LMI-Based $H_\infty$ State-Feedback Control for T-S Time-Delay Discrete Fuzzy Bilinear System

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Abstract—This paper presents a robust $H_\infty$ fuzzy controller for a class of T-S time-delay discrete fuzzy bilinear system (DFBS). Firstly, a discrete robust $H_\infty$ fuzzy controller is proposed to stabilize the T-S time-delay DFBS with disturbance. Secondly, based on the Schur complement and some variable transformation, the stability conditions of the overall fuzzy control system are formulated by linear matrix inequalities (LMIs). Finally, a numerical example is utilized to demonstrate the validity and effectiveness of the proposed control scheme.

Keywords—Discrete fuzzy bilinear system (DFBS), robust control, linear matrix inequalities (LMIs).

I. INTRODUCTION

In recent years, bilinear systems and controls have been widely applied to a wide variety of fields, for example, bioengineering, biochemistry, nuclear engineering and socioeconomics [1]–[5]. There are two advantages of using bilinear models to describe nonlinear models. One advantage is that bilinear systems are an adequate approximation than linear models to describe nonlinear models. The main contribution of this paper is to design a fuzzy controller for the T-S DFBS with disturbance under consideration.

II. SYSTEM DESCRIPTION AND CONTROLLER DESIGN

The T-S fuzzy dynamic model is described by fuzzy IF-THEN rules, which locally represent input-output relations of nonlinear systems. Similar to [7, 10], the $i$th rule of the time-delay T-S DFBS with disturbance can be represented by the following form:

Plant Rule $i$:

IF $s_i(k)$ is $F_{i1}$ and ... and $s_g(k)$ is $F_{ig}$

THEN $x(k+1) = (A_i + \Delta A_i)x(k) + B_i u(k) + N_i x(k)u(k)$

For stability analysis, LMI techniques [4], [7], [9], [10] have been developed to find solutions for the T-S fuzzy control system under consideration.

It is known that time-delay phenomenon commonly exists in dynamic systems due to measurement, transportation, transport lags, and computational delays. Therefore, stabilization of time-delay systems is increasing being attentions in the literature [7, 9]. Recently, the T-S fuzzy model approach has been shown effective and powerful in dealing with nonlinear systems with time-delays. Besides, different methodologies have been also proposed for analysis and synthesis of T-S fuzzy systems with time-delay [8]–[11].

The main contributions of this paper are i) designing a fuzzy $H_\infty$ controller for the T-S DFBS with disturbance; ii) investigating a robust $H_\infty$ fuzzy controller for the T-S Time-Delay DFBS with disturbance; and iii) Describing all the stability conditions for the T-S Time-Delay DFBS with disturbance via LMIs.

This paper is organized as follows. In Section II, the DFBS is established and its fuzzy controller is also designed. The controller design method for robust $H_\infty$ stabilization of the T-S DFBS with disturbance is derived in Section III. A numerical simulation is illustrated in Section IV. Finally, conclusions are given in Section V.
where $k$ is iteration instant, $F_{i}$ is the fuzzy set and $r$ is the number of IF-THEN rules; $x(k) \in \mathbb{R}^{n}$ is the state, $u(k) \in \mathbb{R}$ is the control input; $z(k) \in \mathbb{R}^{n}$ is the disturbance input which is belong to $l_{2}[0, + \infty)$; $s_{1}(k), s_{2}(k), \ldots s_{r}(k)$ are the premise variables. $A_{i} \in \mathbb{R}^{n \times m}, B_{i} \in \mathbb{R}^{m \times n}, N_{i} \in \mathbb{R}^{n \times n}, A_{d} \in \mathbb{R}^{n \times n}, D_{i} \in \mathbb{R}^{n \times m}, C_{i} \in \mathbb{R}^{p \times n}, D_{2i} \in \mathbb{R}^{n \times n}$, and $\Delta A_{i} \in \mathbb{R}^{n \times n}$ are real-valued unknown matrices representing time-varying parameter uncertainty, and are assumed to be of the form, in the following Assumption.

