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Study on the Model of Grey Matrix Game Based on Grey Mixed Strategy

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***Abstract*—It often results in sublevel grey full rank square matrix to become non-feasible solution square matrix when there are null-strategy variables and redundancy constraints in the grey matrix solving method of grey matrix game. We can not directly find the optimum grey game solutions in the non-feasible solution square matrix. However, we define concepts of null-strategy variables and redundancy constraint equations and study relationships of them in the non-feasible solution square matrix. Further, we prove existence theorem of the optimum grey game solutions in non-square matrix $\tilde{A}(\otimes)_{m\times n}$ with constructive methods. In the end, we design the basic steps of solving grey matrix game problems with methods of grey matrix.**

***Keywords*—Grey matrix game,Methods of grey matrix,Null-strategy variables, Redundancy constraint equations**

I. INTRODUCTION

For a grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, if the grey elements of the column and row of the solution can not satisfy non-negative requirement, they are not the solution of this game even its grey inverse matrix of the grey full rank expansion square matrix exists. The paper mainly discusses the solution of grey matrix method when using only grey full rank expansion square matrix can not directly solve for the optimum grey game strategy and the optimum grey game value of the players in the grey matrix game.

II. NULL- STRATEGY VARIABLES IN NON-FEASIBLE SOLUTION SQUARE MATRIX

According to the inverse matrix of non-feasible solution square matrix, we can not find the grey optimum solution of the grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$. The other problem is, under more general circumstances, how to solve a grey game using non-feasible square matrix with information of optimum solution of the game.

Definition 1 For a grey matrix game problem $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, if one or some grey optimum strategy variables of the optimum grey mixed strategy $S_1^{\otimes} = \{x_1^{\otimes}, x_2^{\otimes}, \cdots, x_m^{\otimes}\}, S_2^{\otimes} = \{y_1^{\otimes}, y_2^{\otimes}, \cdots, y_n^{\otimes}\}$ for Player 1 and 2 equal to zero, then we call this or these variables as null-strategy variables.

Theorem 1 The generic grey expansion square matrix of a grey matrix game problem $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ is $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1, 2, \cdots, C_n^m$ or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1, 2, \cdots, C_m^n$. If one (or some) generic grey expansion square matrix is non-feasible solution square matrix, then there exists null-strategy variables in the constraints of the grey inequalities that the non-feasible solution square matrix represents.

Proof. Generally, assume that for any given grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, its generic grey expansion square matrix of is $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1, 2, \cdots, C_n^m$ (or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1, 2, \cdots, C_m^n$. The proof is similar.) We have to prove that if some square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ (the proof for other square matrixes are similar) of $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1, 2, \cdots, C_n^m$ is non-feasible solution square matrix, then there exists null-strategy variables in the constraints of the grey inequalities that the non-feasible solution square matrix represents.

During the grey matrix game, each generic grey expansion square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ corresponds to a grey profit and loss matrix $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\cdots,C_n^m$. In fact, we can take $\tilde{G}_i(\otimes)=\{S_1, S_2; \tilde{A}_i(\otimes)_{m\times m}\}, \quad i=1,2,\cdots,C_n^m$ decided by $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\cdots,C_n^m$ as the ith sub-game of $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$.

According to the necessary and sufficient condition of the unique grey expansion square matrix being the optimum solution square matrix as presented in Gray Game Theory and Its Application on Economic.

Research—compound standard grey number and the grey non-feasible, feasible and optimum solution square matrix of $\tilde{G}(\otimes)$, if each $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ is the unique feasible solution square matrix in the ith sub-game, then it corresponds to the optimum grey game solution of the ith sub-game. Obviously, for a given grey matrix game problem $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, the optimum solution should be the optimum grey game strategy and value after considering the optimum grey game solution of sub-games decided by all $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$.

If $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ of $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ is non-feasible solution square matrix, then there exists negative grey numbers in the columns and rows of the solution for $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$. It indicates that from $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$ we can not directly find the optimum solution of the grey matrix game that $\tilde{A}_K(\otimes)_{m\times m}$ corresponds.

