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Constructions of Equiangular Tight Frames with Genetic Algorithms

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***Abstract*— Equiangular tight frames have applications in communications, signal processing, and coding theory. Previous work demonstrates that few real equiangular tight frames exist for most pairs (n,d), where the frame $\Phi_{n,d}$ is a $d \times n$ matrix with $d \leq n$. This work proposes a genetic algorithm as a solution to the frame design problem. Specifically, the problem of designing real equiangular tight frames by minimizing the subspace minor angle sum-squared error. Numerical experiments show that the proposed method is successful for pairs (n,d) with d less than nine.**

Keywords*— *Genetic Algorithms, Optimization, Frames, Equiangular Tight Frames

I. Introduction

Frame theory was originally introduced in Duffin and Schaeffer [4] in the context of non-harmonic Fourier series. Young [15] in 1980 formally defined a frame in terms of vectors. In 1986, Daubechies, Grossman, and Meyer [3], presented examples of frames for contexts outside of non-harmonic Fourier series analysis, and more recently in [2],[7],[12],[13],[14]. They were reintroduced to the general signal processing community by Kovačević and Chebira [8]. Frame applications range from coding theory and numerical analysis to communications and sparse approximations [7],[13],[14].

Equiangular tight frames (ETFs) are considered an important class of finite-dimensional frames [14] with numerous practical applications and they said to be are a natural choice for combining the advantages of orthonormal bases with the concept of redundancy provided by frames [7],[8],[12],[14]. So the problem here is: How do you construct a real ETF $\Phi_{n,d}$ for a specific pair (*n,d*)?

A. Contributions

This paper proposes a genetic algorithm (GA) [5] and [6] based technique for the construction of equiangular tight frames. Previous construction techniques of equiangular frames are known [9], from the ordinary regular simplex, to frames based on perfect difference sets, quadratic residues, conference matrices [1],[10], Hadamard matrices [12], and Paley tournaments [9], however these are algebraic methods and are considered arduous, specifically for Grassmanian frames [10]. Two known numerical methods are from Sloane [10] and Tropp et al. [14]. These methods involve the use sphere packing in Grassmanian spaces and an alternating projection method, respectively, to find ETFs. No other numerical methods are known to exist. Therefore, it will be demonstrated that with a simple fitness function a GA based solution allows for the construction of real ETFs.

B. Outline

The remainder of this paper is as follows: Section 2 will provide some background information on frames, introduce the mathematics behind frames, and identify the uniqueness of equiangular tight frames. Section 3 will describe the GA based ETF construction method and experimental variables. Section 4 discusses the numerical experiments and the results. Section 5 follows with a conclusion and a few thoughts on future work.

II. Frames

Let V be a Hilbert space. A frame of V is a finite ordered set $\{e_i\}$ of vectors in V which satisfies the frame condition: There exists positive real numbers A and B with $0 < A \leq B < \infty$ such that

$$A\|v\|^2 \leq \sum_i \left|\langle v, e_i \rangle\right|^2 \leq B\|v\|^2 \tag{1}$$

for all $v \in V$. The numbers A and B are called the lower and upper frame bounds. If $A = B$, then one says that the frame is tight and if $A = B = 1$, then the frame is said to be a normalized tight frame (also called a Parseval frame). Like a basis, a frame spans the vector space V ; however, a frame may be linearly dependent.

If V is a finite dimensional real inner product space, then it is convenient to study frames from the point of view of matrices and singular values. Suppose $V = \Re^d$ is an inner product space with the standard inner product $\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i$ and that $E = [e_1 \cdots e_n]$ is an $d \times n$ matrix with $d \leq n$. When is the set of vectors $\{e_1, \cdots, e_n\}$ a frame? To answer this question, note that

$$\sum_{i=1}^{n} \left|\langle v, e_i \rangle\right|^2 = v^T E E^T v, \tag{2}$$

which readily follows from the fact that $[\langle v, e_1 \rangle \cdots \langle v, e_n \rangle] = v^T E.$ The term on the right $v^T E E^T v$ is a quadratic form and is hence bounded above and below by the maximum and minimum

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eigenvalues of the symmetric matrix EE^T, which are given by the squares of the maximum and minimum singular values of E, i.e.,

$$\sigma_d^2(E)\|v\|^2 \le \sum_{i=1}^{n} \left|\langle v, e_i \rangle\right|^2 \le \sigma_1^2(E)\|v\|^2. \qquad (3)$$

