a goal that she or he would like to earn the probability more than p_1 . Furthermore, taking account of the vagueness of human judgment and flexibility for the execution of a plan in many real decision cases, we give a fuzzy goal to the target probability as the fuzzy set characterized by a membership function. In this subsection, we consider the fuzzy goal of probability $\mu_{\tilde{G}}(p)$ which is represented by,

$$\mu_{\tilde{G}}(\omega)=\begin{cases} 0 & \omega \le p_0 \\ g_p(\omega) & p_0 \le \omega \le p_1 \\ 1 & p_1 \le \omega \end{cases} \tag{7}$$

where $g_p(\omega)$ is a strictly increasing continuous function. Furthermore, using the concept of possibility measure, we introduce the degree of possibility as follows:

$$\Pi_{\tilde{P}}(\tilde{G}) = \sup_p \min\left\{\mu_{\tilde{P}}(p),\ \mu_{\tilde{G}}(p)\right\} \tag{8}$$

Using this degree of possibility, we consider the following possibility maximization model for the probability maximization model:

$$\begin{aligned} &\text{Maximize} \quad \Pi_{\tilde{P}}(\tilde{G}) \\ &\text{subject to} \quad \boldsymbol{x} \in X \end{aligned} \tag{9}$$

This problem is equivalently transformed into the following problem by introducing a parameter h:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \Pi_{\tilde{P}}(\tilde{G}) \ge h,\ \boldsymbol{x} \in X \end{aligned} \tag{10}$$

In this problem, each constraint $\Pi_{\tilde{P}}(\tilde{G}) \ge h$ is transformed into the following inequality:

$$\begin{aligned}
&\textstyle\prod_{\tilde{P}}(\tilde{G}) \ge h \\
\Leftrightarrow\ & \sup_p \min\left\{\mu_{\tilde{P}}(p), \mu_{\tilde{G}}(p)\right\} \ge h \\
\Leftrightarrow\ & \exists p:\ \mu_{\tilde{P}}(p) \ge h,\ \mu_{\tilde{G}}(p) \ge h \\
\Leftrightarrow\ & \exists p: \sup_{\boldsymbol{s}} \min_{1\le j\le n}\left\{\mu_{\tilde{M}_j}(s_j) \left| \begin{array}{l} p=\Pr\left\{\omega \middle| \bar{u}(\omega)\ge f\right\},\ \bar{u}=\sum_{j=1}^n \bar{\gamma}_j x_j, \\ \bar{\gamma}_j \sim T_j(s_j,\sigma_j^2),\ s_j = d_j^1+\tilde{m}d_j^2 \end{array}\right.\right\} \ge h, \\
& \mu_{\tilde{G}}(p) \ge h \\
\Leftrightarrow\ & \exists p,\ \exists \boldsymbol{s},\ \exists \bar{u}: \sup_{\boldsymbol{s}} \min_{1\le j\le n} \mu_{\tilde{M}_j}(s_j) \ge h,\ s_j = d_j^1+\tilde{m}d_j^2, \\
& \qquad p=\Pr\left\{\omega \middle| \bar{u}(\omega)\ge f\right\},\ \bar{u}=\sum_{j=1}^n \bar{\gamma}_j x_j,\ \bar{\gamma}_j \sim T_j(s_j,\sigma_j^2), \\
& \qquad \mu_{\tilde{G}}(p) \ge h \\
\Leftrightarrow\ & \exists p,\ \exists \boldsymbol{s},\ \exists \bar{u}: \min_{1\le j\le n} \mu_{\tilde{M}_j}(s_j) \ge h,\ s_j = d_j^1+\tilde{m}d_j^2, \\
& \qquad p=\Pr\left\{\omega \middle| \bar{u}(\omega)\ge f\right\},\ \bar{u}=\sum_{j=1}^n \bar{\gamma}_j x_j,\ \bar{\gamma}_j \sim T_j(s_j,\sigma_j^2), \\
& \qquad p \ge g_p^{-1}(h) \\
\Leftrightarrow\ & \exists p,\ \exists \boldsymbol{s},\ \exists \bar{u}: \mu_{\tilde{M}_j}(s_j) \ge h,\ (j=1,\dots,n),\ s_j = d_j^1+\tilde{m}d_j^2, \\
& \qquad \bar{u}=\sum_{j=1}^n \bar{\gamma}_j x_j,\ \bar{\gamma}_j \sim T_j(s_j,\sigma_j^2), \\
& \qquad \Pr\left\{\omega \middle| \bar{u}(\omega)\ge f\right\} \ge g_p^{-1}(h) \\
\Leftrightarrow\ & \exists \bar{u}: \Pr\left\{\omega \middle| \bar{u}(\omega)\ge f\right\} \ge g_p^{-1}(h), \\
& \qquad \bar{u}=\sum_{j=1}^n \bar{\gamma}_j x_j,\ \bar{\gamma}_j \sim T\left(d_j^1+\left(m+R^*(h)\alpha\right)d_j^2,\sigma_j^2\right)
\end{aligned} \tag{11}$$

