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Relaxed LMI Based designs for Takagi Sugeno Fuzzy Regulators and Observers Poly-Quadratic Lyapunov Function approach

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***Abstract*-This paper addresses the analysis and design of the state and output feedback fuzzy control for the stabilization of the closed loop continuous time Takagi – Sugeno fuzzy system. The approach utilized in this paper is the so called poly Lyapunov function or fuzzy Lyapunov function. An ordinary parallel distributed compensation design technique is used for the state feedback and the output feedback stabilization control problems. Sufficient conditions for both cases in terms of linear matrix inequality are derived. The proposed design procedure subjected to fuzzy Lyapunov function cures the limitation of the previous results in dealing with the time derivative of premise membership functions and use the natural PDC control. The derived stability condition is more relaxed than the previous results. Moreover, the regulator and observer based stabilization problems are cast as a linear matrix inequality feasibility problem instead of the conventional bilinear matrix inequality. This problem is amenable to solve in the framework of the convex optimization. Finally, a simulation example demonstrates the advantage of the given technique.**

***Keywords*—T-S fuzzy control, Lyapunov function, LMI**

I. INTRODUCTION

Drawing on the development on the stability analysis and systematic design for the well known Takagi-Sugeno fuzzy control [1], which is a convenient and flexible tool for handling complex nonlinear systems where its consequent parts are linear systems connected by IF–THEN rules [2], interest in finding relaxed stability and stabilizability conditions for such a fuzzy control system has been increasing last decades.

In this case, state feedback control theory allows designing fuzzy controllers and fuzzy observers ([3]and [4]) by the way of the so-called parallel distributed compensation (PDC) [5]. Having the property of universal approximation, this approach includes the multiple model approach and can be seen also as Polytopic Linear Differential Inclusions (PLDI) [6].

The study of the stability of these models is done using a common Lyapunov function. Most of the works propose the use of a common quadratic Lyapunov function, i.e. $V(x(t))=x^T(t)Px(t)$ with $P = P^T > 0$ [7]. All these results can be stated as linear matrix inequalities (LMIs) that can be efficiently implemented and solved. The power of these results stems from the fact that the search for controller gains and Lyapunov function can be stated as a convex optimization problem in terms of linear matrix inequalities (LMI's). Such optimization problems can be solved efficiently using publically available software [19]. Unfortunately, the standard LMI conditions for quadratic stability are often found to be conservative when applied to fuzzy systems [8]. Although this approach allowed us to apply convex optimization for solving the problems, it was mostly found to be conservative because of the "common" (or strict) structure of the Lyapunov function independent from the fuzzy weighting functions. Moreover, when a large number of subsystems are involved, common Lyapunov functions are inadequate to establish stability or synthesize controllers, by virtue of their conservativeness.

Recently, the relaxed stability and stabilizability conditions for fuzzy control systems were reported to release the conservatism of the conventional conditions by considering the interactions among the fuzzy subsystems[8]-[9]. The relaxed conditions admitted more freedom in guaranteeing the stability of the fuzzy control systems and were found to be very valuable in designing the fuzzy controller, especially when the design problem involves not only stability, but also the other performance requirements such as the speed of response, constraints on control input and output, and so on [10].

Several approaches have been developed to overcome the above mentioned limitations. Piecewise quadratic Lyapunov functions were employed to enrich the set of possible Lyapunov functions used to prove stability[11],[12] and [15]. Multiple Lyapunov functions have been paid a lot of attention due to avoiding conservatism of stability and stabilizability [13],[14] and [16].

In [13], The stability of the continuous time open loop T-S fuzzy system is discussed using the so called multiple or fuzzy Lyapunov function. The problem of the time derivative of premise membership function for the fuzzy Lyapunov function that turn the stability condition for that system to not generally solved analytically or even numerically is solved by converting the upper bound constraint of the time derivative of the premise membership function into LMIs. The main limitation in this paper is that the initial states $x(0)$ should be known in these new LMIs. Furthermore, for different initial states, we need to solve the LMI again, thus the stability conditions given in this paper are initial state dependent.

In [14] and [16], the stabilizability of the continuous time closed loop T-S fuzzy system is studied using the same

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approach. Two solutions are proposed for solving the problem of the time derivative of premise membership function for the fuzzy Lyapunov function. The first one is the solution offered in [13] and the second one is a new PDC design in the case where the time derivative of the premise membership function $h_f(z(t))$ can be calculated from the states, measurable external variables and/or time. The new PDC controller feedbacks the time derivative of the premise membership function. The drawbacks of this approach are the time derivatives are assumed to be always computable from the states and the controller is transformed to be non-PDC controller.

Although some techniques result in a much more powerful, flexible analysis and a relaxed stability condition over their common quadratic Lyapunov function counterpart, the corresponding controllers synthesis were hardly found via non-convex optimization, moreover, the so called PDC design procedures which is conceptually simple and straightforward is transformed into non-PDC procedures and this is on the cost of controller construction and its simplicity.

It is well known that the observer design is a very important problem in control systems, however, in the context of fuzzy control systems using fuzzy Lyapunov function, the fuzzy output feedback analysis and design is hardly addressed in the present research.

The primary motivation for this work is to avoid the limitations appeared in [13][14] and [16] by designing an ordinary PDC controller, which has already proven that it is very simple and nature and does not contain any derivative term that may be lead to instability, without using the initial state dependent LMI conditions in dealing with the upper bound of the time derivative of the premise membership functions for both the fuzzy state feedback control and the fuzzy observer based control. This paper is towards further defining relaxed stabilitzability conditions for the continuous time closed loop model based T-S fuzzy systems in both cases; the state feedback and the observer based control (output feedback).

II. Ordinary Fuzzy State & Output feedback Control

A. Preliminaries

In the paper, the pairs (A_i, B_i) & (A_i, C_i) $i \in \{1,...,r\}$ are supposed to be both controllable & observable respectively. To facilitate the paper, we assume that the reader is familiar with the following; basic fuzzy setup consisting of r fuzzy rules whose consequent parts are characterized by the following Takagi-Sugeno continuous fuzzy system [1]:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)) \tag{1}$$

and for the nonlinear plant represented by (1), the fuzzy controller is designed to share the same IF parts with the plant, the overall nonlinear controller is represented as follows:

$$u(t) = \sum_{i=1}^{r} h_i(z(t))\{-F_i x(t)\} \tag{2}$$

Moreover, the closed loop augmented T-S fuzzy Luenberger observer FLO (The FLO is usually used for estimating the system states $x(t)$) is represented as follows [17]:

$$\dot{\tilde{x}} = \sum_{j=1}^{r}\sum_{k=1}^{r} h_j(z(t))h_k(z(t))A_{jk}\tilde{x}(t) \tag{3}$$

where $A_{jk} = \begin{bmatrix} A_j - B_j F_k & B_j F_k \\ 0 & A_j - L_j C_k \end{bmatrix}$ and $\tilde{x}(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$ (4)

L_i is the observer gain for the i^{th} observer rule, F_j (for $j = 1,2,...,r$) is the controller gain for the j^{th} observer rule. $e(t)$ denotes the estimation error between $x(t)$ & $\hat{x}(t)$. $z(t)= [z_1(t), z_2(t),...,z_s(t)]$ is the premise vector that may be states, measurable external variables and/or time and $h_i(x(t))$ is the normalized weight for each rule, i.e. $h_i(z(t)) \geq 0$ and $\sum_{i=1}^{r} h_i(z(t)) = 1$

B. Main Stability Condition

The main stability condition for the closed loop continuous time fuzzy system (CLCTFS) is found in the literatures [9], [20] as follows:

Theorem 1:

The fuzzy system (1) can be stabilized via the PDC controller (2) if there exists a common positive definite matrix P such that

$$P \succ 0 \tag{5a}$$

$$G_{ii}^T P + P G_{ii} \prec 0 \qquad (i = 1,...,r) \tag{5b}$$

$$\left(\frac{G_{ij} + G_{ji}}{2}\right)^T P + P\left(\frac{G_{ij} + G_{ji}}{2}\right) \leq 0 \qquad (i \prec j, h_i(x)h_j(x) \neq 0) \tag{5c}$$

C. Relaxed Stability Condition

The following relaxed stability condition is reported in the literature [9],[20]to release the conservatism of the (5a-c).

Theorem 2:

The fuzzy system (1) can be stabilized via the PDC controller (2) if there exists a common positive definite matrices P and Q such that

$$P \succ 0, \; Q \geq 0 \tag{6a}$$

$$G_{ii}^T P + P G_{ii} + (s-1)Q \prec 0 \qquad (i = 1,...,r) \tag{6b}$$

$$\left(\frac{G_{ij} + G_{ji}}{2}\right)^T P + P\left(\frac{G_{ij} + G_{ji}}{2}\right) - Q \leq 0 \quad (i \prec j, h_i(x)h_j(x) \neq 0) \tag{6c}$$

where s is the maximum of the number of the fuzzy subsystems that are fired at an instant.

However, it is well known that in a lot of cases, a common positive definite matrix P does not exist whereas the T-S model (1) is stable. In this paper, the following fuzzy Lyapunov function is employed, for the Takagi–Sugeno fuzzy system (1) to relax the stabilization conditions (5) and (6):

$$V(x(t)) = \sum_{i=1}^{r} h_i(z(t))x^T(t)P_i x(t) \tag{7}$$

Where P_i is a positive–definite matrix. This candidate Lyapunov function satisfies 1) V is C^1, 2) $V(0) = 0$ and $V(x(t)) > 0$ for $x(t) \neq 0$ and 3) $\|x(t)\| \to \infty \Rightarrow V(x(t)) \to \infty$. The fuzzy Lyapunov function shares the same membership functions with the Takagi–Sugeno fuzzy model of a system. Applying fuzzy Lyapunov function (7) on (1) instead of a single Lyapunov function $V(x) = x^T(t)Px(t)$ will result in the following theorem

Theorem 3 [14]***:*** Assume that

$$\left|\dot{h}_\rho(z(t))\right| \prec \phi_\rho \tag{8}$$

for $\rho = 1,2,...,r-1$ where $\phi_\rho \succ 0$

The fuzzy system (1) can be stabilized via PDC fuzzy controller (2) if there exist $\varepsilon \succ 0, \phi_\rho$, scalars $s_1, ..., s_r$, Positive definite matrices $P_1, P_2, ..., P_r$ and matrices $F_1, F_2, ..., F_r$ such that

$P_i \geq P_r, \quad i = 1,2,...,r-1, \; P_i \geq s_i I, \quad i = 1,2,...,r$

$s_i \geq 1, \quad i = 1,2,...,r$

$$\begin{bmatrix} \left(\begin{matrix} \frac{1}{3\varepsilon^2}(s_i + s_j + s_k) I_{n\times n} \\ -\sum_{\rho=1}^{r-1} \phi_\rho (P_\rho - P_r) \end{matrix} \right) & \Omega_v \\ \Omega_v^T & 6I_{6n\times 6n} \end{bmatrix} \succ 0 \tag{9}$$

$\forall i,j,k \in \{1,2,...,r\} \quad s.t. \quad i \leq j \leq k$

Where $\Omega_v = \begin{bmatrix} \Omega_{ijk} & \Omega_{ikj} & \Omega_{jik} & \Omega_{jki} & \Omega_{kij} & \Omega_{kji} \end{bmatrix}$

$\Omega_{ijk} = \varepsilon (A_i - B_i F_j)^T + \frac{1}{\varepsilon} P_k$

Proof: see [13] and [14].

***Remark 1*:**

The authors in [13] and [14] comment on the difficulty for the selecting problem of ϕ_ρ in practical control problems and give a solution for this problem by converting the constraint (8) into LMI. Hence, simultaneous solutions for theorem 3 and theorem 4 (given below) as a feasibility problem offer a solution for the stabilization of the fuzzy system (1).

Theorem 4 [14]

Assume that *x(0)* and *z(0)* are known. The constraint (8) holds if there exist positive definite matrices $P_1, P_2, …, P_r$ satisfying

$$\begin{bmatrix} 1 & x^T(0)P_1 & \cdots & x^T(0)P_r \\ P_1 x(0) & \frac{1}{h_1(z(0))}P_1 & & 0 \\ \vdots & & \ddots & \\ P_r x(0) & 0 & & \frac{1}{h_r(z(0))}P_r \end{bmatrix} \geq 0 \tag{10}$$

$$\begin{bmatrix} \phi_\rho P_i & \varepsilon_{\rho\ell} A_i^T \\ \varepsilon_{\rho\ell} A_i^T & \phi_\rho I \end{bmatrix} \geq 0 \quad \forall\, i, \ell, \rho \tag{11}$$

Proof: see[13] and [14]

Remark 2:

1) The initial states *x(0)* should be known in Theorem 4. In general, for different initial states, we need to solve the LMIs again. Thus, the conditions in Theorem 4 are initial states dependent. In the following section, for the design problem including some different initial states, the initial state independence condition can be deduced.
2) In [14] and [16] the problem of selecting of ϕ_ρ's is solved by proposing a new PDC design in the case where the time derivatives of $h_\rho(z(t))$ can be calculated from the states. The new PDC controller is of the following form $u(t) = -\sum_{i=1}^{r} h_i(z(t)) F_i x(t) - \sum_{i=1}^{r} \dot{h}_i(z(t)) T_i x(t)$ where T is an arbitrary matrix. In the following section, we use an ordinary PDC controller instead of the above one.

III. PDC State feedback control for CLCTFS

A. More Relaxed stability condition

Gurra & Vermeiren in [18] said a striking quotation; "It is necessary to change something either the control law or the Lyapunov function or both of them in order to relief the conservatism of the common quadratic Lyapunov function technique". In this paper we change the Lyapunov function to be a fuzzy Lyapunov function and keep the PDC control law in order to get more relaxed stability condition and very simple and straightforward controller synthesis.

Theorem 5: Given the closed loop system (1), if there exist $\varepsilon \succ 0, \; \gamma \rangle 0$, scalars $s_1, s_2, ..., s_r$, and positive definite symmetric matrices $P_1, P_2, ..., P_r$ such that

$P_i \geq S_i I$, $S_i \geq I$, *i=1, 2, …,r*

$$\begin{bmatrix} \left(\frac{1}{3\varepsilon^2 (r-1)}(s_i + s_j + s_k) I_{nxn} + \frac{1}{\gamma^2}(s_i + s_j) I_{nxn} - \mu_{\rho m}(P_\rho - P_r) \right) & \Omega_v \\ \Omega_v^T & 6(r-1) I_{6nx6n} \end{bmatrix} \rangle 0 \tag{12}$$

where

$$\Omega_v = \begin{bmatrix} \Omega_{ijk} & \Omega_{ikj} & \Omega_{jik} & \Omega_{jki} & \Omega_{kij} & \Omega_{kji} \end{bmatrix}$$

$$\Omega_{ijk} = \varepsilon \left(A_j - B_j F_k \right)^T + \frac{1}{\varepsilon} P_i$$

Then the closed loop system (1) is quadratically stable.

Proof:

The ordinary state feedback PDC controller can be expressed as follows:

$$u(t) = -\sum_{i=1}^{r} h_i F_j x(t) \tag{13}$$

Substitute (13) into (1), $\dot{V}(x(t))$ can be obtained as follows:

$$\dot{V}(x(t)) = \sum_{i=1}^{r} \dot{h}_i(z(t)) x^T(t) P_i x(t) + \sum_{i=1}^{r} h_i(z(t)) (\dot{x}^T(t) P_i x(t) + x^T(t) P_i \dot{x}(t))$$

$$\dot{V}(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(z(t)) h_j(z(t)) h_k(z(t)) x^T(t) \left[\sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) P_\rho + (A_j - B_j F_k)^T P_i + P_i (A_j - B_j F_k) \right] x(t)$$

$$\dot{V}(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(z(t)) h_j(z(t)) h_k(z(t)) x^T(t) [\sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) P_\rho + W_{ijk}] x(t)$$

$$\dot{V}(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(z(t)) h_j(z(t)) h_k(z(t)) x^T(t) [\sum_{i=1}^{r} \dot{h}_\rho(z(t)) P_\rho + \frac{1}{6}\hat{W}_{ijk}] x(t)$$

(14)

where

$W_{ijk} = G_{jk}^T P_i + P_i G_{jk}$

$\hat{W}_{ijk} = W_{ijk} + W_{ikj} + W_{jik} + W_{jki} + W_{kij} + W_{kji}$

$G_{jk} = A_j - B_j F_k$

$\sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) = 0 \quad \forall\, z(t)$

$\dot{h}_r(z(t)) = -\sum_{\rho=1}^{r} \dot{h}_\rho(z(t))$

$\sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) = \dot{h}_r(z(t)) + \sum_{\rho=1}^{r-1} \dot{h}_\rho(z(t))$

$\sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) P_\rho = \dot{h}_r(z(t)) P_r + \sum_{\rho=1}^{r-1} \dot{h}_\rho(z(t)) P_\rho$

$$\dot V(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(z(t))h_j(z(t))h_k(z(t))x^T(t)[\sum_{\rho=1}^{r-1}\dot h_\rho(z(t))P_\rho + \sum_{\rho}^{r-1}\dot h_r(z(t))P_r + \frac{1}{6}\hat W_{ijk})]x(t)$$

$$\dot V(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(z(t))h_j(z(t))h_k(z(t))x^T(t)[\sum_{\rho=1}^{r-1}\dot h_\rho(z((t))((P_\rho - P_r) + \frac{1}{6}\hat W_{ijk})]x(t) \quad (15)$$

Assume that $\dot h_\rho(z(t)) = \sum_{m=1}^{2} w_{\rho m}(z(t))\mu_{\rho m}$

where $\sum_{m=1}^{2} w_{\rho m}(z(t))\mu_{\rho m} = 1 \quad \& \; w_{\rho m}(z(t)) \ge 0$. Then we have eq. (14)

$$\dot V(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(z(t))h_j(z(t))h_k(z(t))x^T(t)[\sum_{\rho=1}^{r-1}\sum_{m=1}^{2} w_{\rho m}(z(t))\mu_{\rho m}(P_\rho - P_r) + \frac{1}{6}\hat W_{ijk})]x(t)$$

$$\dot V(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{\rho=1}^{r-1}\sum_{m=1}^{2} h_i(z(t))h_j(z(t))h_k(z(t))w_{\rho m}(z(t))x^T(t)[\mu_{\rho m}(P_\rho - P_r) + \frac{1}{6(r-1)}\hat W_{ijk}]x(t) \quad (16)$$

Note that, although the eq.(16) is complete the proof if it is negative definite, it is Bilinear Matrix Inequality (BMI) condition that is hardly to solve using ordinary procedures (congruent transformation, change of variables and Schur complement). The BMI is not convex and may have multiple local solutions. Chadli et al in [15] suggested the path-following method for solving BMI problem, the major weakness of this method is the choice of the initial values guaranteeing the existence of a solution, if any. In our analysis, to change (16) into an LMI, same procedures suggested in [16] is used here. The idea is instead of dealing with the nonlinear condition in P, we separate the matrix P and Transform the product term in P into an additive term of P, so that the condition changed to be linear condition in P (it transformed from BMI into an LMI) that is easily solved.

The eq. (16) is negative if there exist $\varepsilon \succ 0$ and $\gamma \succ 0$ such that,

$$\mu_{\rho m}(P_\rho - P_r) + \frac{1}{6(r-1)}(\hat W_{ijk} + \varepsilon^2 G_{ij}^T G_{ij} + \varepsilon^2 G_{ik}^T G_{ik} + \varepsilon^2 G_{ji}^T G_{ji} + \varepsilon^2 G_{jk}^T G_{jk} + \varepsilon^2 G_{ki}^T G_{ki} + \varepsilon^2 G_{kj}^T G_{kj}) \prec 0 \quad (17)$$

The left hand of (17) can be rewritten as:

$$\mu_{\rho m}(P_\rho - P_r) + \frac{1}{6(r-1)}(S_{ijk} + S_{ikj} + S_{jik} + S_{jki} + S_{kij} + S_{kji}) \quad (18)$$

where $S_{ijk} = U_{ijk} - \frac{1}{\varepsilon^2}P_k P_k$, $U_{ijk} = (\varepsilon G_{ij}^T + \frac{1}{\varepsilon}P_k)(\varepsilon G_{ij}^T + \frac{1}{\varepsilon}P_k)^T$

Assume that $s_i \le P_i$ & $1 \le s_i$ then the following eq. holds:

$$(16) \le \mu_{\rho m}(P_\rho - P_r) - \left\{\frac{1}{3\varepsilon^2(r-1)}(s_i + s_j + s_k) + \frac{1}{\gamma^2}(s_i + s_j)\right\}I + \frac{1}{6(r-1)}\left\{U_{ijk} + U_{ikj} + U_{jik} + U_{jki} + U_{kij} + U_{kji}\right\} \quad (19)$$

$\dot V \prec 0$ if (19) is negative definite matrix, (19) can be rewritten as:

$$\begin{bmatrix} \frac{1}{3\varepsilon^2(r-1)}(s_i + s_j + s_k) + \frac{1}{\gamma^2}(s_i + s_j) - \mu_{\rho m}(P_\rho - P_r) & \Omega_v \\ \Omega_v^T & 6(r-1)I \end{bmatrix} \succ 0 \quad (20)$$

where $\Omega_v = [\Omega_{ijk} \;\; \Omega_{ikj} \;\; \Omega_{jik} \;\; \Omega_{jki} \;\; \Omega_{kij} \;\; \Omega_{kji}]$ (21)

$$\Omega_{ijk} = \varepsilon(A_i - B_i F_j)^T + \frac{1}{\varepsilon}P_k \quad (22)$$

The effectiveness of the suggested stability condition (12) compared with the basic well known stability conditions (5) and (6) can be illustrated by the following example.

B. Example 1: Consider the following fuzzy model:

$$\dot x(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t))$$

Where $h_1(x_1(t)) = \frac{1 + \sin x_1(t)}{2}$ $\quad h_2(x_1(t)) = \frac{1 - \sin x_1(t)}{2}$

$$A_1 = \begin{bmatrix} a & -4 \\ -1 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix} B_1 = \begin{bmatrix} 0 \\ b \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$y(t) = Cx(t) \quad$ *where* $C = [0 \;\; 1]$

It is assumed that $|x_i(t)| \le \frac{\pi}{2}$ for *i=1,2*. A fuzzy controller is designed to have the local feedback gains with *-1* and *-2* being the eigenvalues. The design parameters $\mu_{\rho m} \;\; \{\rho = 1,2,\ldots,r-1 \;\&\; m = 1,2\}$ can be calculated from (24,25, 26 and 27) as follows: The time derivatives of membership functions is

$$\dot h_\rho(x_1(t)) = \frac{\partial h_\rho(x_1(t))}{\partial x_1}\dot x_1(t), \; \{\rho = 1,2,\ldots,r-1\} \quad (23)$$

By substituting $\dot x_1(t) = [1 \;\; 0] \times \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t))$ into (23), we obtain

$$\dot h_1 = -\cos x_1\left(\frac{a+2}{4}\right)x_1 - 2x_2\cos x_1 + \left(\frac{-a+2}{4}\right)x_1 \sin x_1 \cos x_1 \quad (24)$$

$$\dot h_2 = \cos x_1\left(\frac{a+2}{4}\right)x_1 + 2x_2\cos x_1 - \left(\frac{-a+2}{4}\right)x_1 \sin x_1 \cos x_1 \quad (25)$$

$$\mu_{\rho 1} = \max_{|x_1| \le \pi/2, |x_2| \le \pi/2} \dot h_\rho(x_1(t)) \quad (26)$$

$$\mu_{\rho 2} = \min_{|x_1| \le \pi/2, |x_2| \le \pi/2} \dot h_\rho(x_1(t)) \quad (27)$$

Based on theorems 1, 2 and 5, the stability of the fuzzy control system is ensured for the parameter region shown in figures (1, 2 and 3).

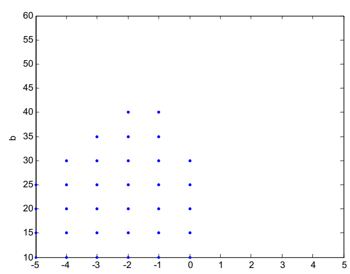

Figure 1 Stability region based on theorem 1

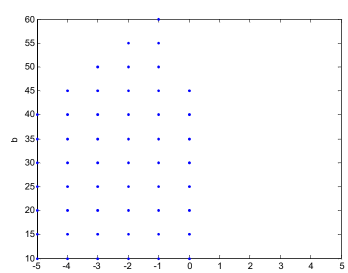

Figure 2 Stability region based on theorem 2

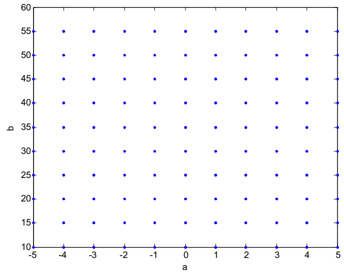

Figure 3 Stability region based on theorem 5

IV. PDC OUTPUT FEEDBACK CONTROL FOR CLCTFS

Theorem 6: Given the augmented system (1), if there exists $\varepsilon \succ 0$, $\gamma \rangle 0$, scalars $s_1, s_2, \ldots, s_r$, a positive definite symmetric matrices $P_1, P_2, \ldots, P_r$ such that

$P_i \geq S_i I$, $S_i \geq I$, $i=1, 2, \ldots, r$

$$\begin{bmatrix} \left(\frac{1}{3\varepsilon^2(r-1)}(\tilde{s}_i+\tilde{s}_j+\tilde{s}_k)I_{2nx2n}+\frac{1}{\gamma^2}(\tilde{s}_i+\tilde{s}_j)I_{2nx2n}-\mu_{\rho m}(\tilde{P}_\rho-\tilde{P}_r)\right) & \tilde{\Omega}_v \\ \tilde{\Omega}_v^T & 6(r-1)I_{12nx12n} \end{bmatrix} \rangle 0$$

(28)

Where $\tilde{\Omega}_v = \begin{bmatrix}\tilde{\Omega}_{ijk} & \tilde{\Omega}_{ikj} & \tilde{\Omega}_{jik} & \tilde{\Omega}_{jki} & \tilde{\Omega}_{kij} & \tilde{\Omega}_{kji}\end{bmatrix}$ (29)

$$\tilde{\Omega}_{ijk} = \varepsilon\left(\tilde{A}_{ij}\right)^T + \frac{1}{\varepsilon}\tilde{P}_k \quad (30)$$

$$\tilde{A}_{ij} = \begin{bmatrix}(A_i - B_iF_j) & B_iF_j \\ 0 & A_i - L_iC_j\end{bmatrix}, \tilde{P}_k = \begin{bmatrix}P_{1k} & 0 \\ 0 & P_{2k}\end{bmatrix}, \quad (31)$$

$$\tilde{P}_\rho(\tilde{P}_r) = \begin{bmatrix}P_{1\rho}(P_{1r}) & 0 \\ 0 & P_{2\rho}(P_{2r})\end{bmatrix}, \tilde{s}_i = \begin{bmatrix}s_{1i} & 0 \\ 0 & s_{2i}\end{bmatrix}$$

Then the augmented system (3) is quadratically stable.

Proof:

$$V(x(t)) = \sum_{i=1}^{r} h_i(z(t))\tilde{x}^T(t)\tilde{P}_i\tilde{x}(t)$$

$\dot{V}(x(t))$ can be obtained as follows:

$$\dot{V}(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_ih_jh_k\tilde{x}^T(t)[\sum_{\rho=1}^{r}\dot{h}_\rho\tilde{P}_\rho + W_{ijk}]\tilde{x}(t)$$

where $W_{ijk} = A_{jk}^T\tilde{P}_i + \tilde{P}_iA_{jk}$

$$\dot{V}(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_ih_jh_k\tilde{x}^T(t)[\sum_{\rho=1}^{r}\dot{h}_\rho(\tilde{P}_\rho - \tilde{P}_r) + \frac{1}{6}\hat{W}_{ijk}]\tilde{x}(t)$$

Then the proof can easily be completed as in theorem 5, hence conditions (28, 29, 30 and 31) follow.

V. DESIGN EXAMPLE

This section provides a simple design example to illustrate the stabilize-ability of a simple PDC controller for the following fuzzy model which is similar to example used in [16]. The stability conditions employed on this example are based on what is called a fuzzy Lyapunov function for both state feedback and output feedback controller.

A. Example 2:

Recall the same fuzzy system as in the previous example with

$$A_1 = \begin{bmatrix}-5 & -4 \\ -1 & -2\end{bmatrix} \quad A_2 = \begin{bmatrix}-2 & -4 \\ 20 & -2\end{bmatrix} \quad B_1 = \begin{bmatrix}0 \\ 10\end{bmatrix} \quad B_2 = \begin{bmatrix}0 \\ 3\end{bmatrix}$$

$$y(t) = Cx(t) \quad \text{where } C = \begin{bmatrix}0 & 1\end{bmatrix}$$

From the membership function, the following values are obtained $\mu_{11} = 3.44$, $\mu_{12} = -3.68$, $\mu_{21} = 3.68$, $\mu_{22} = -3.44$

These design parameters can be found using (26 and 27). (for more details see [16]). In the following, a PDC state feedback and PDC output feedback controllers are designed by solving the conditions in theorem 5 and 6.

B. PDC-State feedback Case

From conditions (20, 21 and 22) P_1 and P_2 can be obtained using MATLAB LMI toolbox [19].

$P_1 = \begin{bmatrix}1.1687 & -0.0056 \\ -0.0056 & 1.8540\end{bmatrix}$ and $P_2 = \begin{bmatrix}1.1668 & -0.0203 \\ -0.0203 & 1.5640\end{bmatrix}$. Then the controller gains are found to be:

$F_1 = \begin{bmatrix}0.1209 & 19.5362\end{bmatrix}$ and $F_2 = \begin{bmatrix}0.7485 & 22.5395\end{bmatrix}$. Figure (4) shows the control result in case of state feedback controller.

