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An Approximate Solution to the Norm Optimal Control Problem

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***Abstract*—The problem of finding the minimum input norm required to bring a dynamical system from an arbitrary initial state to a given end-point constraint set under specified state constraints, in a finite time interval, is considered. Even when there exists a control input which achieves the constraints, there may not be one with a minimum norm. For input-affine systems, it is shown that there is an admissible bang-bang control input whose norm is arbitrarily close to the minimum-norm. Since bang-bang functions are completely specified by their amplitude and switching times, the optimal norm can be numerically estimated by performing a finite dimensional search.**

***Index Terms*—norm-optimal control, bang-bang approximation, state constraints**

I. Introduction

Conventional concepts such as controllability [1] and constrained controllability [8, 9, 10] address the problems of existence and characterization of bounded control inputs for dynamical systems that can bring any initial state to any final constraint set in finite time. This article attempts to compute the minimum input amplitude required to accomplish the above objective. We consider only input/state systems which are affine in the input. Loosely, the problem addressed can be described as follows: Consider the state variable description of a dynamical system evolving over the finite time interval $[t_0, t_f]$, of the form:

$$\dot{x}(t) = f(x,t) + g(x,t)u(t) \quad x(t_0) = x_0 \text{ for } t_0 \le t \le t_f, \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$ for each $t \in [t_0, t_f]$; and $f : R^n \times [t_0, t_f] \to R^n$ and $g : R^n \times [t_0, t_f] \to R^{n \times m}$ are respectively vector valued and matrix valued functions with suitable properties (to be detailed later). Assume that an end point constraint set $C \subset R^n$ and a time varying state constraint set $G(t)$ are specified where $G(t) \subset R^n$ for each $t \in [t_0, t_f]$. Then find

$$\inf_{u(t)} \sup_{[t_0, t_f]} \|u(t)\|_\infty \tag{2}$$

such that $x(t) \in G(t)$ for all $t \in [t_0, t_f]$ and $x(t_f) \in C$. Here $\|\cdot\|_\infty$ denotes the standard l_∞ norm given by $\max_{i=1,\dots,m} |u_i(t)|$. The input that minimizes the norm while satisfying the constraints, is called the norm-optimal control. A precise definition of this problem is included in Section II.

If the problem is specified without state constraints i.e. when $G(t) = R^n$ for all $t \in [t_0, t_f]$, then it is identical to the version treated, among others, by Fattorini in [11] and the authors referenced therein, where the focus has been mainly on linear time invariant (LTI) systems. We aim to provide an approximate solution to the more general problem described above, for possibly nonlinear systems, which are however required to be affine in the input.

This problem arises in diverse practical applications. For example, mathematical models of diseases and infections in humans are being used widely for optimal drug dose calculation and treatment. In particular, mathematical models have been proposed and used widely for the understanding the interaction between the human immuno-deficiency (HIV) virus and the human immune system [7, 6], for the glucose-insulin interaction in diabetic patients [4] or for determining better protocols for chemotherapy in cancer patients [15]. Moreover, optimal control theory has been applied frequently on such models to find out the optimal drug dosage and their expected benefits and shortcomings [6, 13]. In such endeavors, finding the minimum dose required to control the disease seems to be a relevant (but *unanswered*) question, since very often enhanced dosage of drugs are associated with unwanted side-effects and high levels of toxicity. One of the possible applications of the theory developed in this paper is to address such questions efficiently and compute the minimum dose required to control the disease into a satisfactory end state.

A linear version of the above problem with no state constraints was shown to be equivalent to the linear time optimal control problem by Fattorini in [11], who used Bellman et al.'s formulation in [12] to prove this equivalence. Using such equivalence, one can show that the solution to the norm optimal problem turns out to be bang-bang for linear time invariant systems. For general non-linear system with state constraints, however, the above theory becomes invalid since the existence of neither the norm optimal control nor the time optimal control can be guaranteed. Even when the norm optimal solution exists, the first order necessary conditions used to characterize the solution are extremely complex in general and of little computational use.

