Proceedings of the 2009 IEEE International Conference on Systems, Man, and Cybernetics
San Antonio, TX, USA - October 2009

Stochastic inverse problems for growth models

Hycham Basta
Business-effiScience, 13, Cours Valmy, Tour Pacific Ouest, 92977 Paris-La-Défense, France
& CEREMADE, Université Paris IX Dauphine, Place du Marchal de Lattre de Tassigny, 75775 Paris, France
h.basta@business-effiscience.com

***Abstract*—Modern macroeconomics is built on the foundation of nonlinear dynamic stochastic general equilibrium (DSGE) models. In particular, the stochastic growth model is one of the most widely used models in all economics, and is the standard model for business cycle analysis. After reviewing some classical results on the existence of optimal solutions to stochastic calculus of variational problems in finite and infinite horizon, we show the connexions between those kind of problems and some classical stochastic optimal capital growth. Finally, we find some first results on the indeterminacy of capital accumulation path with uncertainty, which generalize the ones obtained by Boldrin and Montrucchio [4].**

***Index Terms*—Stochastic Calculus of Variations, HJB Equation, Stochastic Growth Model, Inverse Problems.**

I. Introduction

Many dynamic models used in current research assume that actions taken by the economic agents are very regular and predictable. It is often claimed that this regularity is a logical consequence of some hypothesis on the maximizing behavior of the agents. Those conclusions were first tested on the standard capital theory model. The dynamic behavior of neoclassical optimal growth models (see Stokey and Lucas [14] and McKenzie [8]) is usually described by a policy function, which is defined as a continuous mapping from any given capital stock to the future capital stock that is optimal according to the intertemporal objective function. Extensive economic litterature has also studied the inverse problem : if we are given a mapping with specific analytic properties on some particular topological space, can we always construct an optimal growth model which produce that mapping as the optimal policy function (see Boldrin and Montrucchio [4] and Montrucchio [11][12]).
Stochastic calculus techniques have been used to study continuous time economic growth under uncertainty. For example, Bourguignon [4] and Merton [8] have extented the neoclassical model of economic growth developped by Solow [12] to incorporate uncertainty in a continuous time framework by extensive use of Itô's lemma. We first present general results on the existence of an optimal solution to stochastic variational problems; we present the historical example of the stochastic Ramsey model to justify the use of diffusion processes. In this model we determine the optimal savings policy under uncertainty, whereas the dynamics of the capital-labour ratio is described by a diffusion-type stochastic process. We also prove some necessary conditions for optimality in infinite horizon.
After having revisited direct problems, a question rises naturally : "Can an arbitrary continuous observed dynamic process be an optimal policy for some value of the discount parameter in a neoclassical optimal accumulation model generalized to the stochastic case ?". We provide here a first, partially affirmative answer to this question.
The proofs are not provided in this short article, but every interested reader will be able to find them in my thesis [1].

June 29, 2009

II. Direct stochastic variational problems

A. The stochastic variational problem in finite horizon

We first begin by recalling not very well-known but existing results on the subject. The problem of extremality in mean of an action function according to an admissible family of Itô semi-martingales has been formulated for the first time by Bismut [2]. Similar techniques as the ones used in the classical celestial mechanics were adopted by him to solve a class of stochastic optimization problems.
As in the case of Bismut's work, we consider a completed filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}^+}, P)$ with a filtration continuous on the right. We define on Ω a standard m-dimensional Wiener process $W = (W^1, ..., W^m)$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. We also consider a family of semi-martingales on Ω with values in $\mathbb{R}^N$ that can be written as Itô processes with the following form :

$$x(t,\omega) = x_0 + \int_0^t \dot{x}(s,\omega)\, ds + \sum_{i=1}^m \int_0^t \Sigma_i(s,\omega)\, dW_s^i \quad (1)$$

where $\dot{x}, \Sigma_1, ..., \Sigma_m$ are adapted measurable processes, and where $\int_0^t \Sigma_i\ dW_s^i$ is the classical stochastic Itô integrale of Σ_i according to W^i for $i \in \{1, ..., m\}$.
For (1) to make sense, we make natural minimale hypotheses. Let $L(t, \omega, x, \dot{x}, \Sigma_1, ..., \Sigma_m)$ be a function defined on $\mathbb{R}^+ \times \Omega \times (\mathbb{R}^N)^{m+2}$ with real values; we suppose that $\forall(x, \dot{x}, \Sigma_1, ..., \Sigma_m)$ fixed in $(\mathbb{R}^N)^{m+2}$, $L(\omega, t, x, \dot{x}, \Sigma_1, ..., \Sigma_m)$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process. Finally, we suppose that the generalized Lagrangian $L(t, \omega, ., ., ., ..., .)$ is hyperregular (for example convex in the last variables).
We also consider a terminal cost function $\Phi(\omega, x)$ defined on $\Omega \times \mathbb{R}^N$ with values in $\mathbb{R}$, and we assume that it is $\mathcal{F}_T$-mesurable in ω for $x \in \mathbb{R}^N$ fixed, and regular enough in x.

