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A Robust SDP Approach to System Identification with Roughly Quantized Data

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***Abstract*—This paper proposes an identification method for linear systems with roughly quantized outputs. Measurement data sampled from low resolution sensors have large quantization errors, which deteriorate the identification accuracy. While the identification problem is formulated into quadratic programming with uncertainty, a proposed method provides an approximate optimal solution via semidefinite programming. Numerical examples demonstrate that we can estimate both plant parameters and true outputs in practical time and show the effectiveness of the proposed method.**

***Index Terms*—system identification, quantization error, least square method, semidefinite programming**

I. Introduction

This paper focuses on plants with low resolution sensors and deals with a system identification with roughly quantized outputs. Although developments of technology allow us to measure with high resolution sensors, low performance and low resolution sensors are still used because of commercial reasons. In commercial items some devices are implemented at low cost, especially in items for automobiles. For example, spatial resolution of sensors in a motor implemented in automobiles is 15 degree ($\pi/12$[rad]). In order to identify plant parameters, we usually implement high resolution sensors instead of low resolution sensors, however, it is sometimes impossible because of mechanical reasons. In this case, we have to identify plant parameters using data sets which have large quantization errors.

Quantization errors cause deterioration of the identification accuracy. In order to increase the identification accuracy, the papers [1], [2] proposed the methods to estimate not only plant parameters but also quantization errors. They estimate both plant parameters and quantization errors simultaneously, and identify a plant more precisely. The method proposed in [2] utilizes the least square method and introduces the objective function with penalty functions to estimate quantization errors. We can estimate plant parameters and quantization errors practically by minimizing the objective function via nonlinear least square method. Unfortunately, the objective function is not convex, and therefore it is difficult to obtain an exact optimal solution. We have another problem that the estimated parameters strongly depend on weight values of penalty functions since penalty functions take a large value when we identify plant parameters using data sets with large quantization errors. It is, therefore, necessary to design the weight value empirically before the identification, however we never know a way how to design it.

This paper proposes a method to estimate both plant parameters and quantization errors based on the least square method. The identification problem is formulated as a semidefinite programming, which is convex optimization problem and whose global optimal solution can be found using modern SDP solvers such as SeDuMi [5] that applies the interior point method.

The organization of the papers is as follows. In Section II, the identification problem with quantized data sets is formulated, and proposes the identification method using semideinite programming. Section III shows numerical examples and demonstrates the effectiveness of the proposing method. In Section IV, conclusions are drawn.

II. Estimation of Plant Parameters and Quantization Errors

A. Identification Problem

The plant to be identified is a single-input and single-output linear discrete-time system whose transfer function is

$$G(z) = \frac{b_m z^m + b_{m-1} z^{m-1} \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0}, \qquad (1)$$

where $a_1, a_2, \ldots, a_{n-1}$ and $b_0, b_1, \ldots, b_m$ are constants. This paper assumes that the orders m, n ($m \leq n$) are known. The input-output difference equation can be described as follows.

$$\begin{aligned} & y_{k+n} + a_{n-1} y_{k+n-1} + \ldots + a_1 y_{k+1} + a_0 y_k \\ & = b_m u_{k+m} + b_{k+m-1} u_{k+m-1} + \ldots + b_1 u_{k+1} + b_0 u_k, \end{aligned} \qquad (2)$$

where $y_{k+n}, \ldots, y_k$ are outputs, $u_{k+m}, \ldots, u_k$ are inputs, and each subscript indicates the corresponding discrete time. In this paper we consider that the output y_k is quantized, and a quantizer is defined by

$$q(x) = \delta \left\lfloor \frac{x}{\delta} + \frac{1}{2} \right\rfloor, \qquad (3)$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and δ denotes a resolution. The above definition implies that the quantizer returns the integer value which is nearest to $l\delta$ for $l = 0, \pm 1, \pm 2, \ldots$. Then let us define a quantized output $\bar{y}_k$ by

$$\bar{y}_k = q(y_k), \qquad (4)$$

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and finally the quantization error is defined as follows.