In view of these difficulties, we propose the following methodology to estimate the optimal norm. In an appropriately defined set of permissible controls (say U), assume that there is at least one input $u(t)$ such that the solution to (1) under $u(t)$

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satisfies both the state constraint and the end-point constraint. Then, there may or may not be a norm optimal control in U. However there is always an input (say $u^a(t) \in U$) with a norm that is arbitrarily close to the optimal norm; such that the solution to (1) under $u^a(t)$ satisfies the state and end-point constraint. If we could compute such an $u^a(t)$ we could get an arbitrarily close estimate of the optimal norm. While it is not known how to compute such a $u^a(t)$, we show that an input equivalent to $u^a(t)$ can be found easily if one slightly enlarges the constraint sets. It turns out that there is a bang-bang input $u^{\pm}(t)$ whose norm is less than or equal to the norm of $u^a(t)$; in addition, the solution of (1) under $u^{\pm}(t)$ lies within an arbitrarily small distance of the original constraint sets. (Bang-bang functions can be loosely defined as functions whose components takes only one of at most two extreme values for almost all of the domain. See (5) for a precise definition). Such a small digression from the specified constraint sets is of little importance in most applications. However, the fact that this input is bang-bang provides us with an easy numerical method for estimating the optimal-norm approximately. Recall that the bang-bang functions are easy to compute as they are completely characterized by their amplitude and switching times. Hence, solution of the norm optimal control problem reduces to a finite dimensional search among the switching times and the amplitudes of the input components, instead of the intractable dynamic optimization implied by (2). The idea of approximating solutions to intractable optimal control problems by bang-bang inputs was introduced in a different context by Chakraborty and Hammer in [2, 3]. For the present problem, this method enables one to estimate the minimum norm and an approximate norm optimal control using bang-bang functions only.

The rest of the article is organized as follows: Section II defines the problem rigorously and introduces most of the notations to be used in this article. The main results, which show that the bang-bang function may be used for approximately computing the optimal norm, are described in Section III, while the application of the theory on a two state example is included in Section IV.

II. Mathematical Preliminaries

The problem described in the introduction will be made precise in this section. Let the dynamical system be defined as in (1). The control input $u(t)$ is assumed to be essentially bounded, measurable and hence belonging to the Banach space $\mathcal{B}_U$ with the norm defined as $\|u(t)\|_{\mathcal{B}} = ess\sup_{[t_0,t_f]}\|u(t)\|_\infty$. Here $\|\cdot\|_\infty$ denotes the standard l_∞ norm given for m-dimensional vectors $p \in R^m$ by $\|p\|_\infty = \max_{i=1,\dots,m}|p_i|$ and for $m \times n$ real matrices $P \in R^{m\times n}$ by $\|P\|_\infty := \max_{i=1,\dots,m;j=1,\dots,n}|P_{ij}|$ where P_{ij} is the (i,j)-element of P. Similarly denote the Banach space of state trajectories with the supremum norm by $\mathcal{B}_X$. The functions $f : R^n \times [t_0,t_f] \to R^n$ and $g : R^n \times [t_0,t_f] \to R^{n\times m}$ are both assumed to be Lipschitz continuous in $x(t)$ and continuous in t. In other words, for any $t \in [t_0,t_f]$, there are real numbers $K,\ L > 0$ such that $\|f(x_1,t) - f(x_2,t)\|_\infty \le K\|x_1 - x_2\|_\infty$ and $\|g(x_1,t) - g(x_2,t)\|_\infty \le L\|x_1 - x_2\|_\infty$ for all $t \in [t_0,t_f]$ and for all $x_1, x_2 \in \mathcal{B}_X$. Moreover, both f and g are uniformly bounded over $[t_0,t_f]$ for each solution to (1). Under these conditions a unique absolutely continuous solution to (1) is guaranteed to exist for every $u(t) \in \mathcal{B}_U$ (e.g. see [14]) given by

$$x(t) = x_0 + \int_{t_0}^{t}[f(x,\tau) + g(x,\tau)u(\tau)]d\tau.$$

Hence the set $\mathcal{B}_X$ consists only of absolutely continuous bounded functions.

Denote by $P(R^n) := \{A : A \subset R^n, A \neq \phi\}$, i.e. the set of all non-empty subsets of R^n. Then we can define a set valued mapping from $[t_0,t_f]$ to some subset of R^n as $G(t) : [t_0,t_f] \to P(R^n)$. We require that the solution to (1) should satisfy the state constraint:

$$x(t) \in G(t) \text{ for all } t \in [t_0,t_f] \tag{3}$$

Moreover, let $C \subset R^n$ be an arbitrary non-empty set and we require that the solution to (1) should satisfy the end-point constraint of the form

$$x(t_f) \in C. \tag{4}$$

Then define the set of *admissible* controls as follows:

Definition 1. Denote the solution to (1) corresponding to input $u(t) \in \mathcal{B}_U$ as $x(t;u(t))$; then the set of *admissible* controls is defined as $U := \{u(t) \in \mathcal{B}_U : x(t;u(t)) \in G(t)$ for all $t \in [t_0,t_f]$ and $x(t_f;u(t)) \in C\}$.