From Definition Gray Game Theory and Its Application on Economic Research—compound standard grey number and the grey non-feasible, feasible and optimum solution square matrix of $\tilde{G}(\otimes)$, we know that the solution of the column and row of $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$ corresponds respectively to the solution of the grey variables $\overset{/\otimes}{x_1}, \overset{/\otimes}{x_2}, \cdots, \overset{/\otimes}{x_m}, \quad \overset{/\otimes}{y_1}, \overset{/\otimes}{y_2}, \cdots, \overset{/\otimes}{y_m}, -\overset{/}{v_\otimes}$ of the grey matrix equation of $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$. However, as we ignore the non-negative requirement of these variables due to structure constraints, the solutions of the matrix equations' variables only equal to that of the inequalities' variables when this requirement is taken into consideration.

Here we find that the solutions decided by non-feasible solution square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ are only the solution of variables of matrix equations $\overset{/\otimes}{x_1}, \overset{/\otimes}{x_2}, \cdots, \overset{/\otimes}{x_m}, \overset{/\otimes}{y_1}, \overset{/\otimes}{y_2}, \cdots, \overset{/\otimes}{y_m}, -\overset{/}{v_\otimes}$ rather than that of variables of non-grey matrix game problem $\overset{\otimes}{x_1}, \overset{\otimes}{x_2}, \cdots, \overset{\otimes}{x_m}, \quad \overset{\otimes}{y_1}, \overset{\otimes}{y_2}, \cdots, \overset{\otimes}{y_m}, -\overset{*}{v_\otimes}$. In other words, these variables in the matrix equations can not be proper valued in interval [0,1](range of grey probability value), to be specific, they must be negative values based on Theorem 2 in Study on the Model of Grey Matrix Game Based on Grey Mixed Strategy (11)—compound standard grey number and the grey non-feasible, feasible and optimum solution square matrix of $\tilde{G}(\otimes)$. However, these strategy variables can only lie in the grey probability value interval therefore if they are to be transformed to feasible solutions of the grey game, the grey strategy variables that the negative variables correspond to must be equal to zero. That is, if the optimum solution of the grey matrix game problem that $\tilde{A}_K(\otimes)_{m\times m}$ corresponds is to be find in non-feasible solution square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$, we must take some variables among $\overset{\otimes}{x_1}, \overset{\otimes}{x_2}, \cdots, \overset{\otimes}{x_m}, \overset{\otimes}{y_1}, \overset{\otimes}{y_2}, \cdots, \overset{\otimes}{y_m}$ of the grey inequalities decided by $\tilde{A}_K(\otimes)_{m\times m}$ as zero.

In this way, we prove that if one (or some) generic grey expansion square matrix is non-feasible solution square matrix, then there exists null-strategy variables in the constraints of the grey inequalities that the non-feasible solution square matrix represents. (Proof ends)

III. REDUNDANCY CONSTRAINT EQUATION AND NULL-STRATEGY VARIABLE OF NON-MATRIX $\tilde{A}(\otimes)_{m\times n}$

Definition 2 For a given grey matrix game $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, substitute the optimum grey solution of its two grey inequalities into the original grey inequalities, then we can get one (or some) strict grey inequality constraint equation which is called as grey redundancy constraint equation.

Theorem 2 For a given grey matrix game $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, there exists a unique optimum solution. If $\tilde{A}(\otimes)_{m\times n}$ is the long column-dimension grey matrix, then there exists redundancy constraint equation in grey inequalities' constraint of Player-1 in this game and null-strategy variable in the strategy variables of Player-2. On the contrary, if $\tilde{A}(\otimes)_{m\times n}$ is the long row-dimension grey matrix, then the redundancy constraint equation exists for Player-2 and null-strategy variable forPlayer-1.

