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MIXED FOURIER TRANSFORMS AND IMAGE ENCRYPTION

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ABSTRACT

In this paper, we discuss a concept of *the mixed Fourier transformation*, when signals are transformed to the time-frequency domain, where the difference between the time and frequency is disappeared. Both cases of continuous-time and discrete-time signals are considered, and properties of the mixed transformations are described. The concept of mixed Fourier transform includes the fractional power of the discrete Fourier transform (DFT) which is referred to as a discrete version of the fractional Fourier transform. Mixed Fourier transformations can be used for calculation of roots of the Fourier and identity transformations. Examples of such root transforms for signal processing and image encryption are given.

1. INTRODUCTION

The representation of signals in frequency domain and application of the Fourier transform and other unitary transforms leads to effective solutions of many problems in signal processing and communication, and many other areas of science as well [1]-[3]. We note the fractional Fourier transform (FRFT) [4], which can be considered as an extension of the Fourier transform in frequency and time domains. This transform was applied in different areas such as quantum mechanics, signal processing, signal compression, image encryption. The transform with angle-order α in its continuous-time form is defined as the integral

$$F_\alpha(\omega) = (\mathcal{F}_\alpha \circ f)(\omega) = \sqrt{\frac{(1 - j\cot(\alpha)}{2\pi}} \int_{-\infty}^{\infty} f(t) e^{j\pi Q_\alpha(\omega,t)} dt$$

where the kernel of the transformation is defined by a quadratic form $Q_\alpha(\omega, t) = B_\alpha\omega^2 - 2C_\alpha\omega t + B_\alpha t^2$ with coefficient-functions $B_\alpha = \cot(\alpha)$ and $C_\alpha = 1/\sin(\alpha\pi/2)$. Such form allows for rotating the signal in the time-frequency domain. The special cases of $\alpha = \pm\pi/2$ lead to the forward and inverse Fourier transformations. The approximation of this concept for the discrete time case has different approaches to preserve the main properties of the FRFT on the rectangular grid and define the complete set of basic discrete-time functions of the transformation [5, 6]. We mention the attempt to define a fractional power of the DFT as a certain linear combination of the identity operation, time reversal operation, and forward and inverse discrete Fourier transformations [7, 8]. Such definition of the discrete FRFT represents a rotation in discrete time-frequency domain and allows for efficient computation of the transform through the fast DFT.

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In this paper, we focus on a general concept of *the mixed Fourier transformations*, when signals are transformed to the time-frequency domain, where the difference between of time and frequency is disappeared. Both continuous and discrete time cases are described. Such transformations allow for effective calculation of different roots of the Fourier and identity transformations, which can be used in signal processing, image filtration and encryption.

2. MIXED TRANSFORMS

In this section we briefly present a concept of mixed transformations which are defined as a transformations in the time and frequency domains simultaneously [9]. We need understand how to unite the time and frequency units for mixed transformations.

We start with the simple example, when a real function $f(t)$ with the Fourier transform $F(\omega)$ is transformed as

$$m_f(t) = f(t) + F^*(\lambda t), \quad t \in (-\infty, +\infty), \tag{1}$$

where λ is a scalar parameter, and * denotes the operation of complex conjugate. In the frequency domain, the function $m_f(t)$ is described as

$$M_f(\omega) = F(\omega) + \frac{2\pi}{\lambda} f\left(\frac{\omega}{\lambda}\right), \quad \omega \in (-\infty, +\infty).$$

By substituting ω/λ by t, we can write the last equation as

$$M_f(\lambda t) = F(\lambda t) + \frac{2\pi}{\lambda} f(t), \quad t \in (-\infty, +\infty). \tag{2}$$

If $\lambda = 2\pi$, equations (1) and (2) can be written together as

$$\begin{cases} m_f(t) &=& f(t) + F^*(2\pi t) \\ M_f(2\pi t) &=& F(2\pi t) + f(t). \end{cases}$$

The transformation

$$f(t) \to m_f(t) = f(t) + F^*(2\pi t), \qquad t \in (-\infty, +\infty), \tag{3}$$

we call *the time-and-frequency mixed transformation*, or simply *the mixed transformation*. The spectrum of the function $m_f(t)$ is expressed thus by the function

$$M_f(2\pi t) = m_f^*(t), \quad t \in (-\infty, +\infty). \tag{4}$$

In other words, the transform of this function coincides with the scaled in time by factor of $1/(2\pi)$ complex conjugate function itself. We define the space of such functions by $\mathcal{M}_{1,1}$. The variable t in the left part of the equality in (4) can be considered as the frequency in Hz, and $2\pi t$ then as the frequency in rad/sec. We thus

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consider t as the time and frequency at the same time, and then state that the Fourier transform of the function $m_f(t)$ is equal to the complex conjugate function itself.

