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# One Sufficient condition and its applications for hamiltonian graph using its spanning subgraph

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***Abstract*—Determinating whether a graph is Hamiltonian is a open difficult problem. In this paper, the problem is converted to determinating whether the graph has a 2-regular Hamiltonian spanning subgraph. We also give the procedures of the method, which can be used directly on computer.**

***Keywords*—Hamiltonian graph; Eulerian spanning; spanning subgraph; sufficient condition**

## I. Introduction

Judge wether a graph is Hamiltonian is to judge if there is a Hamiltonian cycle on the graph [1]. Although it is a long history open problem and some efforts have been done recently[1-12], it is not solved efficiently yet, even for small size graphs, no method could judge Hamilton directly. Beside this, most of existing methods focus on how to judge but do not focus on how to find a Hamilton cycle on the graph, which is very important in applications .

In this paper, we convert the judgment to whether the graph has a 2-regular Hamiltonian spanning subgraph. This method is based on an ordered searching scheme, which provides not only the judge results but also the Hamitonian cycles. We also give some examples to proof the validlity and power for our method. .

The remainder of our paper is arranged as: the section 2 will introduce backgrouds; the section 3 will give the framework of the algorithm; and on section 4, we will present some examples; finally, the future work and conclusion will be given.

## II. Backgrouds

On this section, we will introduce related backgrounds about this paper.

### A. *Terminology [13, 14]*

A graph G=(V, E) consists of a finite nonempty set V of vertices and a set E of 2-element subsets of V called edge. If e=uv is an edge of G, then u and v are adjacent vertices. A graph in which edges have no orientation is called undirected graph.

A graph H is called a subgraph of a graph G, written $H\subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If a subgraph of G has the same vertex set as G, then it is a panning subgraph of G.

A u-v walk W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjancent, that is, we can express W as:

$$W: u=v_0, v_1, \ldots, v_k=v \tag{1}$$

where $k\geq 0$ and $v_i$ and $v_{i+1}$ are adjacent for $i=1,2, \ldots, k-1$.

If u=v, then the walk is closed; while $u\neq v$, the W is open.

The number of edges encountered in a walk (including multiple occurrences of an edge) is called the length of the walk.

A u-v trail in agraph G to be a u-v walk in which no edge is traversed more than once. A circuit in a graph G is a closed trail of length 3 or more. A circuit that repeats no vertex, except for the first and last, is a cycle.

A loop (also called a self-loop) is an edge that connects a vertex to itself.

A simple graph is an undirected graph that has no loops and no more than one edge between any two different vertices.

For an undirected graph, the degree of a vertex is equal to the number of adjacent vertices.



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- Definition 1 (Hamiltonian cycle) A cycle in a graph G that contains every vertex of G is called Hamiltonian graph.

Thus a Hamiltonian cycle of G is a spanning cycle of G.

- Definition 2 (Hamiltonian graph) Hamiltonian graph is a graph that contains a Hamiltonian cycle.

A Hamiltonian cycle (or Hamiltonian circuit) is a cycle in an undirected graph which visits each vertex exactly once and also returns to the starting vertex. Determining whether such paths and cycles exist in graphs is the Hamiltonian path problem which is NP-complete.

*B. Sufficnet condition*

On this subsection, we will present the sufficient condition. Firstly, we will give the main theorem in this paper.

**Theorem 1 (sufficient condition) If a grapg G exists a 2-regular spanning Hamiltonian subgraph H, G is Hamiltonian.**

Proof:

H is Hamiltonian⇒There is a cycle in H which visits each vertex of H exactly once and also returns to the starting vertex (1)

H is a spanning subgraph⇒G and H have the same vertex set. (2)

Combing (1) and (2):

There is a cycle in G which visits each vertex of G exactly once and also returns to the starting vertex

Therefore, G is hamiltonian. #

The proof of theorem 1 is very simple and straight. However, it is not the target for this paper. In this paper, we try to find a new constructive proof for theorem 1, which can be used not only for theorem proof but also for finding Hamiltonian cycle for the graph. So we should discuss some problems as follow:

**Why condition is 2-regular spanning subgraph?**

It seems that the conditions of theorem 1 is too stronge, which can be modified to a more weak condition-"a spanning Hamitonian subgraph". We will proof these two conditiona are equivalent. The other advantage for our statement is it can provide searching scheme for us.

**Lemma 1: ∃H is subgraph of $H_1$, H is a 2-regular spanning Hamiltonian subgraph of G⇔$H_1$ is a spanning Hamitonian subgraph of G.**

Proof:

⇒ H is Hamiltonian, there is a cycle in H which visits each vertex of H exactly once and also returns to the starting vertex;

$H \subseteq H_1$ and H is subgraph of $H_1$, the cycle of H is also the cycle of $H_1$, so $H_1$ is Hamiltonian.

⇐ $H_1$ is Hamiltonian, there is a cycle in $H_1$ which visits each vertex of H exactly once and also returns to the starting vertex. Therefore, for each vertex in the cycle, there are two edges connecting the vertex to two other different vertex. All these edges and all vertex of $H_1$ form a Hamitonian subgraph of $H_1$. Besides this, since $H_1$ is spanning subgraph of G, H is spanning subgraph of G too. #

From the proof of Lemma1, the 2-regular graph is not a redundant condition and it also tells us the 2-regular graph is composed by edges of Hamiltonian cycles and all vertex of G. Therefore, finding this 2-regular graph is very important not only for the proof but also for determing the Hamiltonian cycle. Then we will dicuss how to find the 2-regular graph, which will lead to finding the Hamiltonian cycle of G.