Proof. Generally, for a given grey matrix game $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix. According to $\tilde{A}(\otimes)_{m\times n}$, the grey inequality constraint equations are as shown in Formula 6 and 7.

Here, we have to prove $\tilde{A}(\otimes)_{m\times n}$ ($n>m$) is long column-dimension matrix first. The row-dimension grey expansion square matrix of $\tilde{A}(\otimes)_{m\times n}$ can be expressed as $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$. According to Theorem 6, we can solve out the solution of each sub-game $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\cdots,C_n^m$ to which $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ correspond, and further more the grey optimum solution of $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$. Due to the optimum solution being unique, suppose that the optimum solution is generated from the grey inequalities(total number of m) decided by the Kth row grey expansion square matrix, denoted as $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ or the Kth row grey profit and loss square matrix, $\tilde{A}_K(\otimes)_{m\times m}$.

For a given grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$, if $\tilde{A}(\otimes)_{m\times n}$ ($n>m$) is long column-dimension grey matrix, then there are n constraint grey inequalities for Player-1. From Theorem 6, we know that the rational behaviors of Player-1generate a group of grey matrix strategies $x_1^{\otimes},x_2^{\otimes},\cdots,x_m^{\otimes}$ with the corresponding grey game value $v_{\otimes}^{*}$ is the minimum one decided by $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$, which depends on the inverse matrix $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$ of $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$.

Based on the above assumption, if $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ is the optimum solution square matrix, then redundancy constraint equation does not exist in grey sub-game of $\tilde{A}_K(\otimes)_{m\times m}$. This means m inequalities for Player 1 and 2 each constitute the optimum solution. Thus, only m of the total n inequalities of Player-1 decided by $\tilde{A}(\otimes)_{m\times n}$ ($n>m$) are used.

In the grey matrix game based on long column-dimension grey matrix $\tilde{A}(\otimes)_{m\times n}$ ($n>m$), Player-2 has $n+1$ strategy variables, namely $y_1^{\otimes},y_2^{\otimes},\cdots,y_n^{\otimes},v_{\otimes}^{*}$. As we can get Player-2's optimum grey game solution(m grey strategy variables denoted as $y_1^{\otimes},y_2^{\otimes},\cdots,y_m^{\otimes},v_{\otimes}^{*}$) based on the existed $m+1$ constraint grey inequalities, thus the other $n-m>0$ strategy variables should equal to zero.

As a result, in the grey matrix game decided by $\tilde{A}(\otimes)_{m\times n}$ ($n>m$)(with unique solution), there exists at least $n-m>0$ strategy variables for Player-2.

Following the same steps, we can prove that if $\tilde{A}(\otimes)_{m\times n}$($n<m$)(with unique solution) is long row-dimension grey matrix there exists $m-n>0$ redundancy constraint equations in the constraint of Player-2's grey inequalities while at least $m-n>0$ null-strategy variables of Player-1's strategy variables. (Proof Ends)

IV. Optimum Grey Game Solution of Non-Square Matrix $\tilde{A}(\otimes)_{m\times n}$

Theorem 3 For a given problem of grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$, if $\tilde{A}(\otimes)_{m\times n}$ ($n>m$) is long column-dimension grey matrix then its optimum grey game value must be the minimum grey game value decided by all grey row expansion square matrixes $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$. If $\tilde{A}(\otimes)_{m\times n}$ ($n>m$) is long row-dimension grey matrix, the result is the maximum value decided by all column grey expansion square matrixes $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$.

Proof. Generally, for a given grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$, we first discuss the situation when all $\tilde{A}(\otimes)_{m\times n}$'s grey expansion square matrixes $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\cdots,C_m^n$ are feasible solution expansion square matrixes (Here, we take $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ as an example, the proof for $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\cdots,C_m^n$ is similar).