**Assumption 1** [9]: The parameter uncertainties considered here are norm-bounded and presented by the form $1()$.

The overall fuzzy control law can be represented by

$$\begin{align*}
    & z(k) = C_{i} x(k) + D_{2i} u(k), \
    & u(k) = \sum_{i=1}^{r} h_{i}(s(k)) \rho D_{i} x(k) / \sqrt{1 + x^T(k) D_{i}^T D_{i} x(k)}.
\end{align*}$$

where $\rho \in \mathbb{R}$ is a scalar to be assigned. Substituting (5) into (3), one can get the closed-loop system, $z(k + 1) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_{i} h_{j} (A_{i} + \Delta A_{i}) + \rho N_{i} \sin \theta_{j} + \rho B_{i} D_{i} \cos \theta_{j} x(k) + A_{d} x(k) + D_{d} u(k)$.

Then the robust fuzzy $H_{\infty}$ control problem to be addressed in this paper can be formulated as follows: give an fuzzy system (3) and a scalar $\gamma > 0$, determine a fuzzy controller such that

1. The closed-loop system (6) is robust asymptotically stable when $\alpha(k) = 0$.
2. Under the zero initial condition, the controlled output $z(k)$ satisfies

$$\|z(k)\| < \gamma \|\alpha(k)\| \quad \text{for all nonzero } \alpha(k) \in l_{2}[0, + \infty)$$

and all admissible uncertainties.

### III. MAIN RESULT

In this section, an LMI approach will be developed to solve the problem of robust $H_{\infty}$ feedback control of fuzzy system formulated in the previous section. Before discussing the proof of the theorem, we first give the following lemma which will be used in the proof of our main results.

**Lemma 1**: [7], [9]: Given any matrices $X$, $Y$ and $\varepsilon$ with appropriate dimensions such that $\varepsilon > 0$ we have

$$X^{T} + XY^{T} \leq \varepsilon X^{T} X + \varepsilon^{-1} Y^{T} Y \quad \text{for all nonzero } X, Y.$$
\[ x(k+1) = \sum_{i=0}^{r} b_i(s(k)) \left( (A_i + \Delta A_i)x(k) + B_i u(k) \right) + N_i x(k) u(k) + A_{ \tau} x(k-\tau(k)) \]  
(11)

Now, we choose a Lyapunov function candidate for this system as follows:

\[ V(x(k)) = x^T(k) Px(k) + \sum_{i=0}^{r} -x^T(i) S \sigma(k) \].
(12)

Then it can be verified that

\[ \Delta V = V(x(k+1)) - V(x(k)) \]

By substituting (11) into (13), we can get (14), shown at the bottom of next two page,

where \( \Xi_0 = ((A_i + \Delta A_i) + (A_j + \Delta A_j)\rho B_j D_j \cos \theta_j + \rho N_j \sin \theta_j + \rho B_j D_j \cos \theta_j + \rho N_j \sin \theta_j) x(k) \)

& \[ + A_{ \tau} x(k-\tau(k)) + A_{ \tau} x(k-\tau(k)) \].

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\[ x(k+1) = \sum_{i=0}^{r} b_i(s(k)) \left( (A_i + \Delta A_i)x(k) + B_i u(k) + N_i x(k) u(k) + A_{ \tau} x(k-\tau(k)) + D_i u(k) \right) \]
(13)
Clearly, if (13) is negative definite uniformly for all nonzero \( x(k) \) and \( k \geq 0 \), then the controlled fuzzy system (11) is robustly asymptotically stable.

At first, we assume that the first sum of the equation in (14) is negative definite, that is,
\[
\begin{bmatrix}
(A_{i} + \Delta A_{i}) + \rho B_{i}D_{i} \cos \theta_{i} + \rho N_{i} \sin \theta_{i} \n
\end{bmatrix} x_{i} + A_{i} x(k - \tau(k))
\]
\[
\times \begin{bmatrix}
(A_{i} + \Delta A_{i}) + \rho B_{i}D_{i} \cos \theta_{i} + \rho N_{i} \sin \theta_{i} \n
\end{bmatrix} x_{i} + A_{i} x(k - \tau(k))
\]
\[
+ x^T (S - P) x - x^T (k - \tau) S x(k - \tau) < 0
\]
Using the Lemma 1, then (15) can be rewritten as
\[
x^T (k) \left[ \left( \rho^2 + 1 + \varepsilon \right) (A_{i} + \Delta A_{i})^T P (A_{i} + \Delta A_{i}) \right] x(k)
\]
\[
+ \rho^2 \varepsilon^T (k) (x^T (k - \tau)) + 1 + \varepsilon \right) A_{i}^T P A_{i} - S - P \right] x(k - \tau(k)) < 0
\]
Applying the Schur complement to (16) results in (17), shown at the bottom of next page.