C. PDC-Output feedback Case

From conditions (28, 29, 30 and 31) P_{11} and P_{22} can be obtained as follows:

$P_{11} = \begin{bmatrix}1.1704 & -0.0000 \\ -0.0000 & 1.2593\end{bmatrix}$ and $P_{22} = \begin{bmatrix}1.2 & -0.0 \\ -0.0 & 4291.7\end{bmatrix}$. Then the controller and the observer gains are found to be:

$F_1 = \begin{bmatrix}-0.1052 & 722.3681\end{bmatrix}$ and $F_2 = \begin{bmatrix}0.6334 & 792.3601\end{bmatrix}$.

$$L_1 = L_2 = \begin{bmatrix}-0.0000 \\ 4.2917\end{bmatrix} \times 10^7$$

Figures (5&6) show the control result in case of output feedback controller for both the original states (x_1 & x_2) and the estimated states ($\hat{x}_1$ & $\hat{x}_2$).

Remark 3:

1) The dimension of the LMI stability condition (12) using ordinary PDC is *7nx7n* while the dimension of the LMI stability condition in [14] and [16] using non PDC controller is *11nx11n*, so our condition is more simple.
2) The derived LMI stability condition (12) doesn't depend on the initial state, thus, the condition in theorem 5 are initial states independent which is not the case in the stability condition of theorem 4.
3) The derived stabilizability condition for the closed loop continuous time TS system in theorem 5 using the polyquadratic Lyapunov function is pure LMI while the stabiliability condition for the same system in [15] is BMI that utilized a first order perturbation approximation method to be linearized and the solution of that BMI depend on the choice of initial values guaranteeing the existence of the solution.
4) Example 1 shows that theorem 5 suggested in this paper is more relaxed than theorem 1 and 2.
5) In this paper, we use an ordinary PDC controller that has already proven to be very simple and nature controller. In [14] and [16]a non PDC controller is used that is depend on the time derivatives of $h_\rho(z(t))$ which cannot always be calculated from the states. The non PDC controller works only for the case where the time derivatives of $h_\rho(z(t))$ can be calculated from the states.

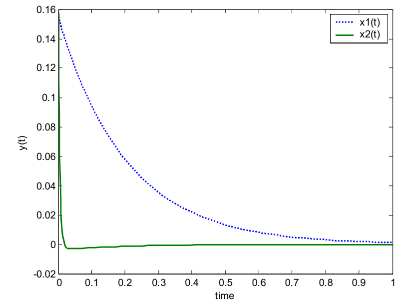

Figure 4 The Control result of x_1 & x_2 in case of state feedback

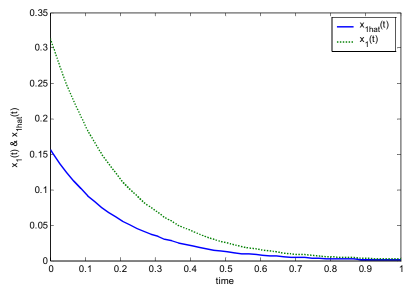

Figure 5 The control result for the orignal and estimated state (x_1) in case of output feedback

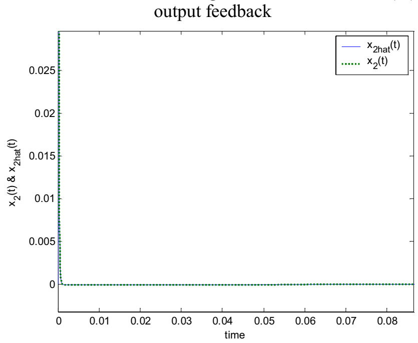

Figure 6 The control result for the orignal and estimated state (x_2) in case of output feedback

VI. Conclusion

This paper considers the stabilization problems for the closed loop continuous time Takagi – Sugeno fuzzy system using the so-called fuzzy Lyapunov function. The proposed design procedure in this paper cures the limitation of the previous results in dealing with the time derivative of premise membership functions and uses the natural PDC instead of the non PDC control for the fuzzy state and output feedback control. The stability conditions in this paper avoid the use of the initial states *x(0)*, initial values for the unknown matrices *P* & *F* and the calculation of the time derivative of $h_\rho(z(t))$ from the states. The derived stability condition is more relaxed than the previous results. Moreover, it enables render the stability condition for the suggested state feedback and output feedback fuzzy control system to be pure LMI that is efficiently solved using the convex optimization techniques instead of the BMI condition that is hardly solved.

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