When $\tilde{A}(\otimes)_{m\times n}$ ($n>m$) is long column-dimension grey matrix, according to the definition of expansion square matrix in thesis Gray Game Theory and Its Application on Economic Research, we could take m grey strategy vectors of Player-1 as row strategy vectors, and m grey vectors from n grey strategy vectors of Player-2 as column vectors to establish a grey square matrix $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\cdots$ which is similar to the form of grey profit and loss. Then we use this row-dimension grey profit and loss matrix $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\cdots,C_n^m$ to construct C_n^m row grey expansion square matrixes $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$.

From the perspective of grey game strategy, both players take rational behaviors during the game with Player-1 and Player-2 respectively using m and n grey strategies. As $m>n$, Player-2 has to take all his n strategies with Player-1. Obviously, he can choose among these strategies on the basis of separate probability. On the other side, Player-1 choose n preferable strategies among the total number of m either randomly or basing on the probability.

The inevitable result of the game is that the strategy that Player-1 chooses randomly among all C_m^n plans generates the minimum grey game value during the game with the n strategies of Player-2.

We know that the row grey expansion square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ include all C_n^m possible strategy combinations during the game of long column-dimension grey matrix $\tilde{A}(\otimes)_{m\times n}$ ($n>m$). What strategies Player-2 will adopt in the game with Player-1 depend completely on their rational behaviors.

Therefore, the optimum grey game value contains the following two parts. On one hand, Player-1 chooses rationally optimum grey probabilities of his m grey strategies $x_1^{\otimes},x_2^{\otimes},\cdots,x_m^{\otimes}$, to obtain maximum $v_{\otimes}^{*}$ under the worst situation. On the other, Player-1 similarly chooses his n grey strategies $y_1^{\otimes}, y_2^{\otimes}, \cdots, y_n^{\otimes}$ out of m to obtain minimum $v_{\otimes}^{*}$ under the worst situation.

Based on the above analysis, during the game when $\tilde{A}(\otimes)_{m\times n}$ ($n > m$) is the long column-dimension grey matrix, the optimum grey game value must be the minimum $v_{\otimes}^{*}$ decided by all the row grey expansion square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1, 2, \cdots, C_n^m$. Meanwhile, $v_{\otimes}^{*}$'s corresponding grey game strategies of players are the optimum grey game strategies.

In conclusion, a grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ ($n \neq m$) can be divided into several grey sub-games according to its dimensions of columns and rows, with total number of C_n^m sub-games of $\tilde{A}_i(\otimes)_{m\times m}, i = 1, 2, \cdots, C_n^m$ if $n > m$ and C_m^n sub-games $\tilde{A}_i(\otimes)_{n\times n}, i = 1, 2, \cdots, C_m^n$ if $n < m$. The grey optimum solution of $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ ($n \neq m$) means the general grey optimum solution after considering each sub-game's solution as local grey optimum solution.

We can prove following the same principles that the optimum grey game value must be the maximum $v_{\otimes}^{*}$ decided by all the column grey expansion square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1, 2, \cdots, C_n^m$ (all are feasible solution square matrixes). Meanwhile, $v_{\otimes}^{*}$'s corresponding grey game strategies of players are the optimum grey game strategies.

Then we discuss the case when one (or some) grey expansion square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1, 2, \cdots, C_n^m$ or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1, 2, \cdots, C_m^n$ of $\tilde{A}(\otimes)_{m\times n}$ is non-feasible solution square matrix(Here we only take $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ as an instance, the proof is similar for $\tilde{B}_L(\otimes)_{(n+1)\times(n+1)}$).

As $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ is non-feasible solution square matrix, we can find neither the feasible solution of each grey sub-game directly in the inverse matrixes of all $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}$ $i = 1, 2, \cdots, C_n^m$ nor the general optimum solution of $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ ($n \neq m$) in a further step.

According to Theorem 1, for non-feasible square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$, if there exists negative grey elements in the column or rows of the solution, then in the grey sub-game $\tilde{G}_K(\otimes) = \{S_{K1}, S_{K2}; \tilde{A}_K(\otimes)_{m\times m}\}$, one or some strategy variables must be null-strategy variables which are to be eliminated.