978-1-4244-2794-9/09/$25.00

The most general formulation of our problem is to minimize the following criteria :

$$E\left[\int_0^T L(t,\omega,x(t,\omega),\dot{x}(t,\omega),\Sigma_1(t,\omega),...,\Sigma_m(t,\omega))\,dt \right. \\ \left. +\ \Phi(\omega,x(T,\omega))\right] \quad (2)$$

where $(x_t)_{0\leq t\leq T}$ is the unique solution of (1).
This problem of stochastic calculus of variations can be viewed as a stochastic control problem where the control variables are $(\dot{x},\Sigma_1,...,\Sigma_m)$. Again, we make suitable hypothesis for (2) to make sense. To solve this problem, Bismut used the same techniques as the ones to derive Euler-Lagrange equations of classical mechanics. He shows that under some specific hypotheses a necessary and sufficient condition for the process $(x_t)_{0\leq t\leq T}$ defined in (1) to be optimal according to criteria (2) is the existence of a semi-martingale $(p_t)_{0\leq t\leq T}$ with values in $\mathbb{R}^N$, that can be written :

$$p_t = p_0 + \int_0^t \dot{p}_s\,ds + \sum_{i=1}^m \int_0^t \Sigma_{i,s}^{'}\,dW_s^i + M_t \quad (3)$$

where $\dot{p},\Sigma_1^{'},...,\Sigma_m^{'}$ are measurable adapted processes, and M is a martingale with null value in 0 not necessarily continuous, orthogonale to W which means that $MW^1,...,MW^m$ are martingales, and the following conditions are verified :

$$(C1)\quad \begin{cases} \dot{p}_t = \dfrac{\partial L}{\partial x}(t,\omega,x_t,\dot{x}_t,\Sigma_{1,t},...,\Sigma_{m,t}) \\ p_t = \dfrac{\partial L}{\partial \dot{x}}(t,\omega,x_t,\dot{x}_t,\Sigma_{1,t},...,\Sigma_{m,t}) \\ \Sigma_{i,t}^{'} = \dfrac{\partial L}{\partial \Sigma_i}(t,\omega,x_t,\dot{x}_t,\Sigma_{1,t},...,\Sigma_{m,t}) \\ p_T = -\dfrac{\partial \Phi}{\partial x}(\omega,x_T) \qquad P\text{-a.s.} \end{cases}$$

Such conditions are satisfactory at first sight as they generalize Euler-Lagrange conditions and reflect many of the other maximum principles on stochastic control.
We can write $(C1)$ as a generalized Hamiltonian system of equations. Indeed, if $H(t,\omega,x,p,\Sigma_1^{'},...,\Sigma_m^{'})$ is the Legendre transform of the function $L(t,\omega,x,\dot{x},\Sigma_1,...,\Sigma_m)$ in the variables $(\dot{x},\Sigma_1,...,\Sigma_m)$, then we can rewrite $(C1)$:

$$(C2)\quad \begin{cases} dx_t = \dfrac{\partial H}{\partial p}(t,\omega,x_t,p_t,\Sigma_{1,t}^{'},...,\Sigma_{m,t}^{'})\,dt \\ \qquad +\ \displaystyle\sum_{i=1}^m \dfrac{\partial H}{\partial \Sigma_i^{'}}(t,\omega,x_t,p_t,\Sigma_{1,t}^{'},...,\Sigma_{m,t}^{'})\,dW_t^i \\ x(0) = x_0 \\ dp_t = -\dfrac{\partial H}{\partial x}(t,\omega,x_t,p_t,\Sigma_{1,t}^{'},...,\Sigma_{m,t}^{'})\,dt \\ \qquad +\ \displaystyle\sum_{i=1}^m \Sigma_{i,t}^{'}\,dW_t^i + dM_t \\ p_T = -\dfrac{\partial \Phi}{\partial x}(\omega,x_T) \qquad P\text{-a.s.} \end{cases}$$

