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Time Series Data Analysis Using Fractional Calculus Concepts and Fractal Analysis

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***Abstract*— A study is conducted on time series data analysis relating the concept of the fractional calculus to fractals and to the fractal dimension. A new definition of the fractal dimension is provided based on a property of the fractional derivative. Using fractional Gaussian noise and fractional Brownian motion, simulations show the information content of a time series can be easily manipulated by the fractional derivative method and generated in real time.**

***Keywords*—Fractals, fractional calculus, complex systems.**

I. Introduction

A challenging problem in the study of complex systems as well as in data analysis from biological systems and nature is understanding why a particular fractal dimension may appear in certain data, Barton and Malinverno [1]. Surprisingly, under slightly modified conditions in the data, another but different fractal dimension may result from an associated analysis. In the study of Geosciences and other related areas one utility of the method presented herein to identify a fractal dimension is that it may be possible to discern a past event in history from evidence of a change in fractal dimension in the data time series.

Also in modern medical applications of time series data analysis there is considerable interest in the information-theoretic properties of the temporal information. This is important for diagnostic decisions, and other means of discerning if some underlying effect has occurred. In medical data one popular methodology is to use approximate entropy (ApEn) [2,3,4] which has been applied to real time data to ascertain if some information quality of the data are changing. Some examples of the success of ApEn in the literature in classifying the state of a process include: discerning differences in heart beat either due to sleep cycle or disease, Yeragani, et al. [5] and Pincus et al. [6]. This procedure has shown applicability in distinguishing hormonal changes and mood swings, [7], and in other complex applications involving very intricate signals. Also examined were EMG and tremor distinction [8], EEGs [9], recognizing epileptic activity [10] and for discriminating cocaine addiction [11].

Thus the various fields of science and medicine share similar concerns related to the analysis of data, in attempting to identify if some underlying effect has occurred in the physical system of interest. In the Geosciences area [1,12] the data may be spatial rather than temporal, but the concepts are similar since fractals can occur in space or time. In this paper the focus, for simplicity, will be on time series data. The theme in this paper involves the measurement of the changing properties of the information content of the real time data being measured and how does that relate to some underlying event that may have occurred previously in the data?

II. Background

A. *Interpetation of Time Series*

A conceptual viewpoint of how a data time series can be viewed is portrayed in Figure (1):

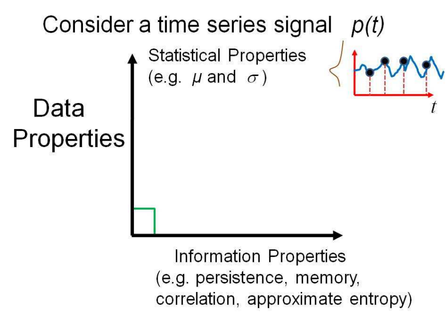

Figure (1) – Interpretation of Time Series Data

In Figure (1), the data properties (mean and variance) of a time series are displayed on the vertical axis. The horizontal axis (x-axis), however, may have information-theoretic characteristics such as memory, correlation, persistence, entropy measures, etc. To demonstrate why these characteristics of the data may be independent, a plot of a stochastic process is shown in Figure (2) which evolves with time, Popoulis and Pillai [13]. This is how a stochastic

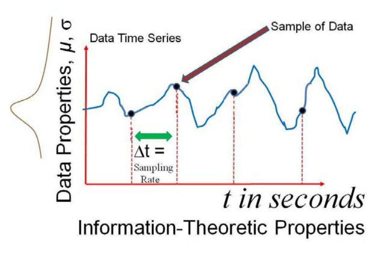

Figure (2) – Interpretation of a Stochastic Process in real time

process can be conceptualized as it changes in time. The relationships between adjacent time samples (persistence or correlation) on the *x* axis may be totally independent of the data characteristics on the *y* axis. In Figure (2) the two separated time samples (shown by black dots) are correlated (both are positive) in the sense they have the same sign indicating a positive degree of persistence. Persistence is defined as the ability of a time series to maintain its same sign on two or more adjacent samples [12]. The term persistence in data is closely allied to concepts in approximate entropy. High positive persistence is consistent with low entropy. Negative persistence is related to high entropy or disorder in the data. Herein the focus will be on the term persistence of a time series. Comparison to approximate entropy will be discussed in future publications, especially as it relates to physiological data or data taken from dynamically changing human-machine systems. To relate times series data to fractals, some basics of this field are now introduced.

B. Some Basics of Fractal Objects

Due to space limitations, a brief description of a specific fractal object (Koch snowflake) and the calculation of the fractal dimension are now introduced. Spatial Fractal objects have an initiator and a generator rule [1]. In Figure (3) the initiator is a line and the generator rule is illustrated as

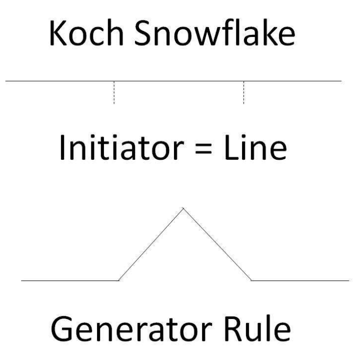

Figure (3) – The Koch Snowflake with Initiator and Generator

constructing an equilateral triangle on the mid portion of the original line. Figure (4) show subsequent iterations with a slightly different initiator (an equilateral triangle) to bring out the properties of this familiar and elementary fractal object.

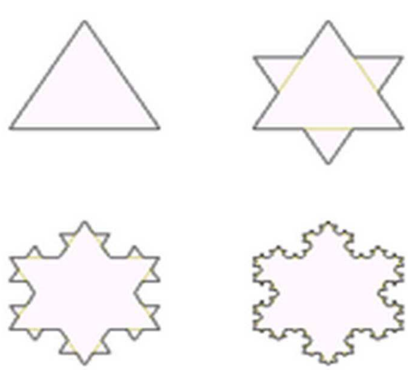

Figure (4) – Generalizations of Figure (3) with a Different Initiator

Figure (5) shows the power law curve that arises. The log of the total length (perimeter) of Figure (4) is displayed on the *y* axis. The reciprocal of the ruler length ($1/\varepsilon$) is shown on the *x* axis on a log scale. The ruler is the smallest unit of measurement in the determination of the perimeter of the object.

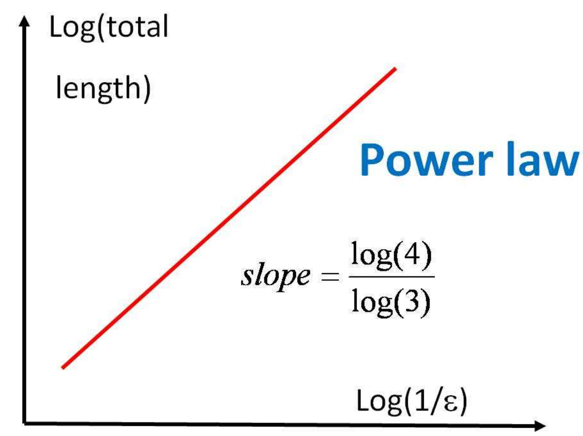

Figure (5) – Power Law Curve for the Koch Snowflake

The following properties of this elementary fractal object are now noted which can easily be shown in Figure (4) [1, 12, 14]:

<u>Four Known Properties of the Koch Snowflake:</u>

(P.1) The length (perimeter) becomes $(4/3)^n$ as $n \to \infty$.

(P.2) The area of the fractal object is finite (bounded and less than a circle that can circumscribe the object).

(P.3) The fractal object is not differentiable with a Euclidean derivative such as d/dx. This derivative is physically a tangent to the object. No tangent would be continuous on Figure (4) as $n \to \infty$.

(P.4) The fractal object is forever continuous.

To recapitulate, the Koch fractal is noted to be forever continuous, nowhere differential (using an Euclidean derivative), has finite area, and infinite perimeter. Fractal objects are ubiquitous in nature because of the optimality they provide, e.g. in the distribution of flow. Having an object with this high level of irregularity, allows for efficient distribution of a flow variable when the area is restricted to be finite and efficient flow distribution is the objective of interest.

The fractal dimension f_D, in this case, can be determined as:

$$f_D = \frac{Log(4)}{Log(3)} > 1 \qquad (1)$$

III. Brief Introduction to Fractional Calculus

Due to space limitations, only basic elements of fractional calculus will be summarized here. The reader is referred to excellent references such as [15] for more elegant and rigorous presentations of the materials. Below is summarized some of these key points, starting first with the gamma function which provides an estimate of a factorial (z!) when z may not be an integer or even positive:

Definition 1: $$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du, \quad (2)$$

Then it follows that $\Gamma(1) = 1, \quad \Gamma(z+1) = z\Gamma(z), \quad (3)$

Thus $$\Gamma(z+1) = z!, \quad (4)$$

provides a definition of z! when z is not an integer or positive.

Definition 2: $$\frac{d}{dx} x^m = m x^{m-1} \quad (5)$$

Which is true for the power law x^m. Thus for the α derivative:

$$\frac{d^\alpha}{dx^\alpha} x^m = \frac{m!}{(m-\alpha)!} x^{m-\alpha} \quad (6)$$

Using equation (4) this can be rewritten more generally:

$$\frac{d^\alpha}{dx^\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha} \quad (7)$$

Where α may not be an integer or even positive. The results now generalize to any time series that could be represented by a power series solution, exponential, trigonometric functions, Fourier series, etc. [15]. Most practical time series functions now fall in this category and the fractional derivative has a well founded definition. Relating this concept to fractals requires the property and a definition of being "scale free" or having no characteristic scale. This gives rise to a fifth property of fractal objects which is rooted in the field of fractional calculus.

IV. The fifth Property of fractal objects

In [1,12,14] one of the principal distinctions of a fractal object is the fact that it has a lack of characteristic scale. This gives rise to a new and fifth property of fractal objects which can be gleaned from the concept within fractional calculus. In mathematical terms [15] it is common to write the lack of characteristic scale as:

Definition 1 – (No Characteristic Scale)

Let the power law relationship be described as:

$$y = f(x) = x^a \quad (8)$$

Then [15] lack of characteristic scale implies:

$$\frac{d^q f(bx)}{[dx]^q} = a^q \frac{d^q f(bx)}{[d(bx)]^q} \quad (9)$$

where q could be a fractional derivative operator. The definition given in equation (9) is somewhat difficult to interpret in a physical sense. A second definition of lack of characteristic scale is more intuitive.

Definition 2 – (No Characteristic Scale)

Again the power law relationship of equation (8) is assumed to hold. The following operations show the scale-free nature of the power law in equation (8).

$$\frac{f(kx)}{f(x)} = \frac{(kx)^a}{x^a} = k^a \frac{x^a}{x^a} = k^a \quad (10)$$

It is noted that in equation (10), the far left hand side is dependent on the scale variable x. However, the far right hand side results in k^a ***which is independent of x!*** This means the same result k^a is always seen at all levels of x. Hence it is consistent with the concept of a fractal object that when viewing it at all scale levels, the object looks the same, independent of the characteristic scale size x.

Definition 3 – (No Characteristic Scale)

Using the definition of a fractional derivative in equation (7), a new and interesting way to define the lack of characteristic scale can now be obtained. Starting with equation (7), if x had a fractional dimension f_D and if the fractional derivative of order f_D was taken on equation (7) the following result would occur:

$$\frac{d^{f_D}}{dx^{f_D}} x^{f_D} = \frac{\Gamma(f_D+1)}{\Gamma(f_D - f_D + 1)} x^{f_D - f_D} = \Gamma(f_D+1) \quad (11)$$

which is constant, independent of x, and lacks characteristic scale. There is great similarity between equations (10) and (11) which only hold for power laws and demonstrate the lack of characteristic scale. This gives rise to the fifth property of a fractal, which can be stated as follows:

Fifth Property of a Fractal:

(P.5) The fractional derivative f_D of a fractal of dimension f_D exists, and is a constant with no characteristic scale.

Remarks:

The concept of a tangent on the Koch snowflake gave rise to the idea that the fractal object was nowhere differentiable if a Euclidean derivative operator was used. However, in equation (11), it is seen that the fractional derivative f_D of a fractal of dimension f_D is a constant with no characteristic scale. Thus the only finite derivatives of fractal objects that can exist are fractional and constant. This may be physically viewed (when taking a fractional derivative) not as a tangent to a curve, but as an expansion or contraction of the view of the size of the object which does not change with time.

The results of equation (11) conclude the theoretical aspects in this paper. The generation and simulation of fractional Gaussian noise and Brownian motion will demonstrate related issues through numerical studies.

V. Simulation of Fractional Noise

Very basic to the study of a time series, which may have a varying content of information, would be to develop some basic test signals to evaluate the calculation of approximate entropy, persistence, and other important quantities. Thus the assessment of the information-theoretic quantities that appear in Figure (1) could be quantified in the data. To develop a fair means to evaluate any time series for its information content, some baseline time series are synthesized here. Starting first with zero mean white Gaussian noise $\zeta(t)$ with unity variance, one could define Brownian motion $Br(t)$ as follows in terms of a fractional calculus operator:

$$Br(t) = \int \zeta(t)\, dt = \frac{d^{-1}}{dt^{-1}} \zeta(t)\ . \qquad (12)$$

Also, a highly studied process which is ubiquitous in nature, is pink noise $P(t)$ having characteristics midway between white Gaussian noise and Brownian motion [16]. Pink noise can be defined in the following manner:

$$P(t) = \text{pink noise} = \frac{d^{-\frac{1}{2}}}{dt^{-\frac{1}{2}}} \zeta(t) \qquad (13)$$

again using the fractional calculus operator. The occurrence of pink noise $P(t)$ is one of the most common signals appearing in nature and from an information-theoretic viewpoint [16] it can be demonstrated that it has unique persistence characteristics. In an information-theoretic view, pink noise will weight recent history with past history of time samples equally in a correlation sense. To generate a class of baseline time series with a wide variety of fractional Gaussian noise and Brownian motion, the following procedure was employed.