$$\gamma_k = y_k - \bar{y}_k. \tag{5}$$

For identification of the discrete-time system (1), if the true outputs y_k are known, the least square method enables us to estimate plant parameters minimizing the objective function J:

$$\begin{aligned} J &= \sum_{i=0}^{N-1} e_{i+n}^2, & (6)\\ e_{i+n} &= (y_{i+n} + \hat{a}_{n-1} y_{i+n-1} + \ldots + \hat{a}_0 y_i) & (7)\\ &\quad -(\hat{b}_m u_{i+m} + \hat{b}_{i+m-1} u_{i+m-1} + \ldots + \hat{b}_0 u_i), & (8) \end{aligned}$$

where $\hat{a}_k$ and $\hat{b}_l$ are parameters to identify, and N is the number of data sets. However, only quantized outputs are known. Even if the quantized outputs $\bar{y}_k$ are used instead of y_k, we can not identify parameters of the plant accurately because each $\bar{y}_k$ has a quantization error. The objective of identification in this paper is to estimate both plant parameters and the true output y_k, and hence the identification problem is formulated as the following problem.

Problem 1: For given $u = [u_0\ u_1\ \ldots\ u_{N-1+m}]$ and $\bar{y} = [\bar{y}_0\ \bar{y}_1\ \ldots\ \bar{y}_{N-1+n}]^T \in \Re^{N+n}$, find $\hat{y} = [\hat{y}_0\ \hat{y}_1\ \ldots\ \hat{y}_{N-1+n}]^T$ such that $q(\hat{y}_k) = \bar{y}_k$ for $k = 0, 1, \ldots, N-1+n$ and that minimizes $\hat{J}$ defined by

$$\begin{aligned} \hat{J} &= \sum_{i=0}^{N-1} \hat{e}_{i+n}^2,\\ \hat{e}_{i+n} &= (\hat{y}_{i+n} + \hat{a}_{n-1} \hat{y}_{i+n-1} + \ldots + \hat{a}_0 \hat{y}_i)\\ &\quad -(\hat{b}_m u_{i+m} + \hat{b}_{i+m-1} u_{i+m-1} + \ldots + \hat{b}_0 u_i),\\ &= \hat{h}_{i+n}^T \hat{\theta}, \end{aligned}$$

where

$$\hat{h}_{i+n} = [\, \hat{y}_{i+n} \quad \ldots \quad \hat{y}_i \quad -u_{i+m} \quad \ldots \quad -u_i \,]^T$$

and

$$\hat{\theta} = [\, 1 \quad \hat{a}_{n-1} \quad \ldots \quad \hat{a}_0 \quad \hat{b}_m \quad \ldots \quad \hat{b}_0 \,]^T .$$

This problem implies that we find valid output $\hat{y}_k$ satisfying that $q(\hat{y}_k) = \bar{y}_k$ and that there exist $\hat{\theta}$ such that $\hat{J}$=0 ideally.

Since the equation $q(\hat{y}_k) = \bar{y}_k$ can be described as the following inequality,

$$\bar{y}_k - \frac{\delta}{2} < \hat{y}_k \le \bar{y}_k + \frac{\delta}{2}, \tag{9}$$

Problem 1 equals the following minimization problem.

$$\text{Minimize } \hat{J} \text{ subject to } \quad \begin{array}{l} \bar{y}_k - \dfrac{\delta}{2} < \hat{y}_k \le \bar{y}_k + \dfrac{\delta}{2} \\ \text{for } k = 0, 1, \ldots, N-1+n, \end{array} \tag{10}$$

where $\hat{\theta}$ and $\hat{y} = [\hat{y}_0\ \ldots\ \hat{y}_{N-1+n}]^T$ are variables. Unfortunately, this is non-convex optimization problem and difficult to solve.