Assume that we eliminate s null-strategy variables from m strategies of Player-1(Proof is similar in others cases of eliminating null-strategy variables) just as eliminate s rows of grey elements from $\tilde{A}_K(\otimes)_{m\times m}$, thus we transform a grey sub-game $\tilde{A}_K(\otimes)_{m\times m}$ into its sub-game with long column-dimension grey profit and loss matrix $\tilde{A}_{Ks}(\otimes)_{(m-s)\times m}$. With the same process of game and obtaining the optimum solution as the above, we can find local optimum solution of this non-feasible solution square matrix and finally the optimum grey game strategy and value of $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$.

Following the same principles, if $\tilde{A}(\otimes)_{m\times n}$ ($n > m$) is long column-dimension grey matrix, its optimum grey game value must be the minimum among all the values based on all row grey expansion square matrixes $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1, 2, \cdots, C_n^m$. On the contrary, with $\tilde{A}(\otimes)_{m\times n}$ ($n < m$) the long row-dimension grey matrix, the result would be the maximum decided by all column grey expansion square matrixes $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1, 2, \cdots, C_m^n$. (Proof Ends).

Theorem 3 actually provides a grey matrix method to solve the problem of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ ($n \neq m$).

V. INFERIOR STRATEGY OF $\tilde{A}(\otimes)_{m\times n}$, REDUNDANCY CONSTRAINT OF INFERIOR STRATEGY AND NULL-STRATEGY VARIABLE

Definition 3 For any given grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, if some grey game strategy is inferior to all the other strategies, then it is called inferior strategy.

Theorem 4 For a given grey matrix game problem $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, if some grey game strategy of a player is grey inferior strategy, then the corresponding grey strategy variable must be null-strategy variable.

Proof. Generally, for a given $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, we first discuss the case when $\tilde{A}(\otimes)_{m\times n}$ is long column-dimension grey matrix and suppose the Hth column grey vector of $\tilde{A}(\otimes)_{m\times n}$ to be Player-2's inferior grey strategy vector(as shown in Formula 1. Similar proof in case for Player-1). According to Formula 1, the constraint grey inequalities of Player-2 are illustrated in Formula 2.

$$\tilde{A}(\otimes)_{m\times n} = \begin{pmatrix} [a_{11}, b_{11}] & \cdots & [a_{1H}, b_{1H}] & \cdots & [a_{1n}, b_{1n}] \\ [a_{21}, b_{21}] & \cdots & [a_{2H}, b_{2H}] & \cdots & [a_{2n}, b_{2n}] \\ \vdots & & \vdots & & \vdots \\ [a_{m1}, b_{m1}] & \cdots & [a_{mH}, b_{mH}] & \cdots & [a_{mn}, b_{mn}] \end{pmatrix} \quad (1)$$

$$\begin{cases} \sum_{i=1}^{n} a_{ij}^{\otimes} x_i^{\otimes} \geq v_{\otimes}^{*}, j=1,2,\cdots,n \\ x_i^{\otimes} = [Lx_i, Rx_i] \geq [0,0], i=1,2,\cdots,n \\ \sum_{i=1}^{n} x_i^{\otimes} = \sum_{i=1}^{n} [Lx_i, Rx_i] = [1,1]_{\gamma_i = c_i, 0 \leq c_i \leq 1, i=1,2,\cdots,n} = 1^{\otimes} \end{cases} \quad (2)$$

For Player-2, as his Hth grey strategy is inferior strategy, he will not adopt this strategy in this game basing on rational behaviors , rather he will make the strategy variable zero, which is the null-strategy variable.

From Formula 1, we know that since the Hth column grey strategy vector is the inferior strategy vector of Player-2, each grey element of the vector must be not less than that of one(or some) column grey strategy vector constituting the optimum strategy of Player-2. For Player-1,we substitute the grey optimum strategy variable value and the optimum grey game value into the grey inequality decided by the Hth column grey strategy vector, and the inequality will finally become the strict grey inequality.