where $L^*(h)$ is a pseudo inverse function defined as $L^*(h)=\sup\left\{t \middle| L(t)\ge h\right\}$. From this transformation, problem (11) is equivalently transformed into the following problem:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \Pr\left\{\omega \middle| \bar{u}(\omega)\ge f\right\} \ge g_p^{-1}(h), \\ &\qquad \bar{u}=\sum_{j=1}^n \bar{\gamma}_j x_j,\ \bar{\gamma}_j \sim T\left(d_j^1+\left(m+R^*(h)\alpha\right)d_j^2,\sigma_j^2\right), \\ &\qquad \boldsymbol{x} \in X \end{aligned} \tag{12}$$

Furthermore, we do the transformation to the stochastic constraint as follows:

$$\begin{aligned}
&\Pr\left\{\omega \middle| \bar{u}(\omega)\ge f\right\} \ge g_p^{-1}(h) \\
\Leftrightarrow\ & \Pr\left\{\omega \left| \begin{array}{l} \sum_{j=1}^n \left(d_j^1+\bar{t}(\omega)d_j^2\right)x_j \ge f, \\ \bar{t} \sim T\left(m+R^*(h)\alpha,\sigma_j^2\right) \end{array}\right.\right\} \ge g_p^{-1}(h) \\
\Leftrightarrow\ & \Pr\left\{\omega \left| \begin{array}{l} \bar{t}(\omega) \ge \dfrac{f-\boldsymbol{d}^1\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}}, \\ \bar{t} \sim T\left(m+R^*(h)\alpha,\sigma_j^2\right) \end{array}\right.\right\} \ge g_p^{-1}(h) \\
\Leftrightarrow\ & \Pr\left\{\omega \left| \begin{array}{l} \bar{t}(\omega)-R^*(h)\alpha \ge \dfrac{f-\boldsymbol{d}^1\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}}-R^*(h)\alpha, \\ \bar{t}-R^*(h)\alpha \sim T\left(m,\sigma_j^2\right) \end{array}\right.\right\} \ge g_p^{-1}(h) \\
\Leftrightarrow\ & \Pr\left\{\omega \left| \begin{array}{l} \bar{t}'(\omega) \ge \dfrac{f-\left(\boldsymbol{d}^1+R^*(h)\alpha\boldsymbol{d}^2\right)\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}}, \\ \bar{t}' \sim T\left(m,\sigma_j^2\right) \end{array}\right.\right\} \ge g_p^{-1}(h) \\
\Leftrightarrow\ & T\left(\frac{f-\left(\boldsymbol{d}^1+R^*(h)\alpha\boldsymbol{d}^2\right)\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}}\right) \le 1-g_p^{-1}(h)
\end{aligned} \tag{13}$$

Consequently, problem (12) is transformed into the following problem:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad T\left(\frac{f-\left(\boldsymbol{d}^1+R^*(h)\alpha\boldsymbol{d}^2\right)\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}}\right) \le 1-g_p^{-1}(h), \\ &\qquad \boldsymbol{x} \in X \end{aligned} \tag{14}$$

This problem is transformed into the following problem by introducing the pseudo inverse function T_i^* for T_i:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \frac{f-\left(\boldsymbol{d}^1+R^*(h)\alpha\boldsymbol{d}^2\right)\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}} \le T^*\left(1-g_p^{-1}(h)\right), \\ &\qquad \boldsymbol{x} \in X \end{aligned} \tag{15}$$

Problem (15) is a nonconvex programming problem, and so an optimal solution of this problem is not necessarily obtained by usual nonlinear programming approaches. However, in the case that the value of parameter h is fixed, constraints of this problem are reduced to a set of linear inequalities. This means that an optimal solution of this problem is obtained by using simplex method of linear programming approach and bisection method for parameter *h*.

Consequently, we construct the following analytical solution method.

Solution procedure

STEP1: Elicit the membership function of a fuzzy goal for the probability with respect to the objective function value.

STEP2: Solve problem (15) in the case $h=1$. If an optimal solution is obtained, it is a strict optimal solution of main problem and terminate this algorithm. Otherwise, go to STEP3.