Under these assumptions, the problem described in the introduction can be precisely stated as follows:

Problem 2. Let the set U be non-empty. Then find $\inf_{u(t)\in U}\|u(t)\|_{\mathcal{B}}$. Compute a norm minimizing input.

The solution to this problem by classical methods of optimal control theory (such as [5]) turns out to be quite complicated. The first question, which is in general difficult to answer, is under what conditions there is at least one control which makes the solution to (1) satisfy the constraints (3) and (4). Secondly, even if we assume that there is at least one such control, it is difficult to guarantee the existence of a norm minimizing control without making further assumptions on the system and state equations. These difficulties motivate us to propose the following method which makes it possible to answer these questions, as well as compute an approximate norm optimal control. The trade off is an arbitrarily small enlargement of the state and terminal constraint sets.

III. Approximate Bang-bang Solution

For the purposes of this article, we define a bang-bang vector valued function as one, whose components take one of at most two distinct values for almost all $t \in [t_0,t_f]$. Formally, let $h(t) : [t_0,t_f] \to R^n$ be a measurable essentially bounded vector valued function with $S^i = \sup_{t\in[t_0,t_f]} h_i(t)$ for $i = 1,\dots,n$ and $I^i = \inf_{t\in[t_0,t_f]} h_i(t)$ for $i = 1,\dots,n$.

Then $h(t)$ is bang-bang if it is of the form

$$h_i(t) = \begin{cases} S^i \text{ for } t \in T_S \subset [t_0, t_f] \\ I^i \text{ for } t \in T_I \subset [t_0, t_f] \end{cases} \quad (5)$$

where the set $\{[t_0, t_f] \setminus \{T_S \cup T_I\}\}$ has zero Lebesgue measure. The time instants when $h(t)$ switches between the two extreme values are called the switching times/instances of the function. If it is known that $h(t)$ has k switching instances in each of its components, then it is completely characterized by the upper and lower bounds $\{S^i, I^i\}$ $i = 1, ..., n$ and the $(nk \times 1)$-vector of switching times.

In this section we will show that the norm optimal control of Problem 2 can be replaced by a bang-bang control function of the form (5) with a finite number of switching times. The amplitude of this bang-bang function will closely approximate the infimum norm of Problem 2. We make the following assumption.

Assumption: *The set U of definition (1) is non-empty.*

Let $\rho = \inf_{u(t)\in U} ||u(t)||_{\mathcal{B}}$ and $\rho_\epsilon = \rho + \epsilon$. Since U is non-empty, for any $\epsilon > 0$, there exists $u^a(t) \in U$ such that $||u^a(t)||_{\mathcal{B}} \leq \rho_\epsilon$. We will consider $u^a(t)$ to be the $\epsilon-$approximate solution to Problem 2. It turns out that $u^a(t)$ can be approximated by a bang-bang function (say $u^{\pm}(t)$) in the sense that the state trajectory generated by $u^a(t)$ matches closely with that generated by $u^{\pm}(t)$. Moreover, $||u^{\pm}(t)||_{\mathcal{B}} \leq \rho_\epsilon$. Recall that the optimal norm ρ and the function $u^a(t)$ are both unknown and cannot be characterized easily. This makes them extremely difficult to compute from the given data. However, $u^{\pm}(t)$ is much easier to compute since, bang-bang functions are completely specified by their upper and lower bounds and their switching times. Therefore, we can search for $u^{\pm}(t)$ instead of $u^a(t)$; and for this purpose it is enough to check among the possible switching times and amplitudes of the bang-bang function $u^{\pm}(t)$. Thus, the infinite dimensional norm optimization problem reduces to a finite dimensional search over the amplitude and the switching instances of $u^{\pm}(t)$.

The following theorem and subsequent corollaries show that given any ϵ-optimal solution to Problem 2, there is a bang-bang input with approximately the same norm, which drives the solution to (1) within arbitrarily close distances of the state and end-point constraints.