The main reason is that the optimum grey game value of Player-1, according to Theorem 3, is decided by the profit and loss matrix $\tilde{A}(\otimes)^{/}_{m\times(n-1)}$ eliminating the Hth column grey strategy vector. To be more specific, the optimum grey game value of $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ is the minimum $v_{\otimes}^{*}$ of all the local optimum solution in the grey sub-game of $\tilde{A}(\otimes)^{/}_{m\times(n-1)}$, and the grey elements of the columns and rows the inverse matrix's solution of $v_{\otimes}^{*}$'s corresponding generic grey expansion square matrix is the optimum grey game strategy. In other words, due to players' rational behaviors, the choice of strategies will make the values of all grey game strategies of Player-1 be tenable for optimum solution matrix's corresponding grey matrix equations. Moreover, from Theorem 2, we know that the grey inequality constraint equation of Player-1 which is decided by the Hth column grey strategy vector is redundancy constraint equation. (Proof Ends)

VI. Solution Steps of the Grey Matrix Method of $\tilde{G}(\otimes)$

To sum up, the process of the grey matrix method solution of grey matrix game problem $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ is as follows:

(1) Inspect and eliminate the inferior strategy variables of players in grey profit and loss matrix $\tilde{A}(\otimes)_{m\times n}$;

(2)Construct the generic grey full rank expansion square matrix of $\tilde{A}(\otimes)_{m\times n}$, namely

$\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ (or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\cdots,C_m^n$);

(3) Solve for the grey inverse matrix $\tilde{B}_i^{-1}(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ (or $\tilde{B}_i^{-1}(\otimes)_{(n+1)\times(n+1)}, i=1,2,\cdots,C_m^n$) of $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ (or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\cdots,C_m^n$);

(4) If all of $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ (or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\cdots,C_m^n$) are feasible solution square matrixes, then the minimum(or maximum) $v_{\otimes}^{*}$ of grey game values that the solution columns and rows of $\tilde{B}_i^{-1}(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ (or $\tilde{B}_i^{-1}(\otimes)_{(n+1)\times(n+1)}, i=1,2,\cdots,C_m^n$) correspond to is the optimum grey game value of $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$. The grey elements of the solution column and row where the grey game value locate corresponds respectively to the player's optimum grey game strategy;

(5) If one (or some) square matrix in $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\cdots,C_n^m$ (or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\cdots,C_m^n$) is non-feasible solution square matrix, then eliminate the null-strategy variables, reconstruct the grey expansion square matrix of this(or these) grey sub-game problem and solve for its grey optimum solution;

(6) Take the minimum(or maximum) $v_{\otimes}^{*}$ as grey optimum solution of $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$. Here, $v_{\otimes}^{*}$ is among all the local optimum solution of grey sub-games decided by all generic grey expansion square matrix in $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$. Then the grey elements in columns and rows of solution matrix where $v_{\otimes}^{*}$ locates corresponds respectively the players' optimum grey game strategy.

Example 1. For a given grey matrix game problem $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, the grey profit and loss matrix is indicated in Formula 5. The grey mixed strategy of Player 1 and 2 is respectively $S_1^{\otimes} = \{x_1^{\otimes}, x_2^{\otimes}\}, S_2^{\otimes} = \{y_1^{\otimes}, y_2^{\otimes}, y_3^{\otimes}\}$. Solve out the grey redundancy constraint equation and the null-strategy variable.

