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A General Growth Model for the Emergence of Power-law Distributions

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***Abstract*—An overwhelming phenomena across natural systems, social systems and ecosystems is discovered in recent years. The phenomena is coined by many terms, such as 1/f noise, Zipf-laws, or scale-free, while power-law distribution of various events or metrics is the fundamental fact that exists in every complex system. It is believed that there exist a mechanism to rule the dynamics of complex system and generate the distribution. In this paper, a general growth model which incorporate Lotka-Volterra dynamics is developed to explain the mechanism of the power-law distribution preliminary. In the model, the influence on power distribution of the spreading rate and the mortality rate can be easily analyzed and explained.**

***Index Terms*—Power-law distribution, Lotka-Volterra equations, Dynamics**

I. INTRODUCTION

There are many power-law distributions in various systems that are discovered from the early of 20th century. They exists in social systems, natural systems and ecosystems. Generally, the phenomena could be expressed by the following equation:

$$y = a \cdot x^{\alpha} \quad (1)$$

In the above equation, x denotes the value or the ranking of a physical parameter, while y denotes the probability of the parameter, α denotes the power, which is different for various systems. For example, the ranking of the city population follows a power law with a power 1 [1]; the degree of the earthquake follows a power paw with a power 1.5 [2]; the evidence from the fossil data have proved that the size of the extinction also follows a power law with an exponent of $\alpha = 2$ [3]; Many biological researchers have pointed out that many parameters of organisms compared to mass also follows power law with different value of exponent. Because the power-law distribution exists in various systems, researchers from different areas have developed many models to explain the source of the regularities. Next we will introduce several main models briefly.

A. Zipf law

In 1947 Zipf studied one hundred largest cities in United States and pointed out that if ranking in the order of decreasing population-size, the rank-frequency distribution follows a power-law with $\alpha = 1$, which also be coined by Zipf law later. He analyzed the formation of the city and pointed out that incorporated "minimum equation", the force of unification, the force of diversification and the force of innovation, the Zipf-law of the density of persons within the domains will emerge. He also presented many other evidences to verify Zipf-law, for example, the number marriage compared to the distance [1].

B. Self-organized criticality

In 1987, Bak et, al presented their discovery from the simulation of "sand pile" [4]. In the simulation, there exist a regular 2-dimensional lattice. At each time, a piece of sand will be added into a site, if the number of the sand at the site exceeds a threshold, the sand will spread to its neighbor sites, and the neighbor sites will also check whether the number of sand exceeds the threshold. If all of the site keep stable, another sand will be added in a randomly selected site.

In the dynamics, the avalanche is defined as the spreading of the sands. The size of the avalanche is defined as the number of the sites that spreading sands. The simulation result indicates that the size of the avalanches fluctuates along with the time. However, the power spectrum density of the fluctuation follows a power law with the exponent equals to 1. Bak insisted that the system will self-organize itself into a critical state. At the state, the size of the avalanches is ruled and characterized.

Besides the self-organize criticality of sand pile model, many other similar models are provided and studied, such as the model on the evolution of the ecology system and the model on the earthquake. Bak published his research on co-evolution of the species in 1993 [5]. In his model, N species are settled on a one-dimensional line with periodic boundary conditions. A random barrier, B_i, is assigned to each species. At each step, the species with the lowest barrier will mutate by assigning a new random number, then the landscape of the neighbors is also changed, the neighbors will also mutate to new random numbers correspondingly. If the new species still have the lowest barrier, they will mutate and change their neighbors barriers further. Then the avalanche will propagate in the system and the size of the avalanche will follow a power distribution, which is also coined by co-evolution phenomenon.

SOC could lead to the power-distribution. However, the exact mechanism that lead to the power distribution and the reason for the different value of the exponent are still not revealed.

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C. Mass Extinction

From 1992, many research indicated that the evolution maybe also a self-organized criticality, there also are several models on the mass extinction. However, Newman presented a very simple model to reproduce the dynamic of mass extinction [6]. In the model, the fitness α of each species is randomly created and will be adjusted slightly. At each step, a stress with level η impacts on the species, then the species whose fitness is lower than the stress will become extinct.