B. Identification method using semidefinite programming

The papers [1], [2] proposed a method to obtain an approximate solution of the problem (10). They introduced a penalty function corresponding to constrained conditions, and defined the modified objective function J_{mod}:

$$J_{mod} = \sum_{i=0}^{N-1} \hat{e}_{i+n}^2 + \xi \sum_{i=0}^{N-1} (\hat{y} - \bar{y})^2 , \tag{11}$$

where ξ is a given constant and a weight value of the penalty function. The second term on the right hand of the above equation may force $\bar{y}_k$ to stay within the range defined in the inequality (9). These methods are available if quantization error is small, however, they do not identify parameters accurately if the outputs are roughly quantized because of two reasons. One is because the objective function J_{mod} is non-convex, which implies that it is difficult to obtain an exact optimal solution. The other is cased by the penalty function. Because the penalty function lets $\hat{y}_k$ be close to $\bar{y}_k$, the true output y_k is not estimated correctly if the value of y_k is far from $\bar{y}$.

In order to overcome these difficulties, this paper proposes a method using semidefinite programming. Let us consider the following minimization problem instead of (10).

$$\text{Minimize } t \text{ subject to } \hat{J} \le t \text{ for all } \hat{y} \in \mathcal{D}, \tag{12}$$

where

$$\mathcal{D} := \left\{ \begin{bmatrix} \hat{y}_0 \\ \hat{y}_1 \\ \vdots \\ \hat{y}_p \end{bmatrix} \in \Re^p : \begin{array}{l} \bar{y}_k - \frac{\delta}{2} \le \hat{y}_k \le \bar{y}_k + \frac{\delta}{2} \\ \text{for all } k = 0, 1, \ldots, p \\ p = N - 1 + n \end{array} \right\},$$

and where $a = [a_0\ \ldots\ a_{n-1}]^T$ and $b = [b_0\ \ldots\ b_m]^T$ are variables to find. This minimization problem minimizes $\hat{J}$ for the worst-case combination of $\hat{y} \in \mathcal{D}$. Therefore, it provides the upper bound of the minimization problem (10). In other words, it gives us the most valid parameters of the plant for all $\hat{y} \in \mathcal{D}$.

From Schur complement, it holds that $\hat{J} \le t$ if and only if

$$\begin{bmatrix} t & \hat{\theta}^T \hat{h} \\ \hat{h}^T \hat{\theta} & 1 \end{bmatrix} \succeq 0 \tag{13}$$

holds because $\hat{J} = \sum_{i=0}^{N-1} \left(\hat{h}_{i+n}^T \hat{\theta} \right)^2 = \hat{\theta}^T \hat{h} \hat{h}^T \hat{\theta}$. In the above inequality, $X \succeq 0$ means that X is a positive semideinite matrix. Since the inequality $\hat{J} \le t$ in the minimization problem (12) can be described by the conic inequality (13) and since the set $\mathcal{D}$ is a hyper rectangle, the constraints of the minimization problem (12) can be described as finite number of conic inequalities using vertices of the hyper rectangle $\mathcal{D}$. Finally we obtain the following minimization problem to identify the plant parameters.

$$\text{Minimize } t \text{ subject to } \begin{bmatrix} t & \hat{\theta}^T \hat{h} \\ \hat{h}^T \hat{\theta} & 1 \end{bmatrix} \succeq 0 \text{ for all } \hat{y} \in \bar{\mathcal{D}} \tag{14}$$

where the set $\bar{\mathcal{D}}$ denotes a set of all vertices in $\mathcal{D}$ and has 2^p elements. This minimization problem is a semidefinite programming (SDP), which is convex and can be efficiently solved using the interior-point optimization technique [3], [4]. Solving the minimization problem, both the plant parameters and the true output are estimated.

Algorithm 1: (Identification Algorithm)

Step1. Solve the minimization problem (14) to estimate the plant parameters $\hat{a}$ and $\hat{b}$.

Step2. Using $\hat{a}$ and $\hat{b}$, simulate the output based on the input-output difference equation (2).

III. Numerical Examples

This section presents numerical examples to demonstrate the effectiveness of the proposed method. We consider a simple model of a motor described by a continuous-time system:

$$G_c(s) = \frac{1}{Ms^2 + Ds}. \quad (15)$$

where M and D denote inertia and viscosity of the motor, respectively. Let $M = 3.5$ and $D = 12.5$, and using MATLAB command `c2d` which discretizes the continuous-time system, we obtain a the discrete-time transfer function $G(z)$ as follows.

$$G(z) = \frac{0.05823z + 0.01952}{z^2 - 1.028z + 0.02812}. \quad (16)$$

Then let us consider the closed loop system with constant output feedback as shown in Fig. 1. The quantized output is feedback to the input, that is,

$$\begin{aligned} y_k &= G(z)u_k, \\ \bar{y}_k &= q(y_k), \\ u_k &= K(u_{dk} - \bar{y}_{k-1}), \end{aligned}$$

where u_{dk} is desired input at step k, the input u_k and quantized output $\bar{y}_k$ are measurable, and static feedback gain K is known.

