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Uncertainty Measures for General Type-2 Fuzzy Sets

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***Abstract*—Five uncertainty measures have previously been defined for interval type-2 fuzzy sets (IT2 FSs), namely centroid, cardinality, fuzziness, variance and skewness. Based on a recently developed α-plane representation technique, this paper generalizes these definitions to general T2 FSs and, more importantly, derives a unified strategy for computing all different uncertainty measures with low complexity. The uncertainty measures of T2 FSs with different shaped *Footprints of Uncertainty* (*FOU*) and different triangular secondary membership functions are computed and are given as examples.**

***Keywords*— α-plane representation, centroid, cardinality, fuzziness, variance, skewness, type-2 fuzzy sets**

I. INTRODUCTION

Type-2 fuzzy sets (T2 FSs) are extensions of type-1 fuzzy sets (T1 FSs), so it is intuitive and consistent for us to define the uncertainty measures of T2 FSs based on their T1 counterparts. The latter have been very well studied, e.g., [4], [12], [14], [24], [28]. It is straightforward to extend these T1 uncertainty measures to a T2 FS by using the *Wavy-Slice Representation Theorem* (*WS RT*) of a T2 FS (also known as *embedded T2 FS* or the *Mendel-John RT*). This was already done by Karnik and Mendel [12], [18] for the centroid of a general T2 FS; however, until very recently there was no practical way to compute the centroid of such a T2 FS, because it would require the explicit enumeration of an extremely large number of embedded T2 FSs.

Different approaches for computing the centroid have been studied. Greenfield et al. [6] compute the centroid of randomly selected embedded T2 FSs, instead of computing the centroid for all embedded T2 FSs. Coupland [3] utilizes the *x*-coordinate of the geometric centroid of the 3D MF of the T2 FS. John and Czarnecki [7] and Lucas el al. [16] use the centroids of all vertical slices of the T2 FS.

Recently, Liu [15] (see, also, [21]) showed how to compute the centroid using an α-plane representation (also called horizontal-slice representation) of a T2 FS. His method uses KM Algorithms, which were developed by Karnik and Mendel [13], [18], [20], [27], for computing the centroid of an interval T2 FS (IT2 FS).

Additionally, Wu and Mendel [26] recently developed the concepts and practical computational procedures for cardinality, fuzziness, variance and skewness of an IT2 FS. They did this by using the *WS RT*. This paper combines the works of Wu, Liu and Mendel and shows how the α-plane approach can be used to compute cardinality, fuzziness, variance and skewness of a general T2 FS. Such uncertainty measures have never been computed before.

II. BACKGROUND

A. Conventional Representation of General T2 FSs

A general T2 FS $\tilde{A}$ can be described by its vertical-slice representation, as:

$$\tilde{A} = \int_{\forall x \in X} \left[\int_{\forall u \in J_x \subseteq [0,1]} f_x(u)/u \right] \Big/ x \tag{1}$$

In (1), *x* is the primary variable; J_x, an interval in [0,1], is the primary membership of *x*; *u* is the secondary variable; and, $f_x(u)$ is the secondary MF at *x*. Mendel and John [22] have provided the following important *WS RT* for $\tilde{A}$:

$$\tilde{A} = \bigcup_{\forall \tilde{A}_e} \tilde{A}_e \tag{2}$$

where $\tilde{A}_e$ is an embedded T2 FS that includes only one primary membership at each *x*. Note that the embedded T1 FS A_e that corresponds to an embedded T2 FS is the primary memberships of that $\tilde{A}_e$. This paper relies heavily on (2) to develop the uncertainty measures for T2 FSs by beginning with the corresponding uncertainty measures of T1 FSs.

B. α-Plane Representation for a T2 FS

The α-plane *RT* of a T2 FS was first stated and proved by Liu [15]. Wagner and Hagras [25] independently arrived at the same representation. Note that an α-plane of a T2 FS is analogous to an α-cut of a T1 FS.