Theorem 3. *Let $u^a(t) \in U$ be an ϵ-optimal solution to Problem 2 and denote the corresponding state trajectory generated by (1) as $x^a(t) := x(t; u^a(t))$. Similarly denote the state trajectory generated by (1) due to the action of the bang-bang input $u^{\pm}(t)$ as $x^{\pm}(t) := x(t; u^{\pm}(t))$. Then for any $\delta > 0$, there is a real number $\phi \in [t_0, t_f]$ and a bang-bang function $u^{\pm}(t) \in \mathcal{B}_U$ such that the following holds:*

(i) $\|u^{\pm}(t)\|_\infty \leq \rho_\epsilon$ for all $t \in [t_0, \phi]$

(ii) $u^{\pm}(t)$ has a finite number of switches in the interval $[t_0, \phi]$.

(iii) the discrepancy between the state trajectories $\|x^a(t) - x^{\pm}(t)\|_\infty \leq \delta$ for all $t \in [t_0, \phi]$.

Proof: Under the assumptions of Section II, there is a unique solution to (1) to any input $u(t) \in \mathcal{B}_U$ over $[t_0, t_f]$ given by:

$$x(t) = x_0 + \int_{t_0}^{t} f(x, \tau)d\tau + \int_{t_0}^{t} g(x, \tau)u(\tau)d\tau \quad (6)$$

Now, let $\eta > 0$ be a real number. Since $g(x^a(t), t)$ is uniformly continuous in t over any finite interval of time, there is a real number $\xi(\eta) > 0$ such that the function $\mu(t', t) := g(x^a(t'), t') - g(x^a(t), t)$ satisfies $\|\mu(t', t)\|_\infty \leq \eta$ whenever $|t' - t| < \xi(\eta)$ and $t', t \in [t_0, t_f]$. By assumption, the matrix valued function g is uniformly bounded for every solution $x(t)$ and for each $t \in [t_0, t_f]$. Hence there exists a real number $N < \infty$ such that $N = \sup_{t\in[t_0,t_f]} ||g(x^a(t), t)||_\infty$.

Next, let $\phi \in (t_0, t_f)$ be a real number (to be chosen later) and let $0 < \gamma \leq \xi(\eta)$ be any number for which the ratio $(\phi - t_0)/\gamma$ is an integer. We build a partition of the interval $[t_0, \phi]$ into segments of length γ, namely, the partition determined by the intervals $[t_0 + q\gamma, t_0 + (q + 1)\gamma]$, $q = 0, 1, 2, ..., ((\phi - t_0)/\gamma) - 1$. We build a bang-bang input function $u^{\pm}(t) = (u_1^{\pm}(t), u_2^{\pm}(t), ..., u_m^{\pm}(t))^T,\ t_0 \leq t \leq \phi$, (where the superscript T denotes the transpose) as follows: for the component $u_i^{\pm}(t)$, select in each interval $[t_0 + q\gamma, t_0 + (q+1)\gamma)$ a switching time θ_{qi} and set

$$u_i^{\pm}(t) : \; = \; \begin{cases} \rho_\epsilon \text{ for } t \in [t_0 + q\gamma, \theta_{qi}) \\ -\rho_\epsilon \text{ for } t \in [\theta_{qi}, t_0 + (q + 1)\gamma) \end{cases} \quad (7)$$
$$q = 0, 1, 2, ..., ((\phi - t_0)/\gamma) - 1$$

$i = 1, 2, ...m$. For each such component function, we have $\int_{t_0+q\gamma}^{t_0+(q+1)\gamma} u_i^{\pm}(\tau)d\tau = \rho_\epsilon \int_{t_0+q\gamma}^{\theta_{qi}} d\tau - \rho_\epsilon \int_{\theta_{qi}}^{t_0+(q+1)\gamma} d\tau = \rho_\epsilon[2(\theta_{qi} - q\gamma - t_0) - \gamma]$. Now, select θ_{qi} to satisfy the equality $\rho_\epsilon[2(\theta_{qi} - q\gamma - t_0) - \gamma] = \int_{t_0+q\gamma}^{t_0+(q+1)\gamma} u_i^a(\tau)d\tau$. Note that θ_{qi} exists due to the fact that $|u_i^a(t)| \leq \rho_\epsilon$ for all $t \in [t_0, t_f]$. For this value of θ_{qi}, we obtain the equality

$$\int_{t_0+q\gamma}^{t_0+(q+1)\gamma} [u_i^a(\tau) - u_i^{\pm}(\tau)]d\tau = 0 \quad (8)$$

for all $i = 1, 2, ...m$ and all $q = 0, 1, 2, ..., ((\phi - t_0)/\gamma) - 1$.

For the following, denote by $\| \cdot \|_\phi := ess \sup_{[t_0,\phi]} \| \cdot \|_\infty$. Now consider the difference in trajectories (6) for the same system sample (i.e. system (1) starting from the same initial condition x_0):

$$\begin{aligned} &||x^a(t) - x^{\pm}(t)\|_\phi \\ &= \bigg\| \int_{t_0}^{t} [f(x^a, \tau) - f(x^{\pm}, \tau)]d\tau \\ &\quad + \int_{t_0}^{t} [g(x^a, \tau)u^a(\tau) - g(x^{\pm}, \tau)u^{\pm}(\tau)]d\tau \bigg\|_\phi \\ &\leq \int_{t_0}^{t} K\|x^a(t) - x^{\pm}(t)\|_\phi d\tau \\ &\quad + \bigg\| \int_{t_0}^{t} [g(x^a, \tau)u^a(\tau) + g(x^a, \tau)u^{\pm}(\tau) \\ &\quad - g(x^a, \tau)u^{\pm}(\tau) - g(x^{\pm}, \tau)u^{\pm}(\tau)]d\tau \bigg\|_\phi \end{aligned}$$

$$
\begin{aligned}
&\le K(t-t_0)\|x^a(t)-x^{\pm}(t)\|_\phi \\
&\quad + \left\| \int_{t_0}^{t} [g(x^a,\tau)-g(x^{\pm},\tau)]u^{\pm}(\tau)d\tau \right\|_\phi \\
&\qquad + \left\| \int_{t_0}^{t} g(x^a,\tau)[u^a(\tau)-u^{\pm}(\tau)]d\tau \right\|_\phi \\
&\le K(t-t_0)\|x^a(t)-x^{\pm}(t)\|_\phi \\
&\quad + \int_{t_0}^{t} L\|x^a(t)-x^{\pm}(t)\|_\phi \|u^{\pm}(t)\|_\phi d\tau \\
&\qquad + \left\| \int_{t_0}^{t} g(x^a,\tau)[u^a(\tau)-u^{\pm}(\tau)]d\tau \right\|_\phi \\
&\le (K+L\rho_\epsilon)(t-t_0)\|x^a(t)-x^{\pm}(t)\|_\phi \\
&\qquad + \left\| \int_{t_0}^{t} g(x^a,\tau)[u^a(\tau)-u^{\pm}(\tau)]d\tau \right\|_\phi
\end{aligned}
$$

Denote $g^a(t) := g(x^a(t),t)$ and $\psi(t) := \frac{1}{1-(K+L\rho_\epsilon)(t-t_0)}$. Now choose $\phi \in (t_0,t_f]$ such that $(K+L\rho_\epsilon)(t-t_0) < 1$ for all $t \in [t_0,\phi]$. Denote the maximum value of $\psi(t) = \frac{1}{1-(K+L\rho_\epsilon)(\phi-t_0)}$ by Ψ; then $\Psi > 0$. Moreover for any given $t \in [t_0,\phi]$, choose an integer $q > 0$ such that $t_0 + q\gamma < t \le t_0 + (q+1)\gamma$. This implies that