**How to find the cycle?**

The main scheme to find the Hamiltonian cycle is to find 2-regular subgraph from G. The necessary condition for this task is that the degree for each vertex must bigger than 1. If one vertex can not satifies the condition, the graph is not Hamitonian. If the graph satifies the necessary condition, it means that degree of each vertex at least 2.

Thus, if we try to obtain 2-regular subgraph, we should select two edges for each vertex.

The selcting should start from the vertex with the least degree and procede according to the increase order of vertex degree.

The reason for this scheme is to guarantee the selecting could be succeed. Since large degree vertex has more edges to select while small degree vertex has less edges to select, selecting edges according to increasing order of vertex degree makes there are enough edges to be selected finally.

For each vertex $v_{i,}$

If degree($v_i$)<2, the cycle is not Hamiltonian

If degree($v_i$)=2, record two adjacent vertex

If degree($v_i$)>2, sort the adjacent vertex, select the vertex with the least degree to procede.

According to three different case, if selecting scheme selects whole vertex, the Hamiltonian cycle can be found by recording the order of whole vertex; if one vertex is case 1, the cycle is not Hamiltonian, then a new cycle should be discussed until fingding a Hamiltonian cycle or judge the graph is not Hamiltonian (all cycles are not Hamiltonian).

In order to explain the whole framework clearly, we will give the data structure of this scheme.

**Data Structure**

The adjacency matrix of a finite undirected graph G on n vertices is the n × n matrix where the nondiagonal entry $a_{ij}$ is the number of edges from vertex $v_i$ to vertex $v_j$, and the diagonal entry $a_{ii}$ is either twice the number of loops at vertex $v_i$

There exists a unique adjacency matrix for each graph (up to permuting rows and columns), and it is not the adjacency matrix of any other graph. In the special case of a finite simple graph, the adjacency matrix is a (0,1)-matrix with zeros on its diagonal. If the graph is undirected, the adjacency matrix is symmetric.

Cycle vertex sequence is a 1-D list for representing the vertex and their relations according to the order of cycle.

## III. 3. FRAMEWORK

The whole framework is as follow:

1. find a simple spanning subgraph H for the graph;

2. sort the vertex of H according to increasing order of their degree and represent as S, write adjacency mtrix A of H, initialize cycle vertex sequence C=[ ];

3. choose a vertex from S according to its order, for each vertex,

If degree($v_i$)<2, the cycle is not Hamiltonian, discuss another cycle

If degree($v_i$)=2, record two adjacent vertex using C, for examlpe, C=[$v_{i-1}$, $v_i$, $v_{i+1}$]

If degree($v_i$)>2, sort the adjacent vertex with the increasing degrees using $S_i$, finding the least degree vertex $v_j$, and C=[$v_{i-1}$, $v_i$, $v_{i+1}$,$v_j$]

4. repeat 3 until can not proceed, if C contains whole vertex of G, and the same start vertex as end vertex, G is Hamiltonian and C is its Hamiltonian cycle; otherwise, discuss another cycle.

5. repeat 3-4, until all path have been discussed. In this case, the graph is not Hamiltonian.

## IV. EXAMPLES

On this section, we present some examples to demonstrate the validity and powerful our method. The judgment of a Hamiltonian cycle is very difficult. Besides this, if we can judge a graph is Hamiltonian, finding a Hmiltonian cycle from the graph also is very difficult. On this section, we use a simple example to show the validity and power of our method.

One example is shown on figure 1. It is a simple graph. Sorting the vertex, S=[v3(2), v5(2), v1(3), v4(3), v6(3), v7(3), v2(4)] where degree of the vertex is represented by a bracket number.

Therefore, select v3, C=[v2,v3,v4]. Since degree of v2 is 4 larger than degree of v4(3), select v4, whose adjacent vertex are v3, v1, v5.

v3 is in C, compare v1(3) with v5(3), since degree v3=degree v5, we can choose arbitrary one, for example, v1, then C=[v2, v3, v4, v1].

For v1, it is connected with v4, v7, v2. Select v7, then C=[v2,v3,v4,v1,v7].

According to these discussion, finally, C=[v2, v3, v4, v1, v7, v5, v6, v2]-

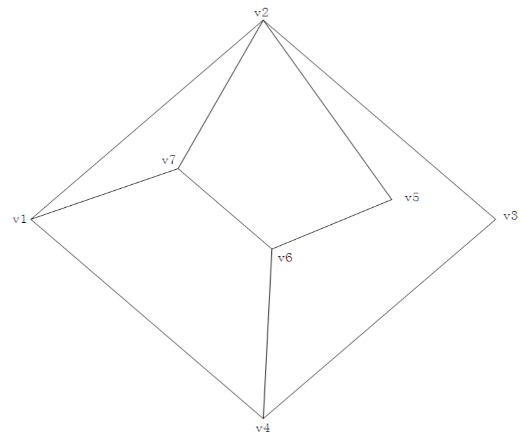


Figure 1

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