Example 1: Let $f(t)$ be the delta function $\delta(t)$. Then, the mixed transform of $f(t)$ and its Fourier transform are calculated by

$$m_f(t) = \delta(t) + 1 \xrightarrow{\mathcal{F}} M_f(\omega) = 1 + 2\pi\delta(\omega).$$

Therefore, $M_f(2\pi t) = 1 + \delta(t) = m_f(t)$.

Example 2: For the cosine wave $f(t) = \cos(\pi t)$ with frequency π, the mixed transform and its Fourier transform are calculated respectively as

$$m_f(t) = \cos(\pi t) + \pi \left[\delta\left(2\pi t - \pi\right) + \delta\left(2\pi t + \pi\right)\right]$$
$$M_f(\omega) = \pi[\delta(\omega - \pi) + \delta(\omega + \pi)] + \frac{1}{2}\left[e^{-j\omega/2} + e^{j\omega/2}\right].$$

Therefore, substituting $\omega = 2\pi t$, we obtain

$$M_f(2\pi t) = \pi\left[\delta(2\pi t - \pi) + \delta(2\pi t + \pi)\right] + \cos(\pi t) = m_f(t).$$

If $f(t)$ is periodic with the fundamental period ω_0, then in the space $\mathcal{M}_{1,1}$, we obtain the following representation of the function

$$f(t) \rightarrow m_f(t) = \sum_{n=0,\pm1,\ldots} c_n \left[e^{jn\omega_0 t} + \delta(t - 2\pi n\omega_0)\right]$$

where c_n are coefficients of the Fourier series of $f(t)$.

To make the mixed transform invertible, we consider this concept in more general sense. Given two parameters $|a| \neq |b|$, let the mixed transform of a real function $f(t)$ be defined as

$$m_f(t) = af(t) + bF^*(2\pi t), \quad t \in (-\infty, +\infty). \tag{5}$$

We denote by $\mathcal{M}_{a,b}$ the space of such functions $m_f(t)$. For the Fourier transform of this mixed transform, the following holds: $M_f(2\pi t) = aF(2\pi t) + bf(t)$, and since $f(t)$ is periodic we obtain

$$M_f^*(2\pi t) = a^*F^*(2\pi t) + b^*f(t). \tag{6}$$

Values of parameters a and b can be taken arbitrary under the condition that $|a|^2 - |b|^2 \neq 0$. The Fourier transform of the function $m_f(t)$ is an element of the space M_{a^*,b^*}. If follows from (5) and (6), that the following matrix equation holds:

$$\begin{pmatrix} m_f(t) \\ M_f^*(2\pi t) \end{pmatrix} = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \begin{pmatrix} f(t) \\ F^*(2\pi t) \end{pmatrix}. \tag{7}$$

Therefore, the original function $f(t)$ and its Fourier transform can be defined by the mixed transform as follows:

$$\begin{pmatrix} f(t) \\ F^*(2\pi t) \end{pmatrix} = \frac{1}{D} \begin{pmatrix} a^* & -b \\ -b^* & a \end{pmatrix} \begin{pmatrix} m_f(t) \\ M_f^*(2\pi t) \end{pmatrix}$$

where the determinant of the matrix in (7) equals $D = |a|^2 - |b|^2$.

We now consider the operation of linear convolution in the space $M_{1,1}$. Let $x(t)$ be a function from this space, which is convoluted with a function $h(t)$. The linear convolution $y(t) = x(t) * h(t)$ in the time-frequency domain equals

$$Y(t) = X(t)H(t) = x^*\left(\frac{t}{2\pi}\right)H(t).$$

If the function $h(t)$ is also from the space $M_{1,1}$, then the linear convolution of these functions equals

$$Y(t) = x^*\left(\frac{t}{2\pi}\right)h^*\left(\frac{t}{2\pi}\right),$$

or $Y(2\pi t) = x^*(t)h^*(t)$. In other words, the linear convolution of functions in this space is reduced to the multiplication of the functions, and there is no much sense in using the linear convolution.

3. DISCRETE MIXED TRANSFORMATIONS

We consider the concept of the mixed transformations with respect to the discrete Fourier transforms [9]. This concept allows us to define the square and high degree roots of the Fourier transformation and other transformations as well.

We first try to define such a mixed transformation whose matrix is represented as a linear combination of the following three matrices $(N \times N)$ $S = aI + bE + cF$ where a, b, and c are coefficients, F is the matrix of the discrete Fourier transformation, and $E = F^2 = F \cdot F$. We call S *the mixed transformation* with respect to the Fourier transformation, or simply MxDFT. Note that $EE = FFFF = I, \quad EF = FFF = FE$.

If we try to find the inverse matrix S^{-1} in a similar form $S^{-1} = pI + sE + tF$, then the direct calculation of the product

$$I = SS^{-1} = (aI + bE + cF)(pI + sE + tF)$$

leads to the following system of four equations with six unknown variables:

$$\begin{cases} ap + bs = 1, \quad as + bp + ct = 0, \\ at + pc = 0, \quad cs + bt = 0. \end{cases} \tag{8}$$

Considering the first and two last equations, we obtain the following:

$$\begin{bmatrix} a & b & 0 \\ c & 0 & a \\ 0 & c & b \end{bmatrix} \begin{bmatrix} p \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where the determinant of the matrix 3×3 equals $-c(a^2 + b^2) \neq 0$. The solution of this system is

$$p = \frac{a}{\Delta}, \quad s = \frac{b}{\Delta}, \quad t = -\frac{c}{\Delta}, \quad (\Delta = a^2 + b^2),$$

and from the second equation of (8), it is not difficult to obtain that $c^2 - 2ab = 0 \rightarrow c = \pm\sqrt{2ab}$. Thus the inverse matrix of $S = aI + bE + cF$ equals

$$S^{-1} = \frac{1}{\Delta}\left(aI + bE - cF\right)$$

with the constraint $c = \pm\sqrt{2ab}$. In this case, the matrices S and S^{-1} are squares of matrices of the mixed transformations,

$$S = \left(\sqrt{a}I \pm \sqrt{b}F\right)^2, \quad S^{-1} = \frac{1}{\Delta}\left(\sqrt{a}I \mp \sqrt{b}F\right)^2.$$

As an example, Figure 1 shows the signal of length 512 in part a, along with real part of the mixed Fourier transform of this signal in b, and the imaginary part of the transform in c. The parameters of the mixed transform are $a = 0.25$, $b = 1 - a = 0.75$, and therefore $c = 0.6124$. One can notice from the figure in part b, that the signal has been rotated by 180° in the MxDFT.

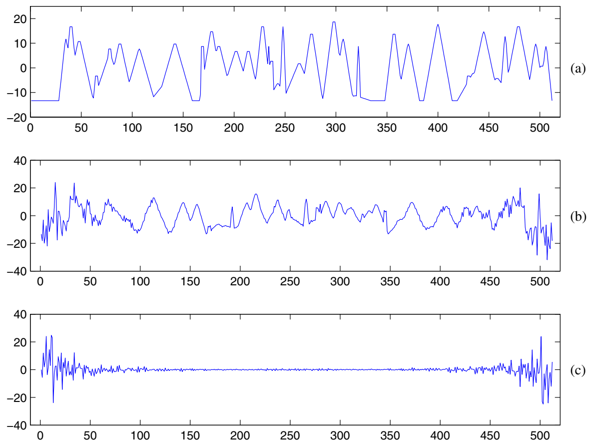

Fig. 1. (a) Original signal of length 512, and (b) the real part and (c) imaginary part of the mixed Fourier transform.

Figure 2 shows the 4-second chirp signal of length 512 in part a, along with real and imaginary parts of the mixed Fourier transform of this signal in b and c, respectively. The spectrum of the mixed transform in absolute scale is shown in d. The parameters of the mixed transform are the same; $a = 0.25$ and $b = 1-a = 0.75$.

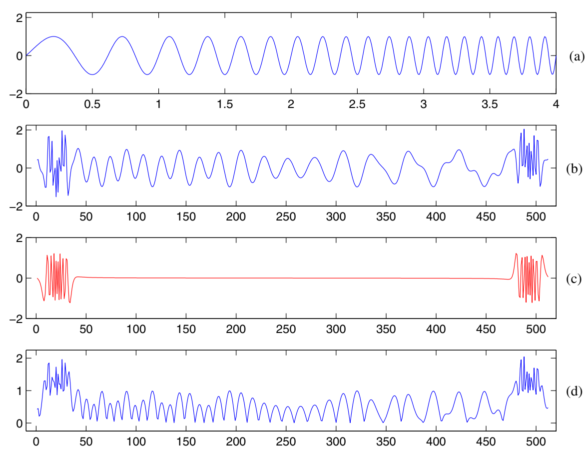

Fig. 2. (a) Original signal of length 512, and (b) the real part, (c) imaginary part, and (d) magnitude of the mixed Fourier transform.

The mixed transform can be applied to the 2-D case for representing and processing images in frequency and spatial domain. We define the 2-D mixed DFT as the separable transform which first is performed by rows and then by columns. The parameters of the 1-D mixed transform are considered unchangeable. Figure 3 shows the results of the 2-D MxDFT of the tree image, when parameters $a = 0.2$ and $b = 1-a$ in part a, $a = 0.75$ and $b = 1-a$ in b, $a = 0.25$ and $b = 1-3a$ in c, and $a = 0.75$ and $b = 1-2a$ in d. The amplitude spectrums of these transforms are shown.

It is clear that by using different parameters of the 1-D mixed transform as keys we can present the original image as a complex

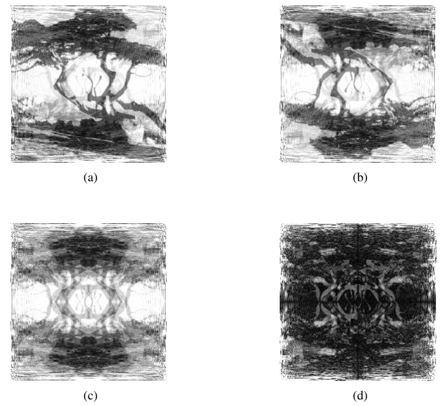

Fig. 3. Amplitude spectrums of the MixDFT of the tree image, when (a) $a = 0.2$ and $b = 0.8$, (b) $a = 0.75$ and $b = 0.25$, (c) $a = 0.25$ and $b = 0.25$, and (d) $a = 0.75$ and $b = -0.5$.

and deep secret image. As an example, we consider 1-D mixed transforms S_1, S_2, and S_3 with parameters $(a, b) = (0.2, -0.5)$, $(0.1, -1.1)$, and $(1.1, -0.1)$, respectively, and then apply them to compose the 2-D mixed transform $S_1(f)$ of the original image and mixed transforms $S_2(S_1(f))$ and $S_3(S_2(S_1(f)))$. Figure 4 shows the tree image in part a, and the amplitude spectrums of three 2-D MxDFTs performed sequentially over the tree image and obtained 2-D MxDFTs in b, c, and d, respectively.

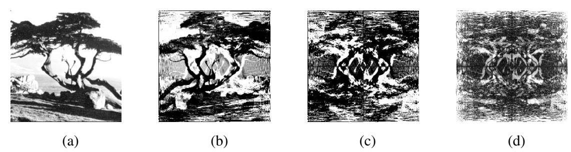

Fig. 4. (a) The tree image f and 2-D mixed Fourier transforms (b) $S_1(f)$, (c) $S_2(S_1(f))$, and (d) $S_3(S_2(S_1(f)))$.

The similar results of the sequential transformation of the Lena image by the same mixed transformations S_1, S_2, and S_3 are given in Figure 5.

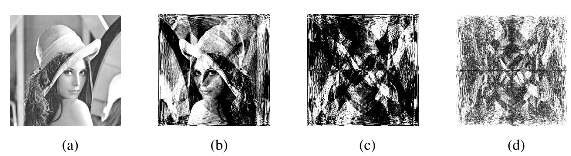

Fig. 5. (a) Lena image f and 2-D mixed Fourier transforms (b) $S_1(f)$, (c) $S_2(S_1(f))$, and (d) $S_3(S_2(S_1(f)))$.

3.1. General concept of the MxDFT

We now consider a general mixed discrete transformation whose matrix is represented as a linear combination of the following four $N \times N$ matrices

$$S = aI + bE + cF + dF^*,$$

where a, b, c, and d are coefficients, I is the identity matrix, F is the matrix of the discrete Fourier transformation, and $E = F^2 = FF$. We call S *the mixed transformation* with respect to the Fourier transformation, MxDFT. Note that $EE = FFFF = I$, and $EF = FFF = F^* = FE$.

We now describe square roots of the Fourier transform from the equation $S^2 = F$, which is considered in the linear space $\mathcal{L}(F)$ spanned on the matrices I, E, F, and F^*. The coefficients a, b, c, and d are calculated by

$$\begin{aligned} a &= \frac{s+w-jp+jt}{4}, \quad & b &= \frac{s+w+jp-jt}{4}, \\ c &= \frac{s-w+p+t}{4}, \quad & d &= \frac{s-w-p-t}{4}. \end{aligned} \tag{9}$$

where the constants s, w, p, and t are defined as $s^2 = 1, w^2 = -1, p^2 = j$, and $t^2 = -j$. The solution is not unique, because of ambiguity of the square roots s, w, p, and t.

Case 1: [+,+,+,+] Consider the following values of the square roots

$$s = 1,\ w = j,\ p = \sqrt{j} = \frac{1+j}{\sqrt{2}},\ t = \sqrt{-j} = \frac{1-j}{\sqrt{2}}.$$

From (9), we obtain the following coefficients:

$$a = \frac{1}{4}(1+\sqrt{2}+j),\ b = \frac{1}{4}(1-\sqrt{2}+j),\ c = a^*,\ d = b^*.$$

We call the transformation defined by this set of coefficients, *the 1st square root discrete Fourier transformation* (1-SQ DFT).

Case 2: [-,+,+,+] For the following values of the square roots

$$s = -1,\ w = j,\ p = \sqrt{j} = \frac{1+j}{\sqrt{2}},\ t = \sqrt{-j} = \frac{1-j}{\sqrt{2}},$$

the coefficients of the transform S are calculated by

$$a = \frac{1}{4}(-1+\sqrt{2}+j),\ b = \frac{1}{4}(-1-\sqrt{2}+j),\ c = a^*,\ d = b^*.$$

The transformation with the matrix $S = aI + bT + cF + dF^*$ corresponding this set of coefficients is called *the 2nd square root discrete Fourier transformation* (2-SQ DFT). It is interesting to note, that by replacing the coefficients c and d in the square root $F^{[1/2]}$, we obtain the square root of the inverse Fourier matrix, i.e.

$$(F^*)^{[1/2]} = aI + bT + dF + cF^*. \tag{10}$$

The Fourier matrix is the 4th order root of the identity matrix, and the matrix $F^{[1/2]}$ is thus the 8th order root of the identity matrix, i.e. $F^{[1/2]} = I^{[1/8]}$. As an example, Figure 6 shows the signal of length 512 in part a, along with amplitude of the Fourier transform of this signal in b and the square root Fourier transform in c. The square root is not symmetric but has small amplitude, when compared with the Fourier transform.

Example 3: To illustrate two defined above square roots of the discrete Fourier transformation, we consider the following sinusoidal signal in the exponential envelop:

$$x(t) = 2e^{-\frac{t}{16}} \sin(\pi t^2/64), \quad t \in [0, 50]. \tag{11}$$

The discrete-time signal of length 512 is composed from $x(t)$ by sampling uniformly in the interval $[0, 50]$. Figure 7 shows the signal $x(t)$ in part a, along with the real parts of the Fourier transform of this signal in b, 1-SQ DFT in c, and 2-SQ DFT in d. Figure 8 shows the phase in radians of the DFT of the signal $x(t)$ in part a, along with the phases of the 1-SQ DFT in b, and 2-SQ DFT in c.

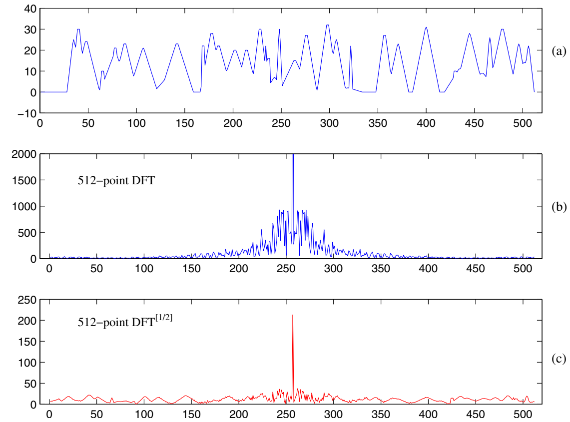

Fig. 6. (a) Original signal of length 512, (b) the Fourier transform, and (c) the square root Fourier transform. (The transforms are plotted in the absolute scale and shifted to the center.)

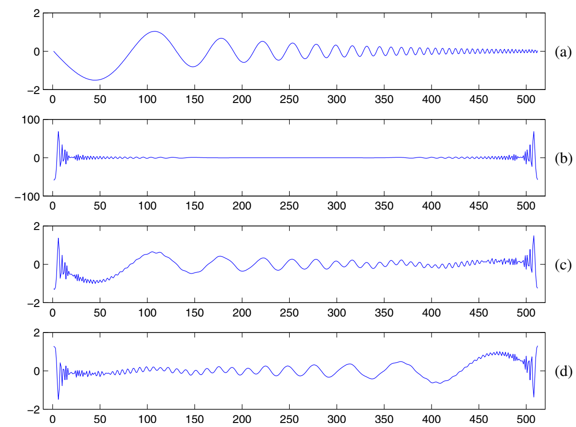

Fig. 7. (a) Original signal of length 512, and the real parts of the (b) DFT, (c) 1-SQ DFT, and (d) 2-SQ DFT.

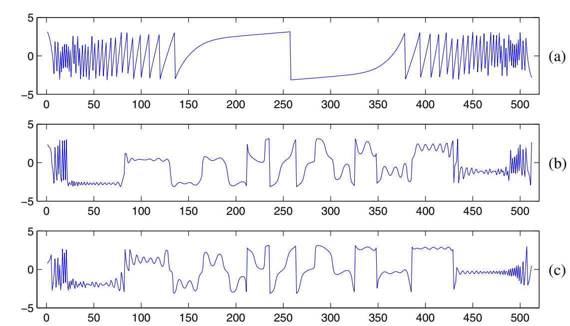

Fig. 8. Phases of the (a) DFT, (b) 1-SQ DFT, and (c) 2-SQ DFT.

3.2. Series of Fourier matrices

In this section we consider a simple method of calculation of the square roots of the Fourier transform, which uses the cyclic convolution. The considered above linear space $\mathcal{L}(F)$ has the following four basic matrices: $I = F^0$, $F = F^1$, $E = F^2$, and $F^* = F^3$, which compose a multiplicative group of order 4 (since $F^4 = I$). A matrix S of $\mathcal{L}(F)$ is represented by

$$S = a_0 I + a_1 F + a_2 F^2 + a_3 F^3$$

with real or complex coefficients a_k, $k = 0, 1, 2, 3$. The square of this matrix can be written as

$$S^2 = \left(a_0 I + a_1 F + a_2 F^2 + a_3 F^3\right)^2 = \sum_{n=0}^{3} b_k F^k,$$

where the coefficients b_n are calculated by the cyclic convolution

$$b_n = \sum_{k=0}^{3} a_k a_{n-k \bmod 4}, \quad n = 0, 1, 2, 3.$$

Let S be a square root of F. Then, the above cyclic convolution is written as

$$\sum_{k=0}^{3} a_k a_{n-k \bmod 4} = \delta_{n;1}, \quad n = 0, 1, 2, 3,$$

where $\delta_{1;1} = b_1 = 1$ and $\delta_{n;1} = b_n = 0$ if $n = 0, 2, 3$. In the frequency domain, this convolution has a form

$$|A_p|^2 = e^{-\frac{2\pi j}{4} p}, \quad p = 0, 1, 2, 3,$$

where A_p is the four-point discrete Fourier transform of the vector-coefficient $\{a_0, a_1, a_2, a_3\}$. Thus, the inverse 4-point DFT of the vector with elements calculated by $A_p = \pm \exp(j\pi p/4)$, i.e.