$$
\begin{aligned}
&||x^a(t)-x^{\pm}(t)||_\phi \\
&\le \Psi \| \int_{t_0}^{t} g^a(\tau)[u^a(\tau)-u^{\pm}(\tau)]d\tau \|_\phi \\
&\le \Psi \left\| \left[\sum_{r=0}^{q-1} \int_{t_0+r\gamma}^{t_0+(r+1)\gamma} g^a(\tau)\left[u^a(\tau)-u^{\pm}(\tau)\right]d\tau \right] \right. \\
&\quad \left. + \int_{t_0+q\gamma}^{t} g^a(\tau)\left[u^a(\tau)-u^{\pm}(\tau)\right]d\tau \right\|_\phi \\
&\le \Psi \left[\left\| \sum_{r=0}^{q-1} g^a(t_0+r\gamma) \int_{t_0+r\gamma}^{t_0+(r+1)\gamma} \left[u^a(\tau)-u^{\pm}(\tau)\right]d\tau \right\|_\phi \right. \\
&\quad + \left\| \sum_{r=0}^{q-1} \int_{t_0+r\gamma}^{t_0+(r+1)\gamma} \mu(\tau,t_0+r\gamma)\left[u^a(\tau)-u^{\pm}(\tau)\right]d\tau \right\|_\phi \\
&\qquad \left. + \left\| \int_{t_0+q\gamma}^{t} g^a(\tau)\left[u^a(\tau)-u^{\pm}(\tau)\right]d\tau \right\|_\phi \right] \\
&\le \Psi \sum_{r=0}^{q-1} \int_{t_0+r\gamma}^{t_0+(r+1)\gamma} \left\{ \sup_{[t_0+r\gamma,t_0+(r+1)\gamma]} \|\mu(\tau,t_0+r\gamma)\|_\infty \right. \\
&\qquad\qquad \left. \left[\|u^a(t)\|_\phi + \|u^{\pm}(t)\|_\phi\right] \right\} d\tau \\
&\quad + \Psi \int_{t_0+q\gamma}^{t} \|g^a(t)\|_\phi \left[\|u^a(t)\|_\phi + \|u^{\pm}(t)\|_\phi\right] d\tau \\
&\le 2\Psi\rho_\epsilon(\eta\phi + N\gamma)
\end{aligned}
$$

Finally, choose the value of η so that $2\Psi\rho_\epsilon\eta\phi < \delta/2$. Then, choose γ so that

$$
0 < \gamma \le \min\{\xi(\eta), \delta/(4\Psi N\rho_\epsilon)\} \tag{9}
$$

and $(\phi - t_0)/\gamma$ is an integer. For these selections, we obtain $\|x^a(t) - x^{\pm}(t)\|_\infty < \delta$ for all $t \in [0,\phi]$, and our proof concludes. ∎

Clearly the above theorem may be used to piece together the approximate response over the entire interval of interest i.e. $[t_0,t_f]$. Consider the following Corollary.

Corollary 4. *Under the notation of Theorem 3, for any $\delta > 0$, there is a bang-bang function $u^{\pm}(t) \in \mathcal{B}_U$ such that the following holds:*

(i) $\|u^{\pm}(t)\|_\infty \le \rho_\epsilon$ for all $t \in [t_0,t_f]$

(ii) $u^{\pm}(t)$ has a finite number of switches in the interval $[t_0,t_f]$.

(iii) the discrepancy between the state trajectories $\|x^a(t) - x^{\pm}(t)\|_\infty \le \delta$ for all $t \in [t_0,t_f]$.

Proof: From theorem 3, we know that for any $\delta_0 > 0$ there is a $\phi \in [t_0,t_f]$ and a bang-bang input with a finite number of switches, say $u^{0,\pm}(t)$ such that $\|x^a(t) - x^{\pm}(t)\|_\infty < \delta_0$ for all $t \in [t_0,\phi]$. Divide the time interval $[t_0,t_f]$ into sub intervals of length less than or equal to ϕ such as defined by the partition $\{t_0, t_0+\phi, t_0+2\phi, ..., t_0+n\phi, t_f\}$ where n is an integer such that $n \ge \frac{t_f-t_0}{\phi} - 1$. Now, by the application of $u^{0,\pm}(t)$ over $[t_0,\phi]$ we have$\|x^a(\phi) - x^{\pm}(\phi)\|_\infty \le \delta_0$.

Next, denote by $\|\cdot\|_{2\phi} := ess\sup_{[t_0+\phi,t_0+2\phi]} \|\cdot\|_\infty$. Now for any $\delta_1 > 0$ we can define the bang-bang input $u^{1,\pm}(t)$ exactly as in Theorem 3, over the interval $[t_0+\phi, t_0+2\phi]$ such that the discrepancy in the trajectories generated by $u^a(t)$ and $u^{1,\pm}(t)$ over the interval $[t_0+\phi, t_0+2\phi]$ is

$$
\begin{aligned}
&||x^a(t)-x^{\pm}(t)||_{2\phi} \\
&= \Big\| [x^a(\phi)-x^{\pm}(\phi)] \\
&\quad + \int_{t_0+\phi}^{t} [f(x^a,\tau)-f(x^{\pm},\tau)]d\tau \\
&\quad + \int_{t_0+\phi}^{t} [g(x^a,\tau)u^a(\tau)-g(x^{\pm},\tau)u^{\pm}(\tau)]d\tau \Big\|_{2\phi} \\
&\le \big\| [x^a(\phi)-x^{\pm}(\phi)] \big\|_{2\phi} \\
&\quad + \Big\| \int_{t_0+\phi}^{t} [f(x^a,\tau)-f(x^{\pm},\tau)]d\tau \\
&\quad + \int_{t_0+\phi}^{t} [g(x^a,\tau)u^a(\tau)-g(x^{\pm},\tau)u^{\pm}(\tau)]d\tau \Big\|_{2\phi} \\
&\le \delta_0 + \delta_1
\end{aligned}
$$