This system of necessary and sufficient conditions is obviously equivalent to the following system of conditions :

$$(C3)\quad \begin{cases} dx_t = \dfrac{\partial \overline{H}}{\partial p}(t,\omega,x_t,p_t,\Sigma_{1,t},...,\Sigma_{m,t})\,dt \\ \qquad +\ \displaystyle\sum_{i=1}^m \Sigma_{i,t}\,dW_t^i \\ x(0) = x_0 \\ dp_t = -\dfrac{\partial \overline{H}}{\partial x}(t,\omega,x_t,p_t,\Sigma_{1,t},...,\Sigma_{m,t})\,dt \\ \qquad -\ \displaystyle\sum_{i=1}^m \dfrac{\partial \overline{H}}{\partial \Sigma_i}(t,\omega,x_t,p_t,\Sigma_{1,t},...,\Sigma_{m,t})\,dW_t^i \\ \qquad +\ dM_t \\ p_T = -\dfrac{\partial \Phi}{\partial x}(\omega,x_T) \qquad P\text{-a.s.} \end{cases}$$

where

$$\overline{H}(t,\omega,x,p,\Sigma_1,...,\Sigma_m) = \sup_{\dot{x}\in\mathbb{R}^d}\{<\dot{x},p> - L(t,\omega,x,\dot{x},\Sigma_1,...,\Sigma_m)\}$$

is nothing else than the partial Legendre Tranform of the random function L in the drift of the Itô process $(x_t)_{0\leq t\leq T}$. The first result is a generalization of the Euler-Lagrange optimality condition to the stochastic case

Theorem 2.1 (Generalized Euler-Lagrange Equation): Consider the following markovian one-dimensionnal diffusion

$$\forall\, 0\leq t\leq T, \qquad x_t^* = x(0) + \int_0^t \overline{b}(x_s^*)\,ds + \sum_{i=1}^m \int_0^t \overline{\sigma}_i(x_s^*)\,dW_s^i$$

where the functions $\overline{b}$ and $\overline{\sigma}_i, i\in\{1,...,m\}$ are twice continuously differentiable with bounded derivatives.
Let L be a Lagrangian defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m$ with values in $C_b^3(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m)$ and a function Φ with values in $C_b^1(\mathbb{R})$. Suppose that $(x_t^*)_{0\leq t\leq T}$ is optimal, i.e. it minimizes the following program :

$$(\mathcal{P}_T) \quad \inf_{x\in\mathcal{X}} E\left[\int_0^T L(x_t,b_t,\sigma_t)dt + \Phi(x_T) \mid x(0) = x\right]$$

Then, $L(.,.,.)$ needs to verify the following Euler-Lagrange system :
$\forall k \in \{1,...,m\}$

$$\begin{aligned}
0 &= \left\{\frac{\partial^2 L}{\partial x \partial b} + \frac{\partial^2 L}{\partial b^2}\overline{b}' + \sum_{i=1}^{m} \frac{\partial^2 L}{\partial b \partial \sigma_i}\overline{\sigma_i}'\right\}\overline{\sigma}_k - \frac{\partial L}{\partial \sigma_k} \\
0 &= \left\{\frac{\partial^2 L}{\partial x \partial b} + \frac{\partial^2 L}{\partial b^2}\overline{b}' + \sum_{i=1}^{m} \frac{\partial^2 L}{\partial b \partial \sigma_i}\overline{\sigma_i}'\right\}\overline{b} \\
&+ \frac{1}{2}\left(\sum_{i=1}^{m} |\overline{\sigma}_i|^2\right)\left\{\frac{\partial^3 L}{\partial x^2 \partial b} + \frac{\partial^3 L}{\partial x \partial b^2}\overline{b}' + \sum_{i=1}^{m} \frac{\partial^3 L}{\partial x \partial b \partial \sigma_i}\overline{\sigma}_i'\right. \\
&+ \frac{\partial^2 L}{\partial b^2}\overline{b}'' + \left(\frac{\partial^3 L}{\partial x \partial b^2} + \frac{\partial^3 L}{\partial b^3}\overline{b}' + \sum_{i=1}^{m} \frac{\partial^3 L}{\partial b^2 \partial \sigma_i}\overline{\sigma}_i'\right)\overline{b}' \\
&+ \sum_{i=1}^{m}\left[\left(\frac{\partial^3 L}{\partial x \partial b \partial \sigma_i} + \frac{\partial^3 L}{\partial b^2 \partial \sigma_i}\overline{b}' + \sum_{k=1}^{m} \frac{\partial^3 L}{\partial b \partial \sigma_i \partial \sigma_k}\overline{\sigma}_k'\right)\overline{\sigma}_i'\right. \\
&+ \left.\left.\frac{\partial^2 L}{\partial b \partial \sigma_i}\overline{\sigma}_i''\right]\right\} - \frac{\partial L}{\partial x} \\
0 &= \frac{\partial L}{\partial b}\left(x_T^*, \overline{b}(x_T^*), \overline{\sigma}(x_T^*)\right) + \frac{\partial \Phi}{\partial x}(x_T^*) \qquad P\text{-a.s.}
\end{aligned}$$