$$A_0 = \pm 1,\ A_1 = \pm\frac{1}{\sqrt{2}}(-1+j),\ A_2 = \pm j,\ A_3 = \pm\frac{1}{\sqrt{2}}(1+j),$$

results in the square root matrices of F. This method can be extended for calculating roots of high degrees of the Fourier matrix, $S^n = F$, for $n \geq 2$.

Figure 12 shows two images in parts a and c, along with the 2-D SQ-DFT of these images in b and d, respectively.

(a) (b) (c) (d)

Fig. 9. (a) The tree image and (b) 2-D SQ-DFT of the image, (c) Lena image and (d) 2-D SQ-DFT of the image.

Example 4: Let the mixed matrix $S \in \mathcal{L}(F)$ be the square root of the identity matrix, i.e.

$$S^2 = \left(\sum_{n=0}^{3} a_k F^k\right)^2 = \sum_{n=0}^{3} b_k F^k = I.$$

It is clear that $S = \pm I$ and $S = \pm E$ are such square roots, but these are trivial cases. We now try to find other solution when there are at least two coefficients a_k are not equal zero. Since $b_1 = b_2 = b_3 = 0$ and $b_0 = 1$, the coefficients a_k can be found from the following equation of the cyclic convolution:

$$\sum_{k=0}^{3} a_k a_{n-k \bmod 4} = \delta_n, \quad n = 0, 1, 2, 3,$$

where $\delta_n = 1$ if $n = 0$, and $\delta_n = 0$ if $n = 1, 2, 3$. In the frequency domain, this convolution has a form $|A_p|^2 = 1$, where $p = 0, 1, 2, 3$. Thus, the coefficients a_k can be defined by the inverse four-point DFT of the sequence(s) $\{A_p = \pm 1;\ p = 0 : 3\}$. The set of coefficients $A = (A_0, A_1, A_2, A_3)$ leads to a square root of the identity matrix, whose coefficients are calculated by

$$A = (-1, 1, 1, 1) \xrightarrow{\mathcal{F}^{-1}} a_0 = 0.5,\ a_1 = a_2 = a_3 = -0.5.$$

We obtain the following square root of the identity matrix:

$$S = I^{[1/2]} = \frac{1}{2}(I - E - F - F^*).$$

In the $N = 4$ case, this matrix equals

$$I^{[1/2]} = \frac{1}{2}\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}.$$

Basis functions of the square root of 8-point identity transformation, S, are illustrated in Figure 10.

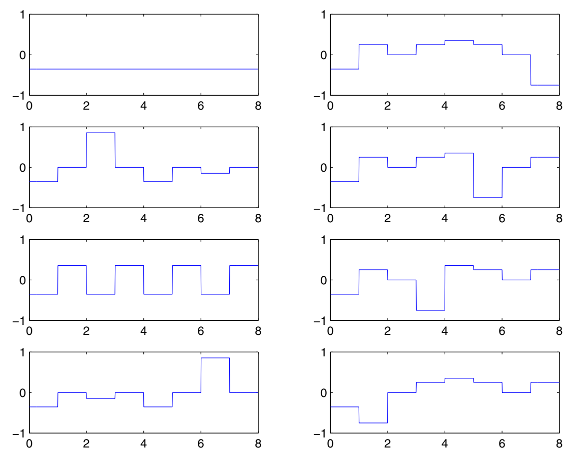

Fig. 10. Basis functions of the square root of the 8-point DIT.

Thus, in the space $\mathcal{L}(F)$ of the mixed transforms, there are six square roots of the identity matrix

$$S_0 = \pm I, \quad S_1 = \pm E, \quad S_2 = \pm\frac{1}{2}(I - E - F - F^*).$$

The discrete transform defined by the mixed matrix S_2 we call *the square root of discrete identity transformation,* or briefly (SR-DIT). The transform $S_1 = E$ is a square root of I, because the Fourier transform is the 4th degree root of the identity transform. The square root S_2 differs greatly from S_1 and contains the cosine transform $C = (F + F^*)/2$,

$$S_2 = \frac{1}{2}(I - E - F - F^*) = \frac{1}{2}(I - E) - \frac{F + F^*}{2} = \frac{1}{2}(I - E) - C,$$

and the difference of transforms equals

$$S_2 - S_1 = \frac{1}{2}(I - 3E) + C.$$

As an example, Figure 11 shows the original signal of length 512 in part a, along with the square root of discrete identity transform in b. For comparison, the real part of the square of the Fourier transform of this signal is given in c.

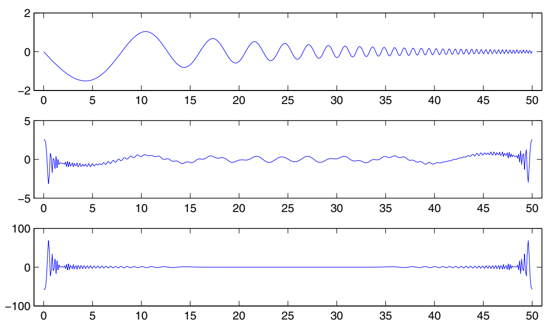

Fig. 11. (a) Signal of length 512, (b) the 512-point SR-DIT and (c) real part of the square of the 512-point DFT of this signal.

Figure 12 shows the tree image in part a, along with the 2-D SR-DIP of the image in b. The result of the second application of the square root over the image in b is shown in c. The second square root is the inverse to itself, and images in a and c are equal.

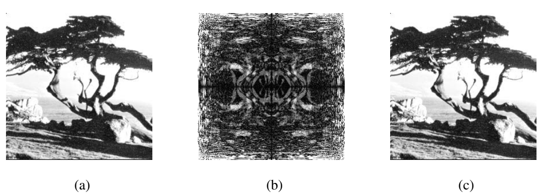

Fig. 12. (a) The tree image and (b) 2-D SR-DIT of the image, and the second (c) 2-D SR-DIT of the image.

In conclusion, we describe the direct method of calculation of the square root of the identity matrix.

Example 3: Let S be a mixed matrix which is a square root of the identity matrix, i.e. $S = aI + bE + cF + dF^*$ and $S^2 = I$. Then, coefficients a, b, and d satisfy the following system of equations $a = 1/2, b = 1/2, d = -c$, and $c = \pm j/2$, and the matrix is defined as

$$S = \frac{1}{2}(I + E) + \pm\frac{1}{2}j(F - F^*) = \frac{1}{2}\big(I + E \pm j(F - F^*)\big).$$

As an example, Figure 13 shows the raw-functions of the matrix S defined for $c = j/2$, in the $N = 8$ case.

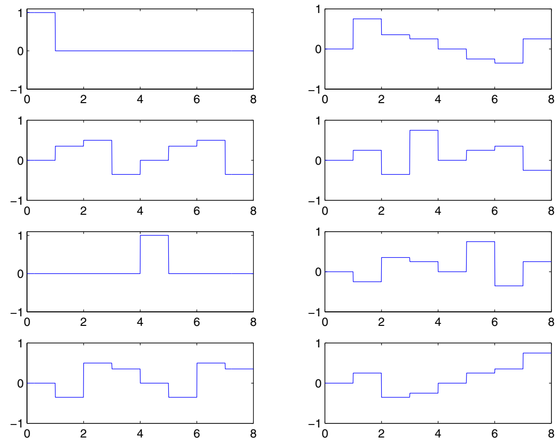

Fig. 13. The basis functions of the square of the identity transform.

4. CONCLUSIONS

In this paper, the concept of the mixed Fourier transform in the continuous and discrete time cases have been considered. The mixed transform represents the signals and images in the time-frequency domain, where the concepts of time and frequency are united. Mixed Fourier transformations can be used for calculating different roots of the Fourier and identity transformations, as well as other transformations, such as Hadamard and cosine transformations. Our preliminary experimental examples show that the described mixed and root transformations can be used for signal and image processing, especially for image encryption.

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