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Inverse Eigenvalue Problem for Generalized Periodic Jacobi Matrices With Linear Relation

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***Abstract*—This paper presents the following inverse eigenvalue problem for generalized periodic Jacobi matrices: Given two unequal real numbers, and nonzero vectors. Find n steps generalized periodic Jacobi matrices J, which is satisfied the conditions that the numbers and the nonzero vectors are the characteristic pairs of J. This paper obtains the existence and uniqueness of this kind of inverse eigenvalue problem, discusses the expression of the problem's solution, and some numerical example is provided.**

***Keywords*—linear relation, generalized periodic Jacobi matrices, characteristic value, inverse problem**

I. INTRODUCTION

The generalized periodic Jacobi matrices are the following form of matrices:

$$J=\begin{pmatrix} a_1 & b_1 & & & c_n \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-2} & a_{n-1} & b_{n-1} \\ b_n & & & c_{n-1} & a_n \end{pmatrix} \quad (1)$$

In it, $a_i,b_i,c_i \in R(i=1,2,\cdots,n)$ .If the second diagonal elements $c_i=kb_i+l(i=1,2,\cdots,n),k,l$ are constants, so the expression（1）is called generalized periodic Jacobi matrices with linear relation. The paper research the inverse eigenvalue problem of expression（1）.

Question A Given two unequal real numbers $\lambda,\mu(\lambda\neq\mu)$ and two nonzero real vectors $x=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}\in R^n$, $y=(y_1,y_2,\cdots,y_n)^{\mathrm{T}}\in R^n$ .Find the $n\times n$ real generalized periodic Jacobi matrices $J(c_i=kb_i+l(i=1,2,\cdots,n),k,l$ are known real number）,let $Jx=\lambda x, Jy=\mu y$.

Let:

$$x=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}, y=(y_1,y_2,\cdots,y_n)^{\mathrm{T}}, \quad (2)$$

$$x_{n+j}=x_j, y_{n+j}=y_j, b_{n+j}=b_j, c_{n+j}=c_j, j=0,1,\cdots, \quad (3)$$

$$D_i=\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1}\end{vmatrix}(i=0,1,\cdots,n), \quad (4)$$

$$d_i=\sum_{s=0}^{i-1}k^s x_{i-s}y_{i-s}(i=1,2,\cdots,n), \quad (5)$$

$$e_i=\sum_{s=0}^{i-1}k^s D_{i-1-s}(i=1,2,\cdots,n). \quad (6)$$

II. THE MAIN RESULTS

Let (λ,x) and (μ,y) are two characteristic pairs of generalized periodic Jacobi matrices J, so

$$c_{i-1}x_{i-1}+a_ix_i+b_ix_{i+1}=\lambda x_i \quad (i=1,2,\cdots,n), \quad (7)$$

$$c_{i-1}y_{i-1}+a_iy_i+b_iy_{i+1}=\mu y_i \quad (i=1,2,\cdots,n). \quad (8)$$

From expression（7），（8）and（4）,can get

$$b_iD_i=(\mu-\lambda)x_iy_i+c_{i-1}D_{i-1} \quad (i=1,2,\cdots,n). \quad (9)$$

For $c_i=kb_i+l \quad (i=1,2,\cdots,n)$,so expression（9）can be converted to

$$b_iD_i=(\mu-\lambda)x_iy_i+kb_{i-1}D_{i-1}+lD_{i-1}(i=1,2,\cdots,n) \quad (10)$$

Note that $b_0=b_n, D_0=D_n$,from expressions（5），（6）and（10）, there are

$$b_iD_i=(\mu-\lambda)d_i+k^ib_nD_n+le_i(i=1,2,\cdots,n). \quad (11)$$

So we can get the follow theorem.

Theorem 1 If the following conditions are satisfied:

（1） $D_i\neq 0(i=1,2,\cdots,n)$;

（2） $k^n\neq 1$

So the solution of Question A is unique existence and

$$b_n=\frac{(\mu-\lambda)d_n+le_n}{(1-k^n)D_n}, \quad (12)$$

$$b_i=\frac{(\mu-\lambda)d_i+k^ib_nD_n+le_i}{D_i}(i=1,2,\cdots,n-1); \quad (13)$$

$$c_i=kb_i+l(i=1,2,\cdots,n); \quad (14)$$

$$a_i=\begin{cases} \lambda-\dfrac{c_{i-1}x_{i-1}+b_ix_{i+1}}{x_i}, x_i\neq 0; \\ \mu-\dfrac{c_{i-1}y_{i-1}+b_iy_{i+1}}{y_i}, y_i\neq 0 \end{cases} (i=1,2,\cdots,n). \quad (15)$$

Prove: In expression（11）, let $i = n$, for the theorem conditions（1）and（2）, there are expression（12）is established. Also from theorem condition（1）and expression（11）, can get expression（13）is established. From the conditions of Question A, known expression（14）is established. Also from theorem condition（1）,known x_i, y_i can not be zero at the same time. From expression（7）and（8）, so expression（15）is established. The end.

Above theorem solved the solution of Question A when $k^n \neq 1$ .When $k^n = 1$,we need to give an additional condition. For example, $b_n = \alpha$ is known, or $b_1 + b_2 + \cdots + b_n = \beta$ （β is known）,we can get the following theorem.

Theorem 2 If the following conditions are satisfied:

（1） $D_i \neq 0 (i = 1, 2, \cdots, n)$;

（2）The equation $z + \sum_{i=1}^{n-1} \frac{(\mu-\lambda)d_i + zk^i D_n + le_i}{D_i} = \beta$ （β is known）. α is the solution.

So Question A has the solution J :

$$b_n = \alpha \quad (16)$$

$$b_i = \frac{(\mu-\lambda)d_i + \alpha k^i D_n + le_i}{D_i} (i = 1, 2, \cdots, n-1); \quad (17)$$

$$c_i = kb_i + l (i = 1, 2, \cdots, n); \quad (18)$$

$$a_i = \begin{cases} \lambda - \dfrac{c_{i-1}x_{i-1} + b_i x_{i+1}}{x_i}, x_i \neq 0; \\ \mu - \dfrac{c_{i-1}y_{i-1} + b_i y_{i+1}}{y_i}, y_i \neq 0 \end{cases} (i = 1, 2, \cdots, n). \quad (19)$$

III. THE NUMERICAL EXAMPLES

Example 1

Give $\lambda = 2, \mu = 3, x = (1,0,1,0)^{\mathrm{T}}, y = (1,1,1,1)^{\mathrm{T}}, k = 2, l = 1$.

It is easy to calculated

$$D_0 = D_4 = -1 \neq 0, D_1 = 1 \neq 0, D_2 = -1 \neq 0,$$
$$D_3 = 1 \neq 0, 1 - k^n = -15 \neq 0.$$

From Theorem 1, the Question A has the unique solution. And

$$d_1 = x_1 y_1 = 1, d_2 = x_2 y_2 + k x_1 y_1 = 2,$$
$$d_3 = x_3 y_3 + k x_2 y_2 + k^2 x_1 y_1 = 5,$$
$$d_4 = x_4 y_4 + k x_3 y_3 + k^2 x_2 y_2 + k^3 x_1 y_1 = 10;$$
$$e_1 = D_0 = -1, e_2 = D_1 + kD_0 = -1, e_3 = D_2 + kD_1 + k^2 D_0 = -3,$$
$$e_4 = D_3 + kD_2 + k^2 D_1 + k^3 D_0 = -5.$$

So

$$b_4 = \frac{(\mu-\lambda)d_4 + le_4}{(1-k^4)D_4} = \frac{10-5}{(-15)(-1)} = \frac{1}{3},$$
$$b_1 = \frac{(\mu-\lambda)d_1 + kb_4 D_4 + le_1}{D_1} = -\frac{2}{3},$$
$$b_2 = \frac{(\mu-\lambda)d_2 + k^2 b_4 D_4 + le_2}{D_2} = \frac{1}{3},$$
$$b_3 = \frac{(\mu-\lambda)d_3 + k^3 b_4 D_4 + le_3}{D_3} = -\frac{2}{3};$$
$$c_1 = kb_1 + l = -\frac{1}{3}, c_2 = kb_2 + l = \frac{5}{3},$$
$$c_3 = kb_3 + l = -\frac{1}{3}, c_4 = kb_4 + l = \frac{5}{3};$$
$$a_1 = \lambda - \frac{c_0 x_0 + b_1 x_2}{x_1} = 2, a_2 = \mu - \frac{c_1 y_1 + b_2 y_3}{y_2} = 3,$$
$$a_3 = \lambda - \frac{c_2 x_2 + b_3 x_4}{x_3} = 2, a_4 = \mu - \frac{c_3 y_3 + b_4 y_5}{y_4} = 3.$$

So

$$J = \begin{pmatrix} 2 & -\frac{2}{3} & 0 & \frac{5}{3} \\ -\frac{1}{3} & 3 & \frac{1}{3} & 0 \\ 0 & \frac{5}{3} & 2 & -\frac{2}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} & 3 \end{pmatrix}$$

and

$$Jx = \lambda x, Jy = \mu y.$$

Example 2

Give $\lambda = \sqrt{3}, \mu = \sqrt{3} + 1, x = (1,-1,1,-1)^{\mathrm{T}}, y = (1,0,1,0)^{\mathrm{T}}, k = 3, l = -1$.