Definition 1. An α-plane for a general T2 FS $\tilde{A}$, denoted $\tilde{A}_\alpha$, is the union of all primary memberships of $\tilde{A}$ whose secondary grades are greater than or equal to α $(0 \le \alpha \le 1)$:

$$\begin{aligned} \tilde{A}_\alpha &= \{(x,u), \mu_{\tilde{A}}(x,u) \ge \alpha \mid \forall x \in X, \forall u \in J_x \subseteq [0,1]\} \\ &= \int_{\forall x \in X} \int_{\forall u \in J_x} \{(x,u) \mid f_x(u) \ge \alpha\} \quad \square \end{aligned} \tag{3}$$

$S_{\tilde{A}}(x \mid \alpha)$ denotes an α-cut of the secondary MF $f_x(u)$, i.e.

$$S_{\tilde{A}}(x \mid \alpha) = [s_L(x \mid \alpha), s_R(x \mid \alpha)] \tag{4}$$

Definition 2. Let $I_{\tilde{A}_\alpha}(x,u \mid \alpha)$ be a 3D indicator function for α-plane $\tilde{A}_\alpha$, where:

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$$I_{\tilde{A}_\alpha}(x,u\,|\,\alpha)=\begin{cases}1, \forall x\in X \text{ and } \forall u\in[s_L(x\,|\,\alpha), s_R(x\,|\,\alpha)] \\ 0, \qquad\qquad\qquad\qquad\qquad \text{otherwise}\end{cases} \quad □ \quad (5)$$

Definition 3. An α-plane *FOU*, $FOU(\tilde{A}_\alpha)$, is:

$$FOU(\tilde{A}_\alpha)=\alpha I_{\tilde{A}_\alpha}(x,u\,|\,\alpha) \quad □ \qquad (6)$$

Observe from (5) that $s_L(x\,|\,\alpha)$ and $s_R(x\,|\,\alpha)$ $\forall x\in X$ denote the *Upper* and *Lower Membership Functions* (*UMF* and *LMF*) for $FOU(\tilde{A}_\alpha)$, respectively, and are used in rest of this paper.

Liu's [15] α-plane RT is:

$$\tilde{A}=\bigcup_{\alpha\in[0,1]} FOU(\tilde{A}_\alpha) \qquad (7)$$

This RT is very useful because each $FOU(\tilde{A}_\alpha)$ can be viewed as an IT2 FS of level α. Consequently, operations involving T2 FSs can be performed by using existing techniques that have already been developed for IT2 FSs.

III. Uncertainty measures for general T2 FS

A. Definitions of Uncertainty Measures for a General T2 FS

The uncertainty measure of a T2 FS is a T1 FS, $M_{\tilde{A}}$, whose MF is $M_{\tilde{A}}(\xi)$, $\forall \xi\in X$. The following two-step procedure is proposed for defining $M_{\tilde{A}}(\xi)$:

1) Choose a well-established definition of an uncertainty measure, $m(A)$, for a T1 FS *A*. This paper uses the uncertainty measure definitions from [26].
2) Use the *WS RT* as:

$$M_{\tilde{A}}(\xi)\equiv\bigcup_{\forall \tilde{A}_e} m(\tilde{A}_e) \qquad (8)$$

where the uncertainty measure of $\tilde{A}_e$, $m(\tilde{A}_e)$, is defined as:

$$m(\tilde{A}_e)\equiv \min_{(x,u)\in\tilde{A}_e}\{f_x(u)\}\big/ m(A_e) \qquad (9)$$

and A_e is the corresponding embedded T1 FS of $\tilde{A}_e$.

Centroid of a general T2 FS: The centroid of a T1 FS *A*, *c*(*A*), is

$$c(A)=\frac{\sum_{i=1}^N x_i\mu_A(x_i)}{\sum_{i=1}^N \mu_A(x_i)} \qquad (10)$$

Definition 4. The centroid of $\tilde{A}$, $C_{\tilde{A}}$, is the union of the centroids of all of its embedded T2 FSs. Its MF, $C_{\tilde{A}}(\xi)$, is:

$$C_{\tilde{A}}(\xi)=\bigcup_{\forall\tilde{A}_e}\left\{\min_{(x_i,u_i)\in\tilde{A}_e} f_x(u)\Bigg/\frac{\sum_{i=1}^N x_i u_i(A_e)}{\sum_{i=1}^N u_i(A_e)}\right\} \quad □ \quad (11)$$

where $u_i(A_e)$ denotes the membership of the embedded T1 FS A_e that corresponds to $\tilde{A}_e$

Cardinality of a general T2 FS: Different definitions of cardinality for T1 FSs have been proposed, e.g., [4], [9], [5], etc. Follwoing [26], the normalized version of De Luca and Termini's [4] definition of cardinality, $p(A)$, is used in this paper:

$$p(A)=\frac{|X|}{N}\sum_{i=1}^N \mu_A(x_i) \qquad (12)$$

where $|X|=x_N-x_1$ is the length of the universe of discourse used in the computation.