We consider two cases that the quantization resolution is 10 degree with $K = 11.3$ and 15 degree with $K = 17.4$. In both cases the desired input is taken as a step input as follows.

$$u_{dk} = \begin{cases} 0 & \text{[degree]} \quad (k < 3) \\ 60 & \text{[degree]} \quad (k \geq 3) \end{cases}$$

Fig. 2 and Fig. 6 show the true output y_k and the quantized output $\bar{y}_k$. The inputs u_k in these cases are shown in Fig. 3 and Fig 7. We assume here that the outputs are 0 [degree] exactly before step input, that is, they have no quantization errors at step $k = 0, 1, 2$. This is attainable if a motor has mechanism for forcing its angle to 0 [degree].

The proposed method is compared with a simple least square method and the modified square method proposed in [2]. In a simple square method, the plant parameters are estimated by minimizing the objective function J defined in equation (6) for quantized outputs $\bar{y}_k$ instead of true outputs y_k, and y_k is estimated by simulating the input-output difference equation (2). In the modified least square method, J_{mod} of equation (11) is the objective function to minimize, where a solution depends on the value of ξ. The optimal solution also depends on an initial value of $\hat{y}_k$ in applying nonlinear least square method because J_{mod} is a non-convex function of $\hat{\theta}$ and $\hat{y}_k$.

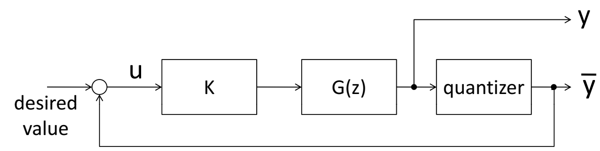

Fig. 1. Closed-loop system considered in numerical examples.

These three methods are applied for the data set $\{u_0, u_1, \ldots, u_{11}, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{12}\}$, and $\xi = 0.03,\ 0.1,\ 1,\ 3$ in the modified objective function J_{mod}. The output sequences to estimate are y_k for $k = 3, 4, \ldots, 12$ because it is assumed that y_0, y_1 and y_2 are known and have no quantization errors. Therefore there are 10 quantized outputs to estimate and $2^{10} = 1024$ conic inequalities in the SDP (14).

Utilizing SeDuMi [5] and YALMIP [6] to solve the SDP in applying the proposed method, and utilizing the MATLAB function `fminunc` of Optimization Toolbox to minimize J and J_{mod} in applying two kinds of least square methods, the plants parameters and outputs are estimated. Fig. 4 and Fig. 8 show the estimated outputs of the modified least square method for several values of ξ. Fig. 5 and Fig. 9 present the estimated outputs of three methods, where the outputs of the modified least square are obtained in the case that $\xi = 3$. The sum of square errors and the maximum error between the true output and the estimated output for each method are shown in Table I and Table II. Table III presents the estimated transfer functions.

We can see that the accuracy of the modified least square method depends on the value of ξ, and that the best value of ξ is 3 from Table I and Table II. In the present examples, the estimated outputs approach the quantized outputs if $\xi > 3$, and zeros if $\xi < 0.03$. Note that we never know the best value of ξ when we apply the modified least square method before identification because we do not know the true output. This implies that we have to design ξ before applying the method.

We can see that the proposed method can provides the estimated outputs which are close to the true outputs. The sum of square errors in Table I and Table II implies that the the proposed method can estimate the outputs much more precisely than other two methods in both two cases of 10 degree resolution and 15 degree resolution.