It is easy to calculated

$$D_0 = D_4 = -1 \neq 0, D_1 = 1 \neq 0, D_2 = -1 \neq 0, D_3 = 1 \neq 0,$$
$$1 - k^n = -80 \neq 0.$$

From Theorem 1, the Question A has the unique solution. And

$$d_1 = x_1 y_1 = 1, d_2 = x_2 y_2 + k x_1 y_1 = 3,$$
$$d_3 = x_3 y_3 + k x_2 y_2 + k^2 x_1 y_1 = 10,$$
$$d_4 = x_4 y_4 + k x_3 y_3 + k^2 x_2 y_2 + k^3 x_1 y_1 = 30;$$
$$e_1 = D_0 = -1, e_2 = D_1 + kD_0 = -2, e_3 = D_2 + kD_1 + k^2 D_0 = -7,$$
$$e_4 = D_3 + kD_2 + k^2 D_1 + k^3 D_0 = -20.$$

So

$$b_4 = \frac{(\mu-\lambda)d_4 + le_4}{(1-k^4)D_4} = \frac{5}{8}, b_1 = \frac{(\mu-\lambda)d_1 + kb_4 D_4 + le_1}{D_1} = \frac{1}{8},$$
$$b_2 = \frac{(\mu-\lambda)d_2 + k^2 b_4 D_4 + le_2}{D_2} = \frac{5}{8},$$
$$b_3 = \frac{(\mu-\lambda)d_3 + k^3 b_4 D_4 + le_3}{D_3} = \frac{1}{8};$$
$$c_1 = kb_1 + l = -\frac{5}{8}, c_2 = kb_2 + l = \frac{7}{8},$$
$$c_3 = kb_3 + l = -\frac{5}{8}, c_4 = kb_4 + l = \frac{7}{8};$$

$$a_1=\lambda-\frac{c_0x_0+b_1x_2}{x_1}=\sqrt{3}+1, a_2=\mu-\frac{c_1y_1+b_2y_3}{y_2}=\sqrt{3}\ ,$$

$$a_3=\lambda-\frac{c_2x_2+b_3x_4}{x_3}=\sqrt{3}+1, a_4=\mu-\frac{c_3y_3+b_4y_5}{y_4}=\sqrt{3}\ .$$

So

$$J=\begin{pmatrix}\sqrt{3}+1 & \frac{1}{8} & 0 & \frac{7}{8}\\ -\frac{5}{8} & \sqrt{3} & \frac{5}{8} & 0\\ 0 & \frac{7}{8} & \sqrt{3}+1 & \frac{1}{8}\\ \frac{5}{8} & 0 & -\frac{5}{8} & \sqrt{3}\end{pmatrix}$$

and

$$Jx=\lambda x, Jy=\mu y\ .$$

Example 3

Give $\lambda=1,\mu=2,x=(0,1,-1,1)^{\mathrm{T}}, y=(1,1,2,3)^{\mathrm{T}}$, $k=1,l=1,\beta=\frac{4}{15}$.

It is easy to calculated

$$D_0=D_4=1\neq 0, D_1=-1\neq 0, D_2=3\neq 0, D_3=-5\neq 0,$$

$$1-k^n=0\ .$$

And

$$d_1=0,d_2=1,d_3=-1,d_4=2\ ;\ e_1=1,e_2=0,e_3=3,e_4=-2\ .$$

And

$$b_4=\alpha, b_1=\frac{(\mu-\lambda)d_1+kb_4D_4+le_1}{D_1}=-\alpha-1\ ,$$

$$b_2=\frac{(\mu-\lambda)d_2+k^2b_4D_4+le_2}{D_2}=\frac{\alpha+1}{3},$$

$$b_3=\frac{(\mu-\lambda)d_3+k^3b_4D_4+le_3}{D_3}=-\frac{\alpha+2}{5}\ .$$

$$b_1+b_2+b_3+b_4=\frac{2\alpha-16}{15}=\beta=\frac{4}{15}\Rightarrow\alpha=10\ .$$

From Theorem 2, the Question A has the unique solution. And

$$b_1=-11, b_2=\frac{11}{3}, b_3=\frac{-12}{5}, b_4=10\ ;$$

$$c_1=kb_1+l=-10, c_2=kb_2+l=\frac{14}{3},$$

$$c_3=kb_3+l=-\frac{7}{5}, c_4=kb_4+l=11\ ;$$

$$a_1=\mu-\frac{c_0y_0+b_1y_2}{y_1}=-20, a_2=\lambda-\frac{c_1x_1+b_2x_3}{x_2}=\frac{14}{3}\ ,$$

$$a_3=\lambda-\frac{c_2x_2+b_3x_4}{x_3}=\frac{49}{15}, a_4=\lambda-\frac{c_3x_3+b_4x_5}{x_4}=-\frac{2}{5}\ .$$

So

$$J=\begin{pmatrix}-20 & -11 & 0 & 11\\ -10 & \frac{14}{3} & \frac{11}{3} & 0\\ 0 & \frac{14}{3} & \frac{49}{15} & -\frac{12}{5}\\ 10 & 0 & -\frac{7}{5} & -\frac{2}{5}\end{pmatrix}$$

and

$$Jx=\lambda x, Jy=\mu y\ .$$

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