Definition 5. The cardinality of $\tilde{A}$, $P_{\tilde{A}}$, is the union of the cardinalities of all of its embedded T2 FSs. Its MF, $P_{\tilde{A}}(\xi)$, is:

$$P_{\tilde{A}}(\xi)=\bigcup_{\forall\tilde{A}_e}\left\{\min_{(x_i,u_i)\in\tilde{A}_e} f_x(u)\Bigg/\frac{|X|}{N}\sum_{i=1}^N u_i(A_e)\right\} \quad □ \quad (13)$$

To-date, studies about the cardinality of a T2 FS have been very limited, e.g., see [8].

Fuzziness of a general T2 FS: Fuzziness (entropy) of a T1 FS is used to quantify the vagueness it has. A T1 FS *C* is most fuzzy when all its memberships equal 0.5. A T1 FS *A* is fuzzier than a T1 FS *B* if *A* is nearer to such a *C* than *B* is. And the fuzziness of a T1 FS may be any function that satisfies the requirements of a *general fuzziness measure* [10]. Yager's [28] definition of a fuzziness measure, $f(A)$, is preferred:

$$f_Y(A)=1-\frac{\left[\sum_{i=1}^N|2\mu_A(x_i)-1|^r\right]^{1/r}}{N^{1/r}} \qquad (14)$$

where *r* is a positive constant.

Definition 6. The fuzziness of $\tilde{A}$, $F_{\tilde{A}}$, is the union of the fuzziness of all of its embedded T2 FSs. Its MF, $F_{\tilde{A}}(\xi)$, is:

$$F_{\tilde{A}}(\xi)=\bigcup_{\forall\tilde{A}_e}\left\{\min_{(x_i,u_i)\in\tilde{A}_e} f_x(u)\Bigg/ 1-\frac{\left[\sum_{i=1}^N|2u_i(A_e)-1|^r\right]^{1/r}}{N^{1/r}}\right\} \quad □ \quad (15)$$

Note that fuzziness of IT2 FSs has also been studied in [2] and [23].

Variance of a general T2 FS: The variance of a T1 FS *A* measures its compactness, i.e., a smaller (larger) variance means *A* is more (less) compact. Lee and Li [14] defined the variance of a T1 FS, $v(A)$, as:

$$v(A)=\frac{\sum_{i=1}^N[x_i-c(A)]^2\mu_A(x_i)}{\sum_{i=1}^N\mu_A(x_i)} \qquad (16)$$

where $c(A)$ is defined in (10).

Definition 7. The variance of $\tilde{A}$, $V_{\tilde{A}}$, is the union of the variances of all of its embedded T2 FSs. Its MF, $V_{\tilde{A}}(\xi)$, is:

$$V_{\tilde{A}}(\xi)=\bigcup_{\forall\tilde{A}_e}\left\{\min_{(x_i,u_i)\in\tilde{A}_e} f_x(u)\Bigg/\frac{\sum_{i=1}^N[x_i-c(A_e)]^2 u_i(A_e)}{\sum_{i=1}^N u_i(A_e)}\right\} \quad □ \quad (17)$$

Skewness of a general T2 FS: The skewness of a T1 FS is a measure of its symmetry. It is negative when the FS skews to the right, is positive when it skews to the left, and is zero when the FS is symmetric.

Different definitions of skewness for a T1 FS have been proposed in [1] and [24]. Because the centroid, variance and skewness of a FS should be viewed as its first-, second- and third-order moments, respectively, their definitions should be consistent; hence, the following definition of skewness, $s(A)$, for a T1 FS has been proposed in [26]:

$$s(A)=\frac{\sum_{i=1}^{N}[x_i-c(A)]^3\,\mu_A(x_i)}{\sum_{i=1}^{N}\mu_A(x_i)} \qquad (18)$$

where $c(A)$ is defined in (10).

Definition 8. The skewness of $\tilde{A}$, $V_{\tilde{A}}$, is the union of the skewness of all of its embedded T2 FSs. Its MF, $V_{\tilde{A}}(\xi)$, is:

$$S_{\tilde{A}}(\xi)=\bigcup_{\forall \tilde{A}_e}\left\{\min_{(x_i,u_i)\in\tilde{A}_e} f_x(u)\Big/\frac{\sum_{i=1}^{N}[x_i-c(A_e)]^3\,u_i(A_e)}{\sum_{i=1}^{N}u_i(A_e)}\right\} \quad \square \qquad (19)$$

B. Computing Uncertainty Measures for a General T2 FS

Direct computations of the above defined uncertainty measures for T2 FSs are not practical because for even reasonable (which is less than "high") accuracy they require enumerations of an extremely large number of embedded T2 FSs. Using the α-plane RT, Liu [15] has already shown that $C_{\tilde{A}}(\xi)$ $(\forall x\in X)$ can be computed by taking the union of the centroids of the α-plane *FOU*s of $\tilde{A}$, i.e.,

$$C_{\tilde{A}}(\xi)=\bigcup_{\alpha\in[0,1]}\alpha\big/C_{\tilde{A}_\alpha}(\xi) \qquad (20)$$

Decompositions like (20) are also valid for the other uncertainty measures, as we state next.

Theorem 1. Let $m(A)$ be a given uncertainty measure for a T1 FS A, and let $M_{\tilde{A}}(\xi)$ $(\forall x\in X)$ be the corresponding uncertainty measure for a T2 FSs $\tilde{A}$, where

$$M_{\tilde{A}}(\xi)\equiv\bigcup_{\forall\tilde{A}_e}m(\tilde{A}_e) \qquad (21)$$

then

$$M_{\tilde{A}}(\xi)=\bigcup_{\alpha\in[0,1]}\alpha\big/M_{\tilde{A}_\alpha}(\xi),\ \forall\xi\in X \quad \square \qquad (22)$$

where $M_{\tilde{A}_\alpha}(\xi)$ is the uncertainty measure of the α-plane *FOU*, $\tilde{A}_\alpha$. And $M_{\tilde{A}_\alpha}(\xi)$ is an interval, i.e.

$$M_{\tilde{A}_\alpha}(\xi)=[m_l(\tilde{A}_\alpha),m_r(\tilde{A}_\alpha)] \qquad (23)$$

where $m_l(\tilde{A}_\alpha)$ and $m_r(\tilde{A}_\alpha)$ denote the left and right end of the interval $M_{\tilde{A}_\alpha}(\xi)$, respectively.

The proof of theorem 1 will be given in the journal version of this paper. Theorem 1 lets us compute our previously defined T2 FS uncertainty measures with reasonable computational costs. Greater accuracy can be obtained by using a larger number of α-planes . The uncertainty measures of each α-plane *FOU* can be computed using algorithms proposed in [26] for computing IT2 FS uncertainty measures. Consequently, our general procedure for computing an uncertainty measure for a general T2 FS is:

1) Decide on how many α-plane *FOU*s will be used, where $\alpha\in[0,1]$. Call that number *k*; its choice will depend on the accuracy that is required. Regardless of *k*, $\alpha=0$ and $\alpha=1$ must always be used.
2) For each α, compute $\tilde{A}_\alpha$.
3) Compute the uncertainty measure of $\tilde{A}_\alpha$, $M_{\tilde{A}_\alpha}(\xi)$, using existing algorithms from [26]. The accuracy of this step depends on the discretization of the primary variable, *x*.
4) Repeat steps 2, 3 for the k α values chosen in step 1.
5) Collect the k $M_{\tilde{A}_\alpha}(\xi)$ to obtain $M_{\tilde{A}}(\xi)$ using (22).

For completeness, algorithms for computing the different uncertainty measures of $\tilde{A}_\alpha$ are reviewed next. Their derivations can be found in [26] by simply replacing $\underline{\mu}(x)$ and $\overline{\mu}(x)$ with $s_L(x\mid\alpha)$ and $s_R(x\mid\alpha)$, respectively.

C. Computational Algorithms for the Uncertainty Measures of an IT2 FS $\tilde{A}_\alpha$

The *centroid* of $\tilde{A}_\alpha$ is:

$$C_{\tilde{A}_\alpha}(\xi)=\bigcup_{\forall A_e(\alpha)}\frac{\sum_{i=1}^{N}x_iu_i(A_e(\alpha))}{\sum_{i=1}^{N}u_i(A_e(\alpha))}=[c_l(\tilde{A}_\alpha),c_r(\tilde{A}_\alpha)] \qquad (24)$$

where $A_e(\alpha)$ are the embedded T1 FSs of $\tilde{A}_\alpha$, and $c_l(\tilde{A}_\alpha)$ and $c_r(\tilde{A}_\alpha)$ can be computed using KM algorithms [12], [18].

The *cardinality* of $\tilde{A}_\alpha$ is can be computed as:

$$P_{\tilde{A}_\alpha}(\xi)=[p_l(\tilde{A}_\alpha),p_r(\tilde{A}_\alpha)]=[p(s_L(x\mid\alpha)),p(s_R(x\mid\alpha))] \qquad (25)$$

where $p(\bullet)$ is defined in (12).

The *fuzziness* of $\tilde{A}_\alpha$ is also an interval and can be computed as:

$$F_{\tilde{A}_\alpha}(\xi)=[f_l(\tilde{A}_\alpha),f_r(\tilde{A}_\alpha)]=[f_Y(A_{e1}(\alpha)),f_Y(A_{e2}(\alpha))] \qquad (26)$$

where $f_Y(A)$ is defined in (14), and $A_{e1}(\alpha)$ and $A_{e2}(\alpha)$ are T1 FSs defined as:

$$\mu_{A_{e1}(\alpha)}(x)=\begin{cases}s_R(x\mid\alpha), & s_R(x\mid\alpha)\text{ is further away from 0.5 than } s_L(x\mid\alpha),\\ s_L(x\mid\alpha), & \text{otherwise.}\end{cases} \qquad (27)$$

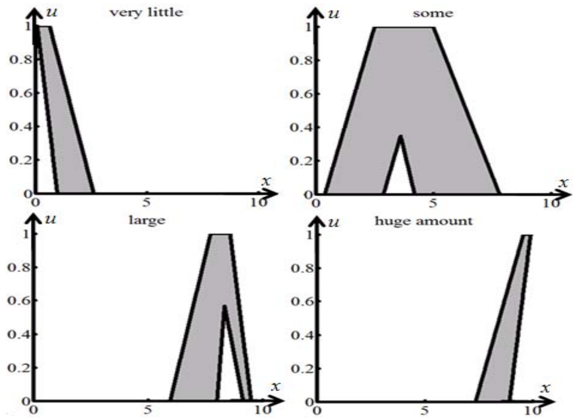

Figure 1. The four selected *FOUs* from the 32-word codebook in [17].

$$\mu_{A_{e2}(\alpha)}(x)=\begin{cases} s_R(x\,|\,\alpha),\ s_R(x\,|\,\alpha) \text{ and } s_L(x\,|\,\alpha)<0.5 \\ s_L(x\,|\,\alpha),\ s_R(x\,|\,\alpha) \text{ and } s_L(x\,|\,\alpha)>0.5 \\ 0.5, \quad \text{otherwise.} \end{cases} \tag{28}$$

The *variance* of $\tilde{A}_\alpha$ is:

$$V_{\tilde{A}_\alpha}(\xi)=\bigcup_{\forall A_e(\alpha)} \frac{\sum_{i=1}^{N}\left[x_i-c(A_e(\alpha))\right]^2 u_i(A_e(\alpha))}{\sum_{i=1}^{N}u_i(A_e(\alpha))} \tag{29}$$

Unlike previous uncertainty measures, there is no practical solution for computing (29). Similar to what was proposed in [26], the defuzzified value of $C_{\tilde{A}}(\xi)$, $c(C_{\tilde{A}}(\xi))$, is used in place of $c(A_e(\alpha))$.

Definition 9. The *relative variance* of an IT2 FS $\tilde{A}_\alpha$ is defined as:

$$\begin{aligned} V_{\tilde{A}_\alpha}(\xi) &= \bigcup_{\forall A_e(\alpha)} \frac{\sum_{i=1}^{N}\left[x_i-c(C_{\tilde{A}}(\xi))\right]^2 u_i(A_e(\alpha))}{\sum_{i=1}^{N}u_i(A_e(\alpha))} \\ &= [v_l(\tilde{A}_\alpha), v_r(\tilde{A}_\alpha)] \end{aligned} \tag{29}$$

where $v_l(\tilde{A}_\alpha)$ and $v_r(\tilde{A}_\alpha)$ can also be computed using KM algorithms, as explained in [26]. □

The skewness of $\tilde{A}_\alpha$ is:

$$S_{\tilde{A}_\alpha}(\xi)=\bigcup_{\forall A_e(\alpha)} \frac{\sum_{i=1}^{N}\left[x_i-c(A_e(\alpha))\right]^3 u_i(A_e(\alpha))}{\sum_{i=1}^{N}u_i(A_e(\alpha))} \tag{30}$$

To avoid the same difficulty as (28), the defuzzified value of $C_{\tilde{A}}(\xi)$, $c(C_{\tilde{A}}(\xi))$, is again used in place of $c(A_\alpha^e)$ in (30).