STEP3: Solve problem (15) in the case $h=0$. If there is no feasible solution, terminate this algorithm. In this case, there is no feasible solution and it is necessary to reset a fuzzy goal for the probability. Otherwise, go to STEP4.

STEP4: Set $U_h \leftarrow 1$ and $L_h \leftarrow 0$.

STEP5: Set $k \leftarrow k+1$ and $h_k \leftarrow \frac{L_h+U_h}{2}$

STEP6: If $\left|h_{k+1}-h_k\right|<\varepsilon$ for the sufficiently small number ε, go to STEP8. Otherwise, go to STEP7

STEP7: Solve problem (15). If an optimal solution is obtained $x(k)$, reset $L_h \leftarrow h$ and return to STEP5. If there is no feasible solution, reset $U_h \leftarrow h$ and return to STEP5.

STEP8: $x(k)$ is an optimal solution and terminate this algorithm.

V. NUMERICAL EXAMPLE

In order to illustrate our considering situation where the proposing solution method is applied, we give a numerical example. We assume that there are four decision variables, three constraints and three scenarios to all parameters. Then, the constant parameters values are given as the following Tables 1. In this numerical example, we assume that random fuzzy variables $\bar{\tilde{t}}_i$ occur according to uniform distributions. Then, mean values and interval values for distribution function of $\bar{\tilde{t}}_i$ are given as the following Table 2 Subsequently, each mean value $\tilde{m}_i$ is a symmetric triangle fuzzy numbers $\langle \gamma, \delta \rangle$ where γ is the center value and δ is the spread value.

TABLE 1. PARAMETERS VALUES OF COEFFICIENTS IN OBJECTIVE FUNCTIONS

	Scenario 1	Scenario 2	Scenario 3
d_{i1}^1	0.3	0.5	0.1
d_{i2}^1	0.1	0.2	0.2
d_{i3}^1	0.2	0.1	0.4
d_{i4}^1	0.2	0.2	0.3
d_{i1}^2	0.4	0.3	0.3
d_{i2}^2	0.4	0.5	0.4
d_{i3}^2	0.2	0.1	0.2
d_{i4}^2	0.4	0.3	0.4

TABLE 2. MEAN VALUES AND DISTRIBUTION FUNCTIONS OF RANDOM FUZZY VARIABLES

	Scenario 1	Scenario 2	Scenario 3
$\tilde{m}_i$	$\langle 0.3, 0.1 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.2, 0.1 \rangle$
Interval	0.1	0.15	0.05

Furthermore, each fuzzy goal for the probability to each scenario is given as follows:

$$\mu_{\tilde{G}_i}(\omega)=\begin{cases} 0 & \omega<0.7 \\ \dfrac{\omega-0.7}{0.2} & 0.7 \le \omega < 0.9,\ i=1,2,3 \\ 1 & 0.9 \le \omega \end{cases}$$

From these data in Tables 1 and 2, we set the target value of total profit which is 0.42, and solve the mini-max problem based on problem (15). Then, we obtain the following optimal solution.

TABLE 3. OPTIMAL SOLUTION OF PROBLEM (15)

x_1	x_2	x_3	x_4
0.080	0.110	0	0.810

Subsequently, comparing the proposed model with the case not including fuzzy variables, each optimal solution is given as the following Table 4.

TABLE 4. OPTIMAL SOLUTIONS OF THREE PROBLEMS

	x_1	x_2	x_3	x_4
Not Fuzzy	0.140	0	0.158	0.702
Problem (15)	0.080	0.110	0	0.810

From the result in Table 4, we find that the rate of portfolios between x_2 and x_3 in problem (15) is opposite to the basic stochastic problem not including fuzzy numbers. This means that decision variables to be the higher rate of d_{ij}^2 tend to be chosen in our proposed model. Therefore, by setting the random fuzzy variable as the market portfolios which are represented the investor's subjectivity, we find that the optimal portfolio largely changes and our proposed model may provide the more appropriate portfolio suited to each investor.

VI. CONCLUSION

In this paper, we have considered random fuzzy portfolio selection problems based on CAPM which had been a useful tool of the investment, and found that the proposed model was solved analytically and efficiently by performing deterministic equivalent transformations and using the linear programming approach with the bisection algorithm on the parameter. In the future, we will apply this random fuzzy portfolio selection problem and solution methods to other asset allocation problems, combinational optimization models and multi-period problems. Nonetheless, this new proposed model of portfolio selection problems and the efficient solution methods will allow us to solve more complicate problems in real situations under more random and ambiguous conditions.

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