The CPU time for solving the SDP (14) is about 14.4 [sec] on Mac Pro with 6.0GB Memory and 2.06GHz CPU. Table IV shows the relationship between CPU time and the number of the outputs to estimate. The CPU time nearly doubles when the number of the outputs to estimate increases by 1 because the number of conic inequalities doubles. This implies that the proposed method enables us to estimate both plant parameters and true outputs in practical time if the number of data sets is not large. The objective of controlling a motor with extremely low resolution sensors as considered in this paper is to track to step inputs of several predetermined value, and a motor has nonlinearity cased by several physical and mechanical factors such as cogging forces. Therefore it is usual in practice that a

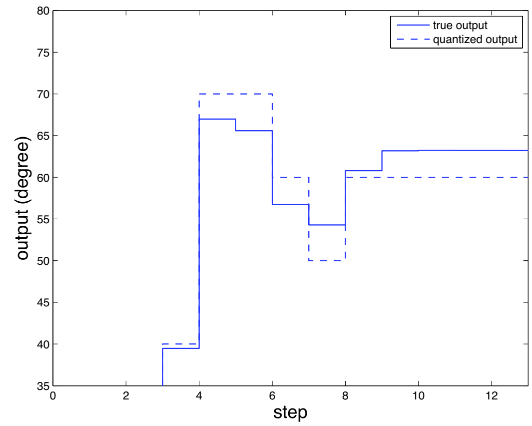

Fig. 2. True output y_k and quantized output $\bar{y}_k$. The quantization resolution is 10 [degree].

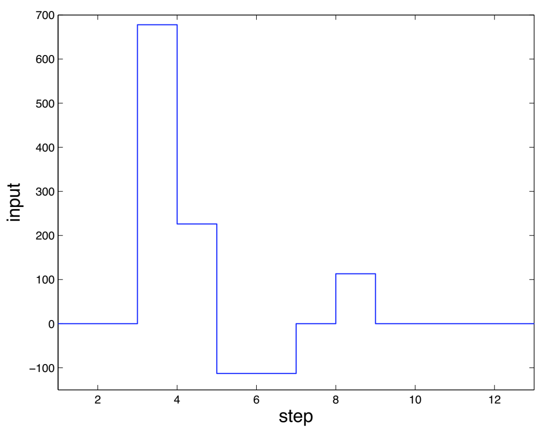

Fig. 3. Input $u(k)$. The quantization resolution is 10 [degree].

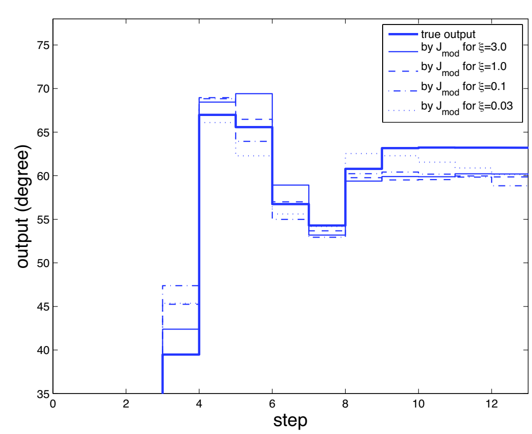

Fig. 4. True output and estimated outputs obtained by minimizing J_{mod} for $\xi = 0.03, 0.1, 1$ and 3 in the case that quantization resolution is 10 [degree].

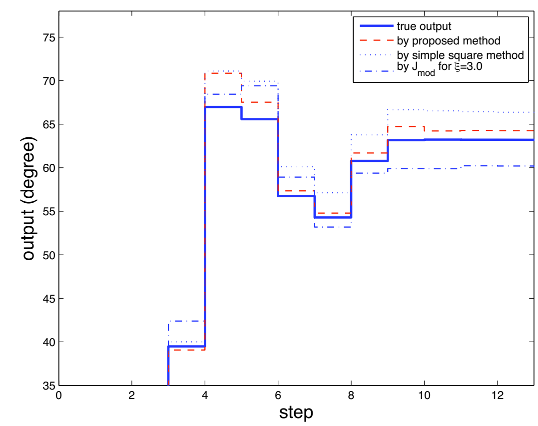

Fig. 5. Results of the case that the quantization resolution is 10 [degree]. Solid line, dashed line, dotted line and dash-dot line denote true output, estimated output by the proposed method, estimated output by a simple least square method and estimated output by the modified least square method using J_{mod}, respectively.