Definition 10. The *relative skewness* of an IT2 $\tilde{A}_\alpha$ is defined as:

$$\begin{aligned} S_{\tilde{A}_\alpha}(\xi) &= \bigcup_{\forall A_e(\alpha)} \frac{\sum_{i=1}^{N}\left[x_i-c(C_{\tilde{A}}(\xi))\right]^3 u_i(A_e(\alpha))}{\sum_{i=1}^{N}u_i(A_e(\alpha))} \\ &= [s_l(\tilde{A}_\alpha), s_r(\tilde{A}_\alpha)] \end{aligned} \tag{31}$$

where $s_l(\tilde{A}_\alpha)$ and $s_r(\tilde{A}_\alpha)$ can also be computed using KM algorithms, as explained in [26]. □

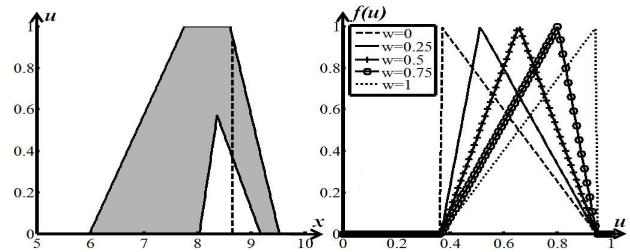

Figure 2(a) *FOU* for the word *large*, and (b) five triangular secondary MFs parameterized by *w* [see (32)] when *x* = 8.65.

IV. EXAMPLES

In this section, uncertainty measures for different T2 FSs are computed using the procedures described above. The specific T2 FSs were selected in the following way:

1) Four representative *FOUs* were selected from the 32-word codebook in [17] to form $FOU(\tilde{A}_0)$. The four words used are *very little*, *some*, *large* and *huge amount*. Their FOUs are depicted in Fig.1 and the parameters of their UMFs and LMFs are given in Table I. These four FOUs were chosen because they have quite different shapes and cover different spans of the primary variable's domain.

2) For each word, its secondary MF was chosen to be a triangle with a base equal to $s_R(x\,|\,0)-s_L(x\,|\,0)$ and apex location, *Apex*(*x*), parameterized as [15] (*w*=0,0.25,0.5,0.75,1) :

$$Apex(x)=s_L(x\,|\,0)+w\left[s_R(x\,|\,0)-s_L(x\,|\,0)\right] \tag{32}$$

An example of the secondary MFs for the word *large* is shown in Fig. 2b at *x*=8.65.

How many discretized α values are needed to produce a close enough approximation to the centroid has been studied in [15] and [21]. Based on those studies, we discretized $\alpha\in[0,1]$ into 21 values, for which $\alpha=0,\ 0.05,\ 0.1,...,\ 0.95,\ 1$.

The five uncertainty measures for *large* are depicted in Fig. 3 for the five values of *w*. Observe that all of the uncertainty measures are T1 FSs, so how to interpret and summarize them are new issues, since these measures are intervals for an IT2 FS. Due to space limitations, we focus here only on how to summarize them. This is done by computing the defuzzified value of each of these T1 FSs. Table II provides numerical results for the centroids of all four words for all five values of *w*. In that table, each word has three columns, labeled T2, IT2 and QT2. The first column provides the defuzzified values of all the uncertainty measures of T2 FS $\tilde{A}$ for the five values of *w*. The second column provides the defuzzified values of all the uncertainty measures of $\tilde{A}_0$ for the five values of *w*, which is equivalent to treating $\tilde{A}$ as an IT2 FS. The third column provides the defuzzified values of a triangle T1 FS formed by

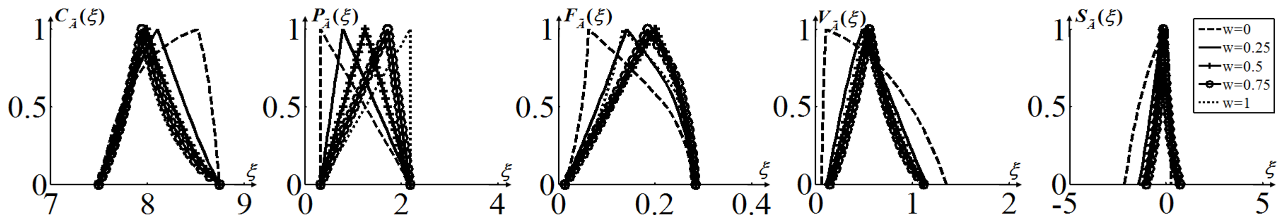

Figure 3. Five uncertainty measures for *large* and its five triangular secondary MFs that are parameterized by *w* [see (32)]. See Fig. 2b for an explanation of which curve is for which value of *w*. Note that the labels for the horizontal axes in these plots differ from plot to plot.

connecting the uncertainty measures of $\tilde{A}_1$ (a point) to the end-points of the uncertainty measures of $\tilde{A}_0$ (an interval). It is equivalent to treating $\tilde{A}$ as a *Quasi-Type-2 FS* (QT2 FS), a set first suggested in [19]. In order to study the differences between the three centroids, the differences (in percentage) (IT2-T2)/T2 and (QT2-T2)/T2 are summarized in Table III.

Comparing the IT2 and QT2 defuzzified values with the T2 defuzzified values can show if it's necessary to use a full-blown general T2 FS. From Table III, it's clear that the QT2 FS always gives better or equal approximations than does the IT2 FS; and, when the errors by using QT2 FSs increase/decrease, the errors by using IT2 FSs also increase/decrease.

It appears that, ***except for the centroid for some words, replacing a general T2 FS by either an IT2 or a QT2 FS gives too large errors.***

V. Conclusions

This paper has extended five well-known T1 FS uncertainty measures to general T2 FSs by using the *WS RT*. It has also shown how Liu's α-plane *RT* can be used to compute the five uncertainty measures of a general T2 FS by using existing algorithms for computing the comparable uncertainty measures for an IT2 FS. This is done one α-plane at a time. Examples were given that showed the five uncertainty measures for four T2 FSs that have five kinds of triangle secondary MFs. They show that, except for the centroid, replacing a general T2 FS by either an IT2 or a QT2 FS is not a good thing to do.

The approach used in this paper can also be applied to define and compute similarity, ranking and subsethood of general T2 FSs, and this work is in progress.

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Table I. *FOU* data for four words. Each LMF and UMF is represented as a trapezoid (a, b, c, d). The fifth parameter for the LMF is its height.

Word	**LMF**	**UMF**
Very little	(0, 0, 0.09, 0.99, 1)	(0, 0, 0.64, 2.63)
Some	(2.88, 3.61, 3.61, 4.21, 0.35)	(0.38, 2.50, 5.00, 7.83)
Large	(8.03, 8.36, 8.36, 9.17, 0.57)	(5.98, 7.75, 8.60, 9.52)
Huge amount	(8.95, 9.93, 10, 10, 1)	(7.37, 9.59, 10, 10)

Table II. Defuzzified values of five uncertainty measures of the T2, IT2 and QT2 FSs for four words, triangular secondary MFs and five *w* values.

		Very little			***some***			***Large***			***huge amount***		
	w	T2	IT2	QT2	T2	IT2	QT2	T2	IT2	QT2	T2	IT2	QT2
Centroid	0	0.492	0.661	0.551	3.799	3.904	3.791	8.286	8.125	8.257	9.507	9.347	9.449
	0.25	0.658	0.661	0.649	3.914	3.904	3.905	8.085	8.125	8.118	9.360	9.347	9.363
	0.5	0.755	0.661	0.697	3.935	3.904	3.916	8.011	8.125	8.082	9.267	9.347	9.316
	0.75	0.821	0.661	0.727	3.945	3.904	3.919	7.971	8.125	8.066	9.203	9.347	9.287
	1	0.869	0.661	0.746	3.951	3.904	3.921	7.945	8.125	8.056	9.155	9.347	9.267
Cardinality	0	0.726	1.091	0.909	1.019	2.601	1.812	0.634	1.259	0.947	0.723	1.044	0.884
	0.25	0.909	1.091	1.000	1.811	2.601	2.206	0.947	1.259	1.103	0.884	1.044	0.964
	0.5	1.091	1.091	1.091	2.601	2.601	2.601	1.259	1.259	1.259	1.044	1.044	1.044
	0.75	1.274	1.091	1.183	3.392	2.601	2.996	1.570	1.259	1.414	1.204	1.044	1.124
	1	1.457	1.091	1.274	4.184	2.601	3.391	1.883	1.259	1.570	1.365	1.044	1.204
Fuzziness	0	0.069	0.093	0.077	0.186	0.310	0.222	0.110	0.150	0.121	0.070	0.094	0.079
	0.25	0.093	0.093	0.090	0.329	0.310	0.301	0.156	0.150	0.147	0.090	0.094	0.090
	0.5	0.110	0.093	0.100	0.431	0.310	0.365	0.189	0.150	0.167	0.106	0.094	0.099
	0.75	0.117	0.093	0.102	0.408	0.310	0.341	0.185	0.150	0.161	0.116	0.094	0.103
	1	0.109	0.093	0.095	0.328	0.310	0.289	0.160	0.150	0.145	0.114	0.094	0.100
Variance	0	0.234	0.430	0.314	2.013	3.232	2.197	0.410	0.713	0.513	0.220	0.402	0.295
	0.25	0.334	0.394	0.364	2.490	3.181	2.884	0.527	0.634	0.585	0.318	0.373	0.345
	0.5	0.368	0.394	0.384	2.555	3.177	2.944	0.552	0.629	0.600	0.356	0.373	0.366
	0.75	0.380	0.403	0.395	2.584	3.176	2.966	0.563	0.633	0.608	0.374	0.382	0.380
	1	0.384	0.414	0.403	2.603	3.176	2.978	0.568	0.638	0.614	0.381	0.393	0.390
Skewness	0	0.209	0.529	0.345	1.904	3.648	2.410	-0.411	-0.939	-0.606	-0.186	-0.481	-0.313
	0.25	0.227	0.319	0.271	0.832	1.919	1.472	-0.288	-0.441	-0.366	-0.219	-0.306	-0.260
	0.5	0.197	0.194	0.194	0.611	1.607	1.233	-0.210	-0.254	-0.233	-0.201	-0.196	-0.197
	0.75	0.169	0.105	0.131	0.510	1.461	1.117	-0.164	-0.149	-0.154	-0.177	-0.115	-0.142
	1	0.146	0.037	0.081	0.452	1.373	1.047	-0.134	-0.081	-0.101	-0.156	-0.052	-0.095