Using similar arguments over all the $n+1$ partitions of $[t_0,t_f]$ and for any sequence of real numbers $\delta_0, \delta_1, ...\delta_n$, there is a sequence of bang-bang control inputs $u^{0,\pm}(t), u^{1,\pm}(t), ..., u^{n,\pm}(t)$, each with a finite number of switches, and defined over the consecutive sub intervals $[t_0, t_0+\phi], [t_0+\phi, t_0+2\phi], ..., [t_0+n\phi, t_f]$ such that

$$
\sup_{[t_0,t_f]} ||x^a(t)-x^{\pm}(t)||_\infty \le \delta_0 + \delta_1 + ... + \delta_n
$$

But for any given $\delta > 0$ we can choose $\delta_i = \frac{\delta}{n+1}$ $(i = 0,1,...,n)$, such that $\sup_{[t_0,t_f]} ||x^a(t) - x^{\pm}(t)||_\infty \le \delta$. Consequently, the required bang-bang function $u^{\pm}(t)$ can be created by the concatenation of the sequence of bang-bang controls $u^{0,\pm}(t), u^{1,\pm}(t), ...u^{n,\pm}(t)$ such that the resulting bang-bang function has the following form

$$u^{\pm}(t) = \begin{cases} u^{0,\pm}(t) & \text{for } t \in [t_0, t_0+\phi] \\ u^{i,\pm}(t) & \text{for } t \in [t_0+i\phi, t_0+(i+1)\phi], \\ & i = 1,...,n-1 \\ u^{n,\pm}(t) & \text{for } t \in [t_0+n\phi, t_f] \end{cases}$$

■

Remark 5. In Theorem 3 and, consequently in corollary 4, the cost of making the error δ smaller is an increase in the number of switches of the bang-bang function $u^{\pm}(t)$. This can be seen by examining inequality (9): to maintain the inequality, γ must be decreased as δ is decreased. According to (7), the number of switches is (in general) $(\phi - t_0)/\gamma$, so that a decrease of γ leads to an increase in the number of switches.

Theorem 3 and Corollary 4 shows that a bang-bang input $u^{\pm}(t)$ can be constructed with the same norm as $u^a(t)$ and that the state trajectories generated by the bang-bang input are arbitrarily close to those generated by $u^a(t)$. Since the trajectories generated by $u^a(t)$ obey the state and the end-point constraint by assumption, it is easy to see that the trajectories corresponding to $u^{\pm}(t)$ will either satisfy the constraints or be in arbitrarily small neighborhoods of the constraint sets. Denote the δ-neighborhood of the state constraint set $G(t)$ as $B_\delta(G(t)) = \{z(t) \in \mathcal{B}_X : \inf_{c(t)\in G(t)} \sup_{[t_0,t_f]} \|c(t) - z(t)\|_\infty \leq \delta\}$ and the δ-neighborhood of the end-point constraint set C by $B_\delta(C) = \{z \in R^n : \inf_{c\in C} \|c - z\|_\infty \leq \delta\}$.

Corollary 6. *Using the notation of Theorem 3, and for any $\delta > 0$, the state trajectory generated by the bang-bang input $u^{\pm}(t)$ satisfies $x^{\pm}(t) \in B_\delta(G(t))$ and $x^{\pm}(t_f) \in B_\delta(C)$ for all $t \in [t_0, t_f]$.*

Proof: From Corollary 4, for any $\delta > 0$, there is a bang-bang input $u^{\pm}(t) \in U$ with a finite number of switches and with $\|u^{\pm}(t)\|_\infty \leq \rho_\epsilon$ such that $\sup_{[t_0,t_f]} \|x^a(t) - x^{\pm}(t))\|_\infty \leq \delta$. But according to hypothesis, $x^a(t) \in G(t) \implies x^{\pm}(t) \in B_\delta(G(t))$ for all $t \in [t_0, t_f]$ and $x^a(t_f) \in C \implies x^{\pm}(t_f) \in B_\delta(C)$ ■