Step 1: Let $n = 2^p$ be the number of data points to represent the input time series where p is an integer.
Step 2: Find the frequency spectra of the input time series.
Step 3: Develop a line with slope $-\beta$ for a power law representation.
Step 4: Multiply the spectra in Step 2 by the line in Step 3. This is the desired output spectra.
Step 5: Now inverse Fourier transform the modulated output spectra from Step 4. This represents the desired Output(t) time series.
Step 6: Step 5 determined the required output time series. It may be desired to window, filter, and slightly modify these data for fine tuning adjustments to account for edges, finite data size, etc.

The above 6 steps provide an accepted means for generating fractional noise where β is the power law curve resulting in the frequency domain of the power spectrum. The spectra power is obtained by squaring the time series signal and performing the Fourier transform. Plotting on a log scale the power spectra versus frequency (also log scale) would yield a power law curve $x^{-\beta}$. The various values of the obtained β can be summarized as follows:

If $\beta = 0 \Rightarrow$ white Gaussian noise with zero persistence and stationary (constant mean and variance).

If $\beta = 2 \Rightarrow$ Brownian motion which is nonstationary.

If $\beta = 1 \Rightarrow$ Pink noise (1/f noise).

For $\beta > 1 \Rightarrow$ Nonstationary signal, strongly persistent.

For $\beta < 1 \Rightarrow$ Stationary signal.

If $0 < \beta < 1 \Rightarrow$ A weakly persistent signal.

If $\beta < 0 \Rightarrow$ An antipersistent signal.

Physically, as β becomes more positive, the persistence of the time series increases (positive) and the approximate entropy (disorder) decreases. Conversely, as β becomes more negative, the persistence decreases (becomes less than zero) and the approximate entropy increases (more disorder). Figure (6) shows the results of the generation of the fractional Gaussian noise and fractional Brownian motion for the following values of β: -1, -0.5, 0 (white Gaussian noise), 0.5, 1 (Pink noise), 1.5, 2 (Brownian motion), 2.5, and 3.0. As described next in equation (14), the relationship between the fractional derivative f_D in the time domain and the β obtained in the power frequency domain is as follows:

$$f_D = -(\beta/2) \qquad (14)$$

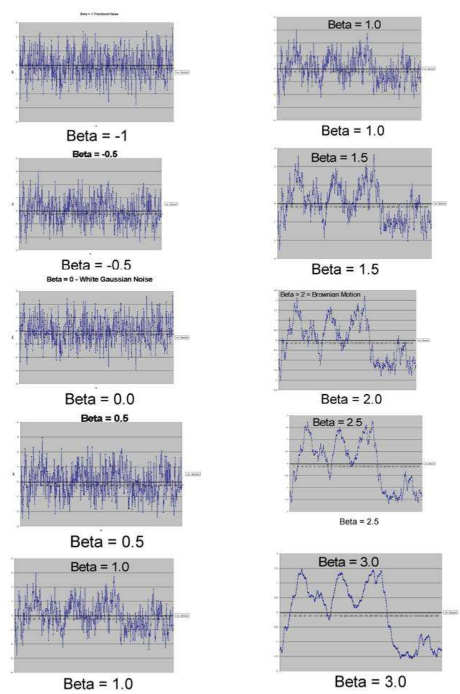

Figure (6) – Numerical Simulations of Fractional Noise

Finally to show the accuracy of the procedure, Figure (7) shows the simulation of pink noise which is known to have 1/f dynamics.

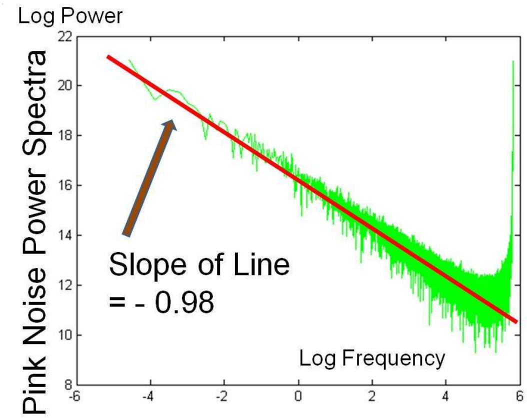

Figure (7) – Simulation of 1/f or Pink Noise

VI. SUMMARY AND CONCLUSIONS

A study was conducted on the relationships between the fractional calculus, fractals, and power laws. The concept of persistence of a real time series was quantified by a β term in a power series plot of fractional noise. The connection to approximate entropy was discussed. A baseline of fractional time series was generated and analyzed to show that the methods used here concur with other approaches in the literature [12].

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