Table III. Percentage Differences between the defuzzified values of five uncertainty measures of the T2 and IT2 FSs and the T2 and QT2 FSs for four words, triangular secondary MFs and five *w* values.

		very little		***some***		***large***		***huge amount***	
	w	(IT2-T2)/T2	(QT2-T2)/T2	(IT2-T2)/T2	(QT2-T2)/T2	(IT2-T2)/T2	(QT2-T2)/T2	(IT2-T2)/T2	(QT2-T2)/T2
Centroid	0	34.45%	11.96%	2.75%	-0.21%	-1.94%	-0.35%	-1.68%	-0.61%
	0.25	0.52%	-1.43%	-0.28%	-0.23%	0.49%	0.40%	-0.13%	0.03%
	0.5	-12.46%	-7.69%	-0.80%	-0.50%	1.41%	0.88%	0.86%	0.53%
	0.75	-19.45%	-11.50%	-1.05%	-0.65%	1.93%	1.18%	1.57%	0.92%
	1	-23.89%	-14.13%	-1.19%	-0.74%	2.26%	1.39%	2.10%	1.23%
Cardinality	0	50.39%	25.27%	155.38%	77.86%	98.41%	49.33%	44.34%	22.24%
	0.25	20.08%	10.05%	43.66%	21.86%	32.93%	16.48%	18.11%	9.06%
	0.5	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
	0.75	-14.33%	-7.17%	-23.31%	-11.67%	-19.85%	-9.94%	-13.29%	-6.65%
	1	-25.10%	-12.59%	-37.83%	-18.96%	-33.15%	-16.62%	-23.50%	-11.79%
Fuzziness	0	34.57%	11.42%	66.57%	19.37%	36.37%	10.03%	35.33%	13.64%
	0.25	0.15%	-2.79%	-5.67%	-8.41%	-4.07%	-5.74%	4.53%	0.17%
	0.5	-15.64%	-9.37%	-28.08%	-15.45%	-20.65%	-11.64%	-11.42%	-6.91%
	0.75	-20.50%	-12.49%	-23.88%	-16.40%	-19.14%	-12.78%	-18.71%	-10.79%
	1	-14.74%	-12.75%	-5.51%	-11.91%	-6.68%	-9.86%	-17.60%	-12.77%
Variance	0	83.52%	33.88%	60.53%	9.14%	74.00%	25.13%	82.65%	34.27%
	0.25	17.92%	8.93%	27.73%	15.81%	20.31%	10.98%	17.35%	8.41%
	0.5	7.23%	4.37%	24.38%	15.24%	13.94%	8.58%	4.70%	2.82%
	0.75	6.14%	4.02%	22.93%	14.80%	12.50%	8.06%	2.16%	1.66%
	1	8.03%	5.15%	22.00%	14.41%	12.22%	8.01%	2.97%	2.14%
Skewness	0	153.23%	65.03%	91.56%	26.55%	128.66%	47.70%	157.75%	68.06%
	0.25	40.59%	19.12%	130.72%	76.97%	53.13%	26.98%	39.69%	18.61%
	0.5	-1.75%	-1.92%	162.90%	101.78%	20.97%	11.17%	-2.29%	-2.06%
	0.75	-37.68%	-22.06%	186.46%	119.14%	-9.05%	-6.09%	-34.77%	-20.00%
	1	-74.75%	-44.12%	203.93%	131.73%	-39.91%	-24.96%	-66.62%	-38.75%