TABLE I

SUM OF SQUARE ERRORS AND MAXIMUM OF ERRORS BETWEEN THE TRUE OUTPUT AND THE ESTIMATED OUTPUTS.

method	$\sum_{k=0}^{12}(y_k - \hat{y}_k)^2$	$\max \lvert y_k - \hat{y}_k \rvert$
proposed method	26.04	4.423
simple least square method	108.12	4.362
minimizing J_{mod} $(\xi = 3)$	108.87	3.831
minimizing J_{mod} $(\xi = 1)$	147.86	5.765
minimizing J_{mod} $(\xi = 0.1)$	205.50	7.905
minimizing J_{mod} $(\xi = 0.03)$	160.30	5.875

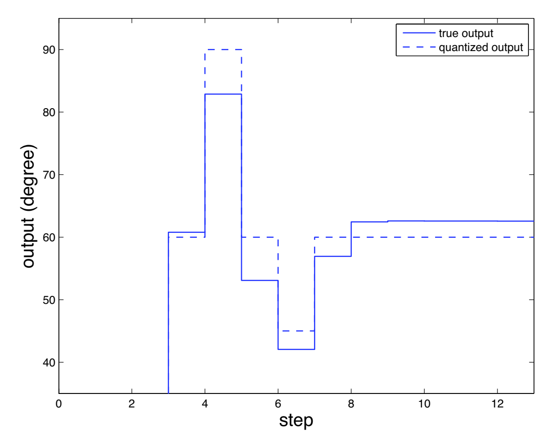

Fig. 6. True output y_k and quantized output $\bar{y}_k$. The quantization resolution is 15 [degree].

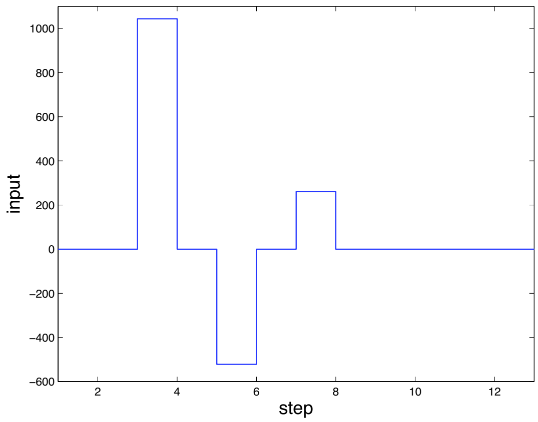

Fig. 7. Input $u(k)$. The quantization resolution is 15 [degree].

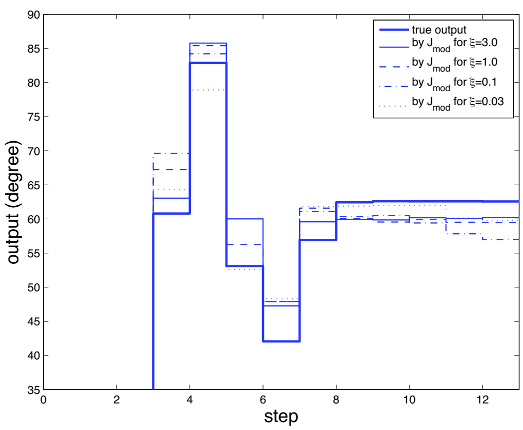

Fig. 8. True output and estimated outputs obtained by minimizing J_{mod} for $\xi = 0.03, 0.1, 1$ and 3 in the case that quantization resolution is 15 [degree].