In Newman's model, there are no interaction among species. However, the simulation result also reproduce the power-distribution of the species extinction, with the power equal to 1

D. Human Dynamics

Many social activities are driven by human behaviors. More and more evidence indicate that human activities are not like a Poisson processes. The dynamic of human behaviors is a burst of activities separated by a long period of dreariness. Barabasi presented a prior selection mechanism to characterize human dynamics [7], [8]. In the model, there is a waiting list for the tasks, each task has its own priority which is determined by its importance. At each step, a new task with a random priority is added into the list, and also the task with the highest priority is selected and executed. After a long period of simulation, the waiting time of the tasks displays a power distribution with the power is 2.

E. Allometric growth

There are also many power laws in the biology, such as the quarter-power allometries, the measurement from the muscle speed to the lifespan all scale closely to 1/4 power of the body mass [9]. West, et, al presented a Growth-Hydraulic model to discover the mechanism that leads to quarter-power allometries in organism-level structure and function. In the model, West gave some hypotheses and attribute the power relationship to the dimension in the growth and energy transmit [10].

F. Consideration

From the above model, we could get that many stochastic process could produce power-law distributions with different exponents. However, there are some common features emerge from different areas when the mechanism of the power-distribution formation is analyzed. Generally, the dynamics of power-distribution phenomena could be viewed as a process of growth, aggregation and corruption which involve individuals, energy and resource. An event is defined as an activity that propagates among the resource space. In the propagation, the energy or the mass of the event will spreading to its neighbors according to a certain rule, then the spreading energy or the mass of its neighbors will aggregate. If the energy is too high, it will collapse and spread to its neighbors too. The description of the event cover many phenomena in social and natural systems. For example, the managers of web sites enhance the influence by expanding the popularity of web sites in the network of peoples. Each person has his own interest for the content of the web site. If he feels the web site is useful (that is the contents of the web site exceed the threshold), he will recommend the web site to his friends. His friends will also evaluate the web site and decide whether to recommend it, then the population of the site will aggregate in a particular group of people. In fact, the dynamics can be viewed as a growth process in a environment with constrained resource.

II. The Formulation

From the above section, It can be concluded that a power distribution comes from a growth process in a constrained environment. In order to explore the dynamics, we introduce a model which incorporated Lotka-Volterra dynamics and Sand pile model to discover the dynamical process and the resource of the power distribution.

A. Lotka-Volterra Dynamics

The Lotka-Volterra model [11] is also referred as predator-prey model. It discovers the dynamics between predators and preys with two differential equations. In the dynamics, predators and prey can influence one another's evolution. It was developed by A, Lotka and V. Volterra independently. The equations are listed below.

$$\frac{dP}{dt} = c \cdot a' \cdot P \cdot N - q \cdot P \tag{2}$$

$$\frac{dN}{dt} = r \cdot N - a' \cdot P \cdot N \tag{3}$$

where P and N denotes the number of predators and preys respectively, t denotes time, r is the growth rate of the prey, a' denotes the searching efficiency of the predators, q is the mortality rate of the predators, while c denotes predator efficiency that turning food to offsprings.

From the equations, the dynamics of predators and preys are discovered. When population of the predators grow, they may consume more preys to survive. However, the population of preys will be decreases deeply due to the predators' flourish. However, the population of predators will also decrease along with the deficiency of preys. Thus the population of the predators and preys influence each other and fluctuate along with time.

B. A Growth Model

A growth model in a constrained environment that is similar to the predator-prey dynamics is developed here. In the model, an event will grow on a N-dimensional environment, the size of the event spreading could be expressed by the below equations.

$$\frac{dN}{dt} = r \cdot N_a - c \cdot N_a \tag{4}$$

$$\frac{dN_a}{dt} = r \cdot N_a \cdot s \tag{5}$$

In the above equations, N denotes the growth population, N_a denotes the number of the active individuals on the sites. r denotes the propagation or spreading rate, which is determined by the topology of the environment and can follow a certain distribution. c denotes the mortality rate for the boundary

Number of events	10,000	I	3
Number of Sites	3000	r	4
InitialState	0.65/0.25/0.1	c	0.4

TABLE I
THE VALUE OF THE PARAMETERS IN EXPERIMENT I.

constraints or other reasons. s is determined by the proportion of critical sites, and s is also dynamical according to the environment state.

In the model, the dynamics of the environment (resource) could be expressed by the following rules.

$$
\begin{aligned}
&N_0' = N_0 + N_a - (r \cdot N_a) \cdot \alpha_0 \\
&N_1' = N_1 + (r \cdot N_a) \cdot \alpha_0 - (r \cdot N_a) \cdot \alpha_1 \\
&N_i' = N_i + (r \cdot N_a) \cdot \alpha_{i-1} - (r \cdot N_a) \cdot \alpha_i \\
&N_I' = N_I + (r \cdot N_a) \cdot \alpha_{i-1} - N_a \\
&\alpha_i = N_i / N_s
\end{aligned}
$$

The state of the sites is calculated according to the above rules. I denotes the number of the state for the sites, while N_i denotes the number of the sites whose state are i. N_s is the total number of the sites. N_I denotes the number of sites which are approaching the active/critical state, while N_a is the number of the avtive/critical sites.

Here the local topology information is neglected by the equations, only the dimension of the environment is incorporated in the model and is referred as propagation rate r. In the model the influence of the topology can be easily analyzed by changing the value or distribution of the spreading rate.

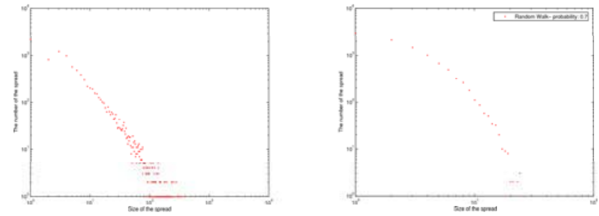
From the beginning, an event with a certain energy will be added into the sites. If the starting site presents critical state, it will spread the energy to its neighbor sites with spreading rate r, and also some energy will lost with the coefficient c for boundary conditions or other reasons. That is, the population of the event will growth according to equation 4. If the neighbor sites also present critical state for the dynamics, they will also spread the energy to their neighbors, its state will become 0, and its neighbors state will also update according to the above rules. If the number of the active sites extinct, the spreading will stop and the size of the propagation will be recorded. At each step, the number of active sites and reproduce sites are calculated with Monte Carlo method.

III. THE EXPERIMENTS

A. The Emergence of the Power Law

In Experiment I, we explore the dynamics of population size. In the model, the value of the parameters is listed in Table I.

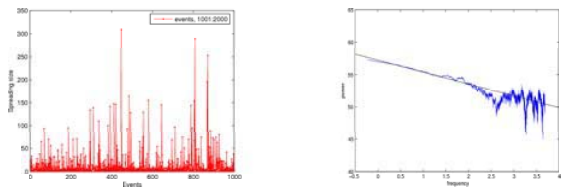
Figure 1 presents the distribution of population size in Experiment I and random-walk model respectively. From the figure, we can see that the distribution of population size in Experiment I follows a power law, which is different from that of random walk model. The result indicates that the developed growth model could produce a power distribution of population size.



(a) L-V model with the parameters listed in Table I.

(b) Random walk model with probability 0.7.

Fig. 1. The distribution of population size in Experiment I and Random Walk model respectively.



(a) The population dynamics along with the time.

(b) The power spectrum of population size distribution

Fig. 2. The dynamics of population in Experiment I.

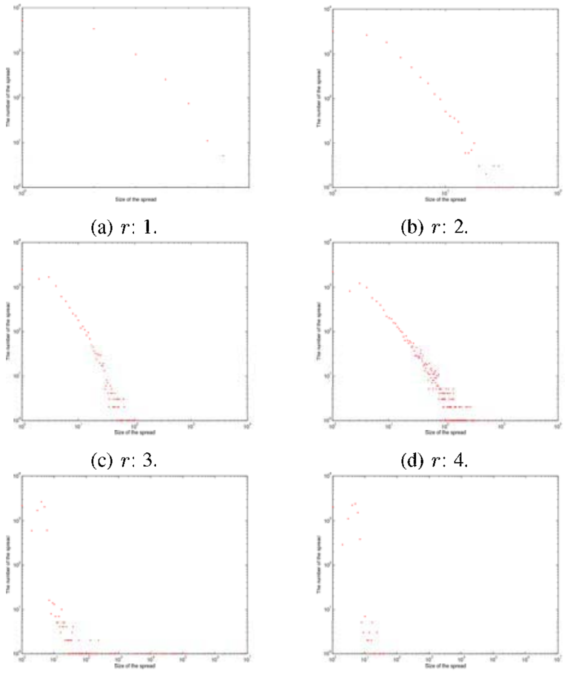
Figure 2 presents the dynamics of population and power spectrum of the population respectively. In Figure 2 A, the dynamics of population size presents a punctuated equilibrium state, which is similar to that of human dynamics [7]. From Figure 2 B, the power spectrum shows a $1/f^2$ distribution, which is similar to the result of sand-pile model. The result indicates that the growth model could also reproduce power distribution that exist in most complex systems.

B. The Influence of Growth Rate r

In order to explore the reason of the power law distribution, we conducted a series experiments, in which the spreading mechanism and the mortality rate are all changed. In Experiment II, the growth rate r is set from 1 to 6, the spreading rate means number of the site that an active site can spread at each step. It reflects the topology and dimension of the environment. The dynamics of the distribution is showed in Figure 3, A∼F. From the figure, it can be concluded that the power distribution of the population emerges when the growth rate r equal to 3 or 4, with the power is about 2.3 and 1.5 respectively. That is, the power distribution is related to the growth rate, and the power is also determined by the growth rate.

C. The Influence of Mortality Rate c

In Experiment III, the mortality rate c changes from 0.2 to 0.9. The dynamics of the population distribution is showed in Figure 4, A∼H. From the figure, it can be concluded that the power distribution of the population emerges when the value of mortality rate c is from 0.3 or 0.6, and the power becomes larger along with the increasing of mortality rate c. When the mortality rate is too small, for example, equals to 0.2, the distribution of the population size becomes irregular. If the



(a) r: 1. (b) r: 2. (c) r: 3. (d) r: 4. (e) r: 5. (f) r: 6.

Fig. 3. The distribution of population size in Experiment II, with the growth rate from 1 to 6.

mortality rate is larger than 0.8, the distribution is similar to that of the random walk model (Figure 1. B).

IV. Conclusion

In the paper, a general growth model that incorporates Predator-Prey dynamics and sand pile model is developed. In the model, the growth of the population is similar to the predator-prey equation, while the update of the environment is similar to the sand pile model. The local information of the sites are omit and the state of the sites are determined according to Monte Carlo method. With the model, a power-law distribution of population size emerges when the growth rate and mortality rate is proper, which indicates that most power distributions in social or natural systems come from a constraint growth process, in which growth rate and morality rate are important to produce the regularities and determine the value of the power in the distribution.

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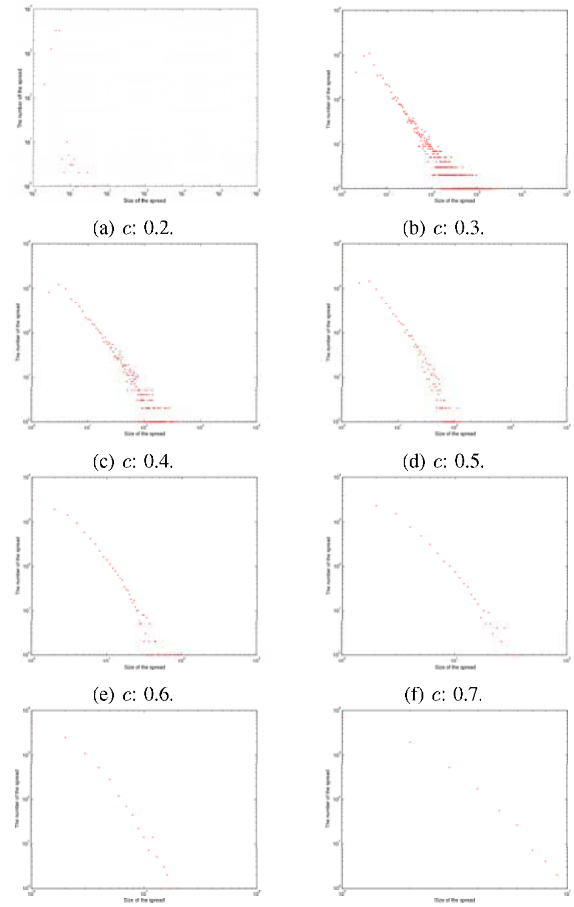

(a) c: 0.2. (b) c: 0.3. (c) c: 0.4. (d) c: 0.5. (e) c: 0.6. (f) c: 0.7. (g) c: 0.8. (h) c: 0.9.

Fig. 4. The distribution of population size in Experiment III, with the mortality rate from 0.2 to 0.9.

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