IV. Numerical Example

We demonstrate the application of the theory developed in the previous sections on the following example. Consider the system described by the equations:

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) + (1-t)u(t) \\ \dot{x}_2(t) &= x_2(t) + (1-t)u(t) \end{aligned} \quad (10)$$

over the time interval $[0, 1]$ seconds. Here the input is $u(t) \in R$ for each t, and $x_1(t)$ and $x_2(t)$ are the two states. The initial values of the two states are $x_1(0) = 1$ and $x_2(0) = -1$. We assume that there are no state constraints, i.e. $G(t) = R^2$ for all $t \in [0, 1]$ and the end point constraint set $C \subset R^2$ is described as $C := \{(y, z) \in R^2 : |y| \leq 0.02, |z| \leq 0.02 \text{ and } |y - z| \leq 0.01\}$. In other words we require that $|x_1(1)| \leq 0.02$ and $|x_2(1)| \leq 0.02$ and $|x_1(1) - x_2(1)| \leq 0.01$. Recall that the set of admissible control inputs U was defined to consist of those inputs for which the solution to (10) satisfies the end-point constraint. The objective is to find the minimum norm input among the elements of U.

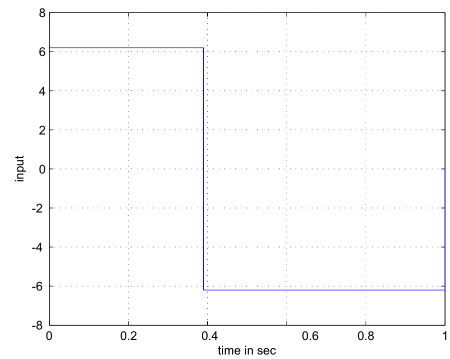

Fig. 1. Approximate norm-optimal bang-bang control input

It is easy to check using standard controllability Gramian tests that a bounded $u(t)$ exists such that the end-point constraint is satisfied. For example, the input $u(t) = \frac{1}{2}(1 - t)(-14.56e^{2t} + 38.66e^{-t})$ is clearly admissible. One can check that with this input, system (10) evolves from the initial values to the final constraint set in the stipulated one second and that the maximum norm of this input is $\sup_{[0,1]} \|u(t)\|_\infty = 12.05$. Hence the set U is non-empty. Thus the theory developed above is applicable and we should search among bang-bang inputs with varying amplitudes and different switching times until we find the bang-bang input with the minimum norm that closely satisfies the end-point constraint. In particular, the time interval of $[0, 1]$ was discretized and for each choice of amplitude, two families of bang-bang functions were created over this discretization with one and two switches respectively. The amplitude was varied between -12.05 and 12.05. It was seen that the minimum norm that was achieved by a single switch bang-bang function was not appreciably improved by increasing the number of switches to two. According to Remark 5, the error should decrease with an increasing number of switches. Since, in this case, the increase of switches from one to two does not lead to an appreciable improvement in the minimum norm required, we choose the input with approximately the minimum magnitude of 6.2 as

$$u^{\pm}(t) = \begin{cases} 6.2 \text{ for } t \leq 0.39 \text{ seconds} \\ -6.2 \text{ for } t > 0.39 \text{ seconds} \end{cases}$$

These results are shown in Figure 1 and Figure 2. Clearly the end-point constraints are satisfied.

V. Conclusion

In summary, this article develops a methodology of estimating the norm-optimal control for control-affine nonlinear systems under arbitrary state and end-point constraints. The theory developed converts the otherwise intractable dynamic optimization into a finite dimensional optimization problem in the amplitude and switching times of the bang-bang approximant. Efficient numerical techniques for the application of this

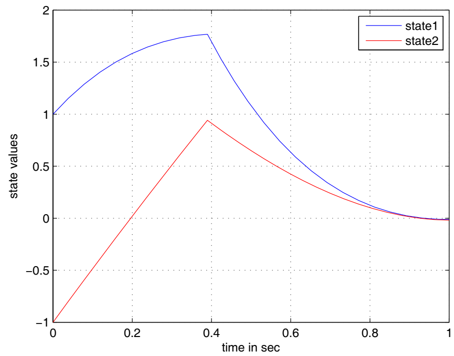

Fig. 2. State Trajectories

theory are being currently developed.

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