In order to use the convex optimization technique, we utilize the Assumption 1, and adopt the transformation matrix:
\[
\text{diag} \left[ I \ P \ I \ P \ P \right]
\]
and take a congruence transformation \( \Theta \) completes the proof of the theorem. Q.E.D.

IV. EXAMPLE

In this section, we apply the proposed method to design a robust \( H_{\infty} \) fuzzy controller for a T-S time-delay DFBS with disturbance. The T-S time-delay DFBS with disturbance is described as follows:

Rule 1: IF \( x_{1} \) is about \(-1\)

THEN \( x(k+1) = \left( A_{i} + \Delta A_{i} \right) x(k) + B_{i} u(k) + N_{i} x(k) u(k) \)

+ \( A_{i} x(k - \tau(k)) + D_{i} \alpha(k) \)

(26a)

where \( V(x(k)) \) is given in (12). By Lemma 1, we have
\[
\Delta V(x(k)) + \rho^2 \varepsilon^T (k) z(k) - \gamma \alpha^2 (k) \alpha(k)
\]
\[
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} h_i h_j h_m \left[ \left( \rho^2 + 1 + \varepsilon \right) A_{i}^T P A_{i} - S - P \right] x(k - \tau(k)) < 0
\]

(22)

Where \( \xi(k) = \left[ x(k) \ x(k - \tau) \ \alpha(k) \right] \), \( \Theta_{a} \) and \( \Theta_{b} \) is shown at the bottom of next two page.

For obtaining the (9) and (10), we assume the (22) is negative definite. Firstly, we consider the (23) and utilize the

Lemma 1, then (25) shown at the bottom of next two page, through a similar procedure, the (10) can also be obtained. Through a similar procedure, the (10) can also be obtained. Thus \( J_{s} < 0 \), which implies \( \xi^T \xi < \gamma \alpha \alpha \) for all nonzero \( \omega \) in \( l_{2} \), if (9) and (10) are satisfied. This completes the proof of the theorem.
where *rand* is a random number taken from a normal distribution over \([-1, 1]\), \(E_{ii} = \begin{bmatrix} 0.3 & 0 \\ 0.3 & 0 \end{bmatrix}\) and \(E_{ii} = \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0.1 \end{bmatrix}\). The membership functions of the state \(x_i\) are defined as \(\mu_i = 0.5(1 + x_i)\) and \(\mu_i = 0.5(1 - x_i)\). Let \(\rho = 0.3\) and choose the controller gain matrices as \(D_1 = \begin{bmatrix} -0.1 & -0.4 \end{bmatrix}\) and \(D_2 = \begin{bmatrix} -0.4 & -0.1 \end{bmatrix}\).

In this example, we choose the \(H_\infty\) performance level \(\gamma = 1\) and applying \(\rho\) and all these matrices to inequalities (9) and (10) in Theorem 1, and utilizing the LMI tool box [7], one can figure out the common positive-definite matrices,

\[
P = \begin{bmatrix} 19.649 & 5.0093 \\ 5.0093 & 12.0597 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 7.3641 & 5.0042 \\ 5.0042 & 6.7715 \end{bmatrix}.
\]

The simulation results of applying the robust \(H_\infty\) fuzzy controller (5) to T-S time-delay DFBS with disturbance (26) with three different initial conditions \(x(0) = [1.5 \ -2]^T\), \(x(0) = [-1.5 \ -1.2]^T\), and \(x(0) = [-1.3 \ 1.2]^T\) and the exogenous \(\alpha(t) = \begin{bmatrix} 0.5e^{-0.006t} \sin(2k) \\ 0.3e^{-0.006t} \cos(2k) \end{bmatrix}\) are illustrated in Figure 1.
From these simulation results, it can be seen the designed fuzzy controller ensures the robust asymptotic stability of the closed-loop system and guarantees a prescribed $H_{\infty}$ performance level.

V. CONCLUSION

This paper has proposed a robust $H_{\infty}$ fuzzy control scheme for a class of T-S time-delay DFBS with disturbance. For stabilizing the T-S time-delay DFBS, some sufficient conditions have been derived to guarantee the stability of the overall fuzzy control system via LMIs. Finally, the design approach has been applied to a numerical example to illustrate the validity and effectiveness of the proposed schemes.

[REFERENCES]