Hence, the tightest values for A and B are $A = \sigma_d^2(E)$ and $B = \sigma_1^2(E)$. This implies that a matrix E with at least as many columns as rows corresponds to a frame if and only if it has full rank. In other words, the vectors $\{e_i\}$ form a frame if and only if they span the complete vector space. If the frame is tight, i.e. $A = B$, then all of the singular values of E are equal and consequently, E is equal to a scalar multiple of a matrix with the property that its rows are orthonormal. Hence, any matrix E corresponding to a tight frame can be found by taking the first d rows of an $n \times n$ orthogonal matrix and multiplying the resulting matrix by a nonzero scalar. Furthermore, given a $d \times n$ matrix E corresponding to a normalized tight frame, one can always augment E with an appropriate set of rows to obtain an orthogonal matrix. Thus the requirement that E correspond to a normalized tight frame induces the additional structure that the rows are orthonormal.

Figures 1 and 2 below show examples of (3,2) and (5,2) frames Φ_1 and Φ_2, where Φ_1 is a normalized tight frame, also called a Parseval tight frame, with $A = B = 1$ and Φ_2 is not.

$$\Phi_1 = \begin{pmatrix} 0.6323 & -0.2430 & -0.7356 \\ 0.4365 & -0.6728 & 0.5974 \end{pmatrix} \text{ with } A = B = 1$$

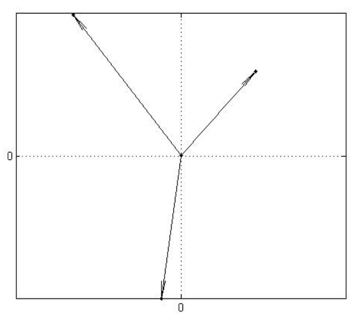

Figure 1. An example of a (3,2) tight frame.

$$\Phi_2 = \begin{pmatrix} 0.6007 & -0.1352 & -0.8331 & -0.6532 & 0.6628 \\ -0.0924 & 0.6506 & -0.7337 & -0.2181 & 0.6067 \end{pmatrix}$$

with $A = 0.613$ and $B = 2.7118$.

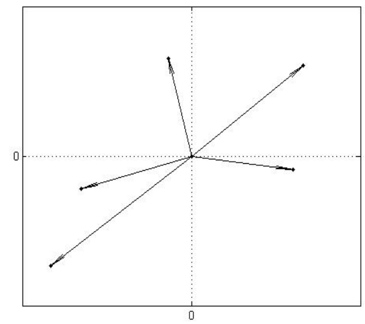

Figure 2. An example of a (5,2) frame.

Equiangular tight frames, optimal Grassmannian frames, or 2-uniform frames, are a family of vectors $\{f_k\}_{k=1}^{n}$ in E^d (where $\mathrm{E} = \mathrm{R}$ or C) that satisfy the following conditions [12]

1. $\|f_k\|_2 = 1$ for $k = 1,...,n$,
2. $\left|\langle f_k, f_l \rangle\right| = c$, for all $k \ne l$ and some constant c ,
3. $\frac{d}{n}\sum_{k=1}^{n} \langle f, f_k \rangle f_k = f$, for all $f \in \mathrm{E}^m$.

These conditions together imply that $\left|\langle f_k, f_l \rangle\right| = \sqrt{\frac{n-d}{d(n-1)}}$, for all $k \ne l$, which is the smallest possible value for c for a set of n equiangular unit-norm vectors in E^d . Figure 3 below shows an example of an (3,2) ETF Φ_3 .

$$\Phi_3 = \begin{pmatrix} 0 & -\sqrt{3}/2 & \sqrt{3}/2 \\ 1 & -1/2 & -1/2 \end{pmatrix} \text{with } A = B = 1$$

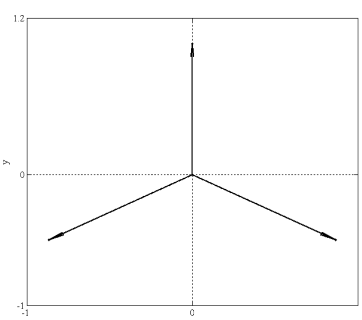

Figure 3. Figure 3: An (3,2) ETF, with A = B = 1 and $\|f_k\|_2 = 1$.

Φ_3 is also known as the *Mercedes-Benz* frame [8], or Peres-Wooters states in quantum information theory [8].

III. GENETIC ALGORITHM EQUIANGULAR TIGHT FRAME CONSTRUCTIONS

In this work each individual solution represents a frame, encoded as a vector for genetic recombination operations. These vectors are recombined using selection pressures presented by the environment (fitness function). The environment here is modeled as a fitness function that will allow us to evolve an ETF for a given (*n,d*) pair. The fitness function is based on the constraints specified below to evaluate the suitability of the subspace angles of each frame solution.

A. Fitness/Cost Function

Minimize the sum squared error function *F*, given that the optimal subspace angle $\theta_{n,d}$ for an ETF(*n,d*) is defined as

$$\theta_{n,d} = \cos^{-1}\left(\sqrt{\frac{n-d}{d(n-1)}}\right) \tag{4}$$

and the subspace angles $\theta_{k,l}$ for a given frame are defined as

$$\theta_{k,l} = \cos^{-1}\left(\left|\langle f_k, f_l\rangle\right|\right), \text{ for all } k,l \text{ with } k \neq l. \tag{5}$$

This means that

$$\left|\langle f_k, f_l\rangle\right| = c_{n,d} = \sqrt{\frac{n-d}{d(n-1)}}, \tag{6}$$

or that the absolute inner products for column vectors *k,l* must be equal to some constant $c_{n,d}$ for all $k \neq l$. Therefore, a suitable cost function to optimize is the sum-squared error between the optimal subspace and $\theta_{n,d}$ and the angles of a particular frame $\theta_{k,l}$. That is,

$$F = \left(\sum_{i,j}^{n}(\theta_{n,d} - \theta_{k,l})^2\right) \tag{7}$$

and thus, goal here is to then to minimize *F* where

$$\min(F) = \min\left(\sum_{k,l}^{n}(\theta_{n,d} - \theta_{k,l})^2\right). \tag{8}$$

B. Genetic Algorithm and Parameters

A uniform-elite based selection on a population of two hundred real (and binary) chromosomes with the top twenty reproducing with the remaining population is used here. In addition, a form of migration occurs every fifty iterations where the bottom twenty percent is replaced with random solutions. Each solution is a vector that is reshaped to (*n,d*) for the frame evaluation. For reproduction a two-point crossover on the vectors is employed and the mutation is Gaussian, whereby a random number, or mutation, is chosen from a Gaussian distribution. The amount of mutation, which is proportional to the standard deviation of the distribution, decreases at each new generation. The basic algorithm is described below in Figure 4:

The Genetic Algorithm:

1) Define the size of the *n* by *d* solution frame
   a) Randomly initialize the population N of length *nd* solutions
2) Evaluate each solution in the population
   a) Using cost function
   b) Rank solutions from highest to lowest scoring
   c) If stopping criteria is true then return best solution, else go to step 3.
3) Create new population by mating current population;
   a) Selection
   b) Crossover
   c) Mutation
   d) Migration
4) Go to step 2.

Figure 4. Genetic algorithm process.

The experimental variables for the GA are listed in Figure 5:

Experimental Variables:

Population: 200, Crossover: Two-point, Mutation: Gaussian(1,1), Selection: Uniform, Elite=20, Crossover Fraction=0.8, Migration: Forward (ξ = 20%, every 50 iterations), Iterations = 5000, *Error F* Tolerance: 0.005.

Figure 5. Genetic algorithm experimental variables.

IV. NUMERICAL EXPERIMENTS

The goal here is to solve for real ETFs for a given (*n,d*) pair by minimizing the cost function *F*. There were two experiments here, the first experiment involved searching for real ETFs over the range $(-\infty, +\infty)$ and the second experiment limited the ETF values to binary values [-1,1]. The first experiment found solutions for $\Phi_{d+1,d}$, $\Phi_{6,3}$, and $\Phi_{10,5}$, however the (10,5) solution is poor. The second experiment found a solution for $\Phi_{16,6}$. Table 1 shows the results for these two experiments. Listed in the table are the optimal subspace angle $\theta_{n,d}$, the optimal norm-inner-product constant $c_{n,d}$, the minimum *F* (*Error F*), the sum-squared error

$$(Error\ C\ = \sum_{k,l}^{n}(\left|\langle f_k, f_l\rangle\right| - c_{n,d})^2) \tag{9}$$

between the optimal $c_{n,d}$ and the solution, and the frame bounds A,B for $\Phi_{n,d}$.

TABLE I. EXPERIMENTAL RESULTS FOR GA-BASED ETF CONSTRUCTIONS.

EFT(n,d)	$\theta_{n,d}$ (º)	$c_{n,d}$	*Error F*	*Error C*	*A,B*
(3,2)	120	0.5	0.00445	4.58e-9	1.5,1.5
(4,3)	109.471	0.3333	0.00759	1.563e-8	1.33,1.33
(5,4)	104.478	0.25	0.00944	2.54-8	1.25,1.25
(6,5)	101.537	0.2	0.0182	9.64e-8	1.2,1.2
(7,6)	99.5941	0.1667	0.0234	1.628e-8	1.66,1.67
(8,7)	98.2132	0.1428	0.0443	5.88e-7	1.42,1.44
(9,8)	97.1808	0.0603	0.125	1.089e-6	1.12,1.12
(10,9)	96.3794	0.105	0.1111	3.292e-6	1.11,1.11
(6,3)	63.4349	0.44721	0.0048	5.694e-9	2.0,2.0
(10,5)	70.528	0.3333	17.75	8.24e-2	1.15,2.69
(16,6)	70.528	0.3333	2.089e-13	1.035e-30	2.67,2.67

As shown in Table 1, ETFs $\Phi_{d+1,d}$ and $\Phi_{6,3}$ are most easily found, however a solution for ETF $\Phi_{10,5}$ was not very fit. In addition, the ETF $\Phi_{16,6}$ was only found by binary search. Generally, the *Error F* increased for increasing pair (d+1,d), which is expected for fixed iterations, however the solutions for $\Phi_{6,3}$ and $\Phi_{16,6}$ were found relatively quickly, $\Phi_{16,6}$ in 100 iterations, and with lower *Error F* than most of the (d+1,d) pairs.

Example solutions are shown in Figure 6 for a (4,3) ETF and in Figure 7 for a (6,3) ETF.

$$\Phi_{4,3} = \begin{pmatrix} 0.7078 & -0.5033 & 0.4261 & -0.6305 \\ -0.5874 & 0.4526 & 0.6897 & -0.5548 \\ 0.3924 & 0.7361 & -0.5855 & -0.5429 \end{pmatrix} \text{with } A = B = 1.33$$

and

$$\Phi_{6,3} = \begin{pmatrix} 0.4481 & 0.7780 & 0.5917 & -0.1471 & 0.0581 & -0.9047 \\ 0.8923 & 0.1479 & -0.7883 & 0.6218 & 0.4166 & -0.0207 \\ -0.0545 & 0.6106 & 0.1690 & 0.7692 & -0.9072 & 0.4254 \end{pmatrix}$$

with $A = B = 2.0$.

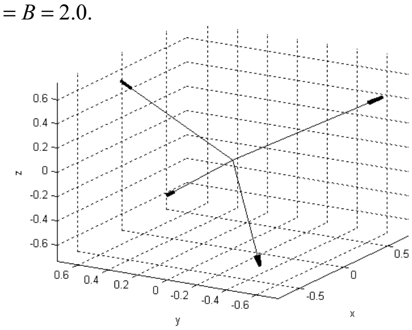

Figure 6. An evolved construction of a (4,3) ETF.

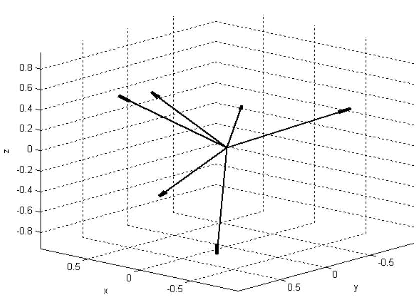

Figure 7. An evolved construction of a (6,3) ETF.

In addition, a resulting solution for a (16,6) EFT by genetic algorithm binary construction is shown below.

$$\Phi_{16.6} = 1/\sqrt{6}\begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The above result shows similarity to results found by Hadamard matrix solutions [13].

V. CONCLUDING REMARKS AND FUTURE WORK

As proposed, a genetic algorithm method for the construction of ETFs has shown to be successful. Some limitations of the algorithm were demonstrated in the solution for the (10,5) real ETF, whereby no acceptable solution was found. Overall however, solutions were found for all other ETF pairs (n,d) with d less than nine. Future research will involve searching for higher dimensional ETFs, complex ETFs, and sparse ETFs for compressed sensing applications.

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