TABLE II
SUM OF SQUARE ERRORS AND MAXIMUM OF ERRORS BETWEEN THE TRUE OUTPUT AND THE ESTIMATED OUTPUTS.

method	$\sum_{k=0}^{12}(y_k - \hat{y}_k)^2$	$\max \lvert y_k - \hat{y}_k \rvert$
proposed method	34.22	3.644
simple least square method	130.28	5.669
minimizing J_{mod} ($\xi = 3$)	186.1991	6.9304
minimizing J_{mod} ($\xi = 1$)	270.20	6.430
minimizing J_{mod} ($\xi = 0.1$)	356.60	8.8241
minimizing J_{mod} ($\xi = 0.03$)	1273.50	6.250

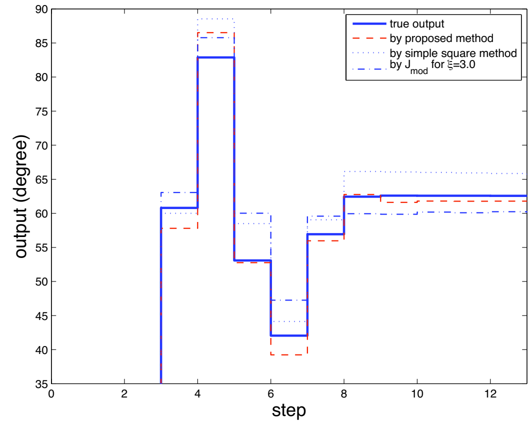

Fig. 9. Results of the case that the quantization resolution is 15 [degree]. Solid line, dashed line, dotted line and dash-dot line denote true output, estimated output by the proposed method, estimated output by a simple least square method and estimated output by the modified least square method using J_{mod}, respectively.

TABLE III
ESTIMATED TRANSFER FUNCTIONS.

resol. [deg]	simple least square method	proposed method	using J_{mod} for $\xi = 3.0$
10	$\frac{0.0590z+0.0274}{z^2-0.9792z-0.0196}$	$\frac{0.0576z+0.0362}{z^2-0.8501z-0.1495}$	$\frac{0.3068z-0.0643}{z^2-1.304z+0.3068}$
15	$\frac{0.0020z+0.0575}{z^2-1.0009z+0.0273}$	$\frac{0.0554z+0.0371}{z^2-0.8226z-0.1775}$	$\frac{0.4364z-0.0500}{z^2-1.432z+0.4364}$

plant is identified not using data sets of M-sequence for open-loop system but using data sets of a step input for closed-loop system, and that a parameter identification is carried out several times for each step input value. In this case, we can not control precisely, and it is often the case that the same data set of input and output is measured for different experiments because of rough quantization. Thus the number of data sets is not large in practice in the identification of a plant with roughly quantized outputs. This means that the proposed method is practically effective from the view point of CPU time.

IV. CONCLUSION

This paper focused on a plant with low resolution sensors, dealt with a system identification for roughly quantized data sets, and proposed a method to estimate both plant parameters and true outputs. Although the identification problem is formulated as non-convex and nonlinear programming, the proposed method provides an approximate solution by solving a semidefinite programming. Numerical examples demonstrated that we can estimate both plant parameters and the true output in practical time, and show the effectiveness of the prosed method.

A disadvantage of the proposed method is that the number of conic inequalities of the SDP (14) increases in proportion to

TABLE IV
CPU TIME VS. NUM. OF $\hat{y}_k$.

num. of $\hat{y}_k$	num. of conic inequalities	computation time [sec]
4	16	0.346866
5	32	0.535757
6	64	0.961295
7	128	1.765356
8	256	3.128351
9	512	6.338576
10	1024	14.405220

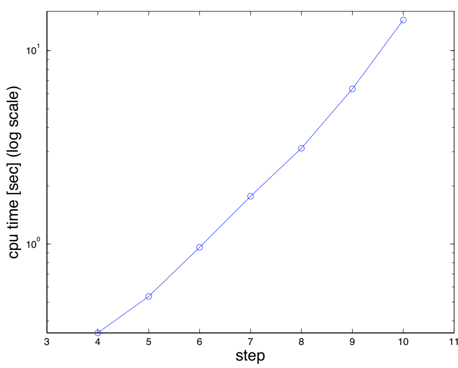

Fig. 10. Plot of Table IV. CPU time is presented on a logarithmic scale.

the square of the number of data sets. This causes exponential increase of computational time if we consider the system identification using more complicated and longer input and output sequences. Some optimization algorithms, such as a sum-of-square approach [7], could be used in order to obtain an approximate solution for large data sets in practical time.

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