$$\tilde{A}(\otimes) = \begin{pmatrix} [1,2] & 3 & 11 \\ 7 & 5 & [2,4] \end{pmatrix} \quad (3)$$

$$\begin{cases} (1+\gamma_{11})x_1^{\otimes} + 7x_2^{\otimes} \geq v_{\otimes}^{*} & 4.1 \\ 3x_1^{\otimes} + 5x_2^{\otimes} \geq v_{\otimes}^{*} & 4.2 \\ 11x_1^{\otimes} + (2+2\gamma_{23})x_2^{\otimes} \geq v_{\otimes}^{*} & 4.3 \\ x_1^{\otimes} + x_2^{\otimes} = 1^{\otimes} & 4.4 \end{cases} \quad (4)$$

$$\begin{cases}(1+\gamma_{11})y_1^{\otimes}+3y_2^{\otimes}+11y_3^{\otimes}\le v_{\otimes}^{*}\\ 7y_1^{\otimes}+5y_2^{\otimes}+(2+2\gamma_{23})y_3^{\otimes}\le v_{\otimes}^{*}\\ y_1^{\otimes}+y_2^{\otimes}+y_3^{\otimes}=1^{\otimes}\end{cases} \quad (5)$$

Solution. According to Formula 3, the grey inequalities of this grey game are demonstrated in Formula 4 and 5. The generic grey expansion square matrixes are as shown in Formula 6,7 and 8.

$$\tilde{B}_1(\otimes)=\begin{pmatrix}[1,2] & 3 & 1\\ 7 & 5 & 1\\ 1 & 1 & 0\end{pmatrix} \quad (6)$$

$$\tilde{B}_2(\otimes)=\begin{pmatrix}[1,2] & 11 & 1\\ 7 & [2,4] & 1\\ 1 & 1 & 0\end{pmatrix} \quad (7)$$

$$\tilde{B}_3(\otimes)=\begin{pmatrix}3 & 11 & 1\\ 5 & [2,4] & 1\\ 1 & 1 & 0\end{pmatrix} \quad (8)$$

$$x_1^{\otimes}=\frac{2\gamma_{23}-3}{2\gamma_{23}-11}=[\frac{1}{9},\frac{3}{11}] \quad 9.1$$

$$x_2^{\otimes}=\frac{-8}{2\gamma_{23}-11}=[\frac{8}{11},\frac{3}{9}] \quad 9.2 \quad (9)$$

$$v_{\otimes}^{*}=\frac{-49+6\gamma_{23}}{2\gamma_{23}-11}=[\frac{49}{11},\frac{43}{9}] \quad 9.3$$

$$\begin{aligned}&(1+\gamma_{11})x_1^{\otimes}+7x_2^{\otimes}=(1+\gamma_{11})\frac{2\gamma_{23}-3}{2\gamma_{23}-11}-\frac{56}{2\gamma_{23}-11}\\ &=\frac{-56+2\gamma_{23}-3+2\gamma_{11}\gamma_{23}-3\gamma_{11}}{2\gamma_{23}-11}=\left[\frac{59}{11},\frac{62}{9}\right]_{\gamma_{11},\gamma_{23},0\le\gamma_{11},\gamma_{23}\le 1}\\ &>v_{\otimes}^{*}=\frac{-49+6\gamma_{23}}{2\gamma_{23}-11}=\left[\frac{49}{11},\frac{43}{9}\right]_{\gamma_{11},\gamma_{23},0\le\gamma_{11},\gamma_{23}\le 1}\end{aligned} \quad (10)$$

$$y_1^{\otimes}=0 \quad 11.1$$

$$y_2^{\otimes}=\frac{2\gamma_{23}-9}{2\gamma_{23}-11}=[\frac{7}{9},\frac{9}{11}] \quad 11.2 \quad (11)$$

$$y_3^{\otimes}=\frac{-2}{2\gamma_{23}-11}=[\frac{2}{11},\frac{2}{9}] \quad 11.3$$

Solving Formula 6, 7 and 8, we can get the solution as indicated in Formula 9. Substitute Formula 9.1 and 9.2 into Formula 4.1, then we find that Formula 4.1 is a strict grey inequality, as shown in Formula 10. So Formula 4.1 is redundancy constraint equation.

Similarly, the optimum grey game strategy of Player-2 is shown in Formula 11 from which we know that $y_1^{\otimes}$ is the null-strategy variable of Playe-2.

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