the first $m+1$ equations need only to be verified along the optimal path $(x_t^*)_{0 \le t \le T}$.

This system of equations generalizes the classical Euler-Lagrange equation for the deterministic calculus of variations. Indeed, if we put $\sigma \equiv 0$ and if L doesn't depend on σ, the previous system simplifies to the Douglas equation with terminal condition (which is similar to the developed form of the one-dimensional euler-lagrange equation) :

$$\begin{aligned}
\frac{\partial^2 L}{\partial x \partial b}\overline{b} + \frac{\partial^2 L}{\partial b^2}\overline{b}\,\overline{b}' - \frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial b}(x_T^*) + \frac{\partial \Phi}{\partial x}(x_T^*) &= 0
\end{aligned}$$

We can also give a necessary and sufficient optimality condition for a smooth convex Lagrangian; the necessary condition has been proved in the theorem (1.1) :

Theorem 2.2: Consider the following markovian diffusion

$$\forall\, 0 \le t \le T, \qquad x_t^* = x + \int_0^t \overline{b}(x_s^*)\, ds + \sum_{i=1}^{m} \int_0^t \overline{\sigma}_i(x_s^*)\, dW_s^i$$

where functions $\overline{b}$ and $\overline{\sigma}_i, i \in \{1,...,m\}$ are twice continuously differentiable with bounded derivatives; consider a Lagrangian $L \in C_b^1(\mathbb{R},\mathbb{R},\mathbb{R}^m)$ convex in all variables and a convex real function $\Phi \in C_b^1(\mathbb{R})$. Then $(x_t^*)_{0 \le t \le T}$ is optimal for the following program $(\mathcal{P}_T)$:

$$\inf_{x \in \mathcal{X}} E\left[\int_0^T L(x_t, b_t, \sigma_t)\, dt + \Phi(x_T) \mid x(0) = x\right]$$

if and only if

$$\frac{\partial L}{\partial b}(x_t^*, \overline{b}(x_t^*), \overline{\sigma}(x_t^*)) = -\int_t^T \frac{\partial L}{\partial x}(x_s^*, \overline{b}(x_s^*), \overline{\sigma}(x_s^*))\, ds$$
$$-\sum_{i=1}^{m} \int_t^T \frac{\partial L}{\partial \sigma_i}(x_s^*, \overline{b}(x_s^*), \overline{\sigma}(x_s^*))\, dW_s^i - \frac{\partial \Phi}{\partial x}(x_T^*) \quad P\text{-a.s.}$$

This theorem suppose that the filtration is the smallest σ-field with respect to which W_t is measurable. We can easily obtain a similar theorem in case of another filtration.
To justify the use of Wiener process to model stochastic growth, we refer to the seminal paper of Merton [9] who derived the stochastic Solow equation when the population size is characterized by a diffusion process, i.e.

$$dN = nNdt + \sigma N dz$$

where n is the expected growth rate of population per unit of time and σ is the instantaneous variance. Both n and σ are exogenously given and assumed constant. This process is obtained as the continuous-time limit of a reasonable stochastic process for the population dynamics as a simple branching process for population growth. The model of capital accumulation is assumed to be a one-sector neoclassical model with a constant returns to scale, strictly concave production function $F(K,N)$, where $K(t)$ denotes the capital stock and $N(t)$ denotes the labour force which is assumed to be proportional to the population size. The capital accumulation equation is classicly written as :

$$\dot{K}(t) = F(K(t), N(t)) - \lambda K(t) - C(t)$$

where λ is the rate of depreciation (assumed to be non-negative and constant) and $C(t)$ is aggregate consumption. If we define

$$\begin{aligned}
k(t) &\equiv K(t)/N(t), \quad \text{the capital-labour ratio} \\
c(t) &\equiv C(t)/N(t), \quad \text{per capita consumption} \\
f(k) &\equiv F(K,N)/N = F(K/N, 1), \quad \text{per capita (gross) output} \\
s(k) &\equiv 1 - c/f(k), \quad \text{(gross) savings per unit output}
\end{aligned}$$

By Itô's lemma, the stochastic Solow equation is given by

$$\begin{aligned}
dk_t &= [s(k_t)f(k_t) - (n + \lambda - \sigma^2)k_t]dt - \sigma k_t dz_t \\
k_0 &= c > 0
\end{aligned}$$

The stochastic Ramsey problem turns out to determine the optimal savings policy under uncertainty. Formally the finite-horizon problem is to find a savings policy, $s^*(k, T-t)$, so as to maximize

$$E\left[\int_0^T U((1 - s_t)f(k_t))dt \mid k(0) = c\right]$$

where $U(.)$ is a strictly concave Von Neumann-Morgenstern utility function of per capita consumption for the representative man. Without showing the technique used to solve the problem (stochastic dynamic programming), we see that this kind of stochastic model belongs to the more general family of stochastic variational problems of kind $(\mathcal{P}_T)$.

Going back to the analysis of Merton [9], because of the non-linearity of the Bellman partial differential equation closed-form solutions are rare. However, in the limiting infinite-horizon ($T \rightarrow \infty$) case of Ramsey, the analysis is substantially simplified because this partial differential equation reduces to an ordinary differential equation. That remark suggests to study directly the infinite horizon stochastic problem.

B. The stochastic variational problem in infinite horizon

Again, we suppose that the dynamics of the admissible processes takes the following form :

$$x(t,\omega) = x_0 + \int_0^t b(s,\omega)\, ds + \sum_{i=1}^m \int_0^t \sigma_i(s,\omega)\, dW_s^i$$

where $W = (W^1, ..., W^m)$ is a standard brownian motion in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$. Let ρ be a strictly positive constant.
Let's give a direct method to prove existence of a strong solution to the variational stochastic calculus problem posed in a reasonable space of admissible processes. Consider the set $\mathcal{U}^\rho$ defined by :

$$\mathcal{U}^\rho = \left\{ (b(.),\sigma(.)) \in \mathbb{R}^N \times \mathbb{R}^{m\times N} / E\left[\int_0^{+\infty} e^{-\rho t} |b(t)|^2 dt\right] + \sum_{i=1}^m E\left[\int_0^{+\infty} e^{-\rho t}|\sigma_i(t)|^2\, dt\right] < \infty \right\}$$

We recall that this set is a real Hilbert space for the following bilinear form :

$$((b(.),\sigma(.)),(d(.),\beta(.))) \longmapsto E\left[\int_0^{+\infty} e^{-\rho t} b(t)\, d(t)\, dt\right] + \sum_{i=1}^m E\left[\int_0^{+\infty} e^{-\rho t}\sigma_i(t)\, \beta_i(t)\, dt\right]$$

The cost function to minimize is :

$$J(b(.),\sigma(.)) = E\left[\int_0^{+\infty} e^{-\rho t} L(x(t), b(t), \sigma(t))\, dt\right] \qquad (4)$$

with $L : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{m\times N} \longrightarrow \mathbb{R}$.

Let's define the (SVC) problem as minimizing the Stochastic Variational Calculus Problem (4) on $\mathcal{U}^\rho$.
We also add the two following hypotheses :

- **(H0.1)** : $\exists\, \delta, K > 0$ so that
$$\forall (x,b,\sigma) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{m\times N}$$
$$L(x,b,\sigma) \geq \delta \left(|b|^2 + \sum_{i=1}^m |\sigma_i|^2\right) - K$$
- **(H0.2)** : function L is globaly convex.

Then, we have the following result :

Theorem 2.3: Suppose we have **(H0.1)** *and* **(H0.2)***; if the problem (SVC) si finite, that is if* $\inf_{\mathcal{U}^\rho} J(b(.),\sigma(.)) > -\infty$*, then we have an admissible optimal path.*

As we suspect that the long-term distribution of the optimal capital accumulation will play an important role, we must ask about the existence of a tranversality kind condition. We see that this condition will permit to avoid problems of integrability by using a weaker notion of optimality :

Definition 2.4: We will say that a process

$$x_t^* = x_0 + \int_0^t b_s^*\, ds + \int_0^t \sigma_s^*\, dW_s$$

is optimal in the sense of Brock for the optimization problem

$$(\mathcal{P}_\infty) \qquad \inf_{x\in\mathcal{X}} E\left[\int_0^{+\infty} e^{-\rho t} L(x_t, b_t, \sigma_t)\, dt \mid x(0) = x\right]$$

if for all admissible process

$$x_t = x_0 + \int_0^t b(s)\, ds + \int_0^t \sigma(s)\, dW_s$$

we have

$$\liminf_{T\longrightarrow+\infty} E\left[\int_0^T e^{-\rho t}\left(L(x_t,b_t,\sigma_t) - L(x_t^*, b_t^*, \sigma_t^*)\right) dt\right] \geq 0$$

Then we have the following proposition :

Proposition 2.5: Let's consider the following one-dimensional markovian diffusion :

$$\forall t \geq 0, \qquad x_t^* = x + \int_0^t \overline{b}(x_s^*)\, ds + \sum_{i=1}^m \int_0^t \overline{\sigma}_i(x_s^*)\, dW_s^i$$

and a Lagrangian $L \in C_b^1(\mathbb{R},\mathbb{R},\mathbb{R}^m)$*. If* $(x_t^*)_{t\geq 0}$ *is optimal in the sense of Brock for the problem* $\mathcal{P}_\infty$*, then, for all processes* $(e_t)_{t\geq 0}$ *of the form :*

$$e_t = \int_0^t \dot{h}_s\, ds + \sum_{i=1}^m \int_0^t h_{i,s}\, dW_s^i$$

we have :

$$\liminf_{\epsilon\to 0}\liminf_{T\to\infty} \frac{1}{\epsilon} E_x\left[\int_s^T e^{-\rho t}\left(L(x_t^* + \epsilon e_t, \overline{b}(x_t^*) + \epsilon h_t, \overline{\sigma}_1(x_t^*) + \epsilon h_{1,t} + \epsilon h_{1,t}, ..., \overline{\sigma}_m(x_t^*) + \epsilon h_{m,t}) - L(x_t^*, \overline{b}(x_t^*), \overline{\sigma}(x_t^*))\right) dt\right]$$
$$\geq E_x\left[-e^{-\rho s}\frac{\partial L}{\partial b}(x_s^*, \overline{b}(x_s^*), \overline{\sigma}(x_s^*))e_s\right]$$

III. THE GENERALIZED DOUGLAS PROBLEM

Let's recall that the original Douglas problem consisted in solving the following problem : if we have a family of curves statisfying the n-dimensional differential equation :

$$\ddot{x} = F(t, x, \dot{x}) \qquad\qquad (ODE)$$

can we determine scalar functions (and if possible all of them) $L(t, \dot{x}, \ddot{x})$ so that any solution of the previous ordinary differential equation is a regular optimum of the variational calculus problem

$$\inf \int L(t, x(t), \dot{x}(t))\, dt$$

where the infimum is taken over all family solutions of the ordinary equation (ODE).
In the following we propose a generalization of this inverse Douglas problem to stochastic Itô processes.

A. General result on rationalizing markovian policy function by stochastic dynamic optimization

We suppose that the following N-dimensional markovian diffusion is completely observable :

$$x_t^* = x_0 + \int_0^t \overline{b}(x_s^*)\, ds + \sum_{i=1}^m \int_0^t \overline{\sigma}_i(x_s^*)\, dW_s^i \qquad (5)$$

where $W = (W^1, ..., W^m)$ is the same standard Wiener process as previously introduced. We make the same classical hypotheses that insure existence and uniqueness of a solution to the stochastic differential equation above :

- **(H1.1)** : the vector $\overline{b}$ and the matrice $\overline{\sigma}$, whose columns are composed by m vectors of dimension N, $(\overline{\sigma}_1, ..., \overline{\sigma}_m)$, will be supposed to be C^2 vector fields.

- **(H1.2)** : $\overline{b}$ and $\overline{\sigma}$ are supposed to be lipschitz, i.e.

$$\exists K > 0 \ / \ \forall (x, y) \in \mathbb{R}^N \times \mathbb{R}^N$$
$$|\overline{b}(x) - \overline{b}(y)| + \| \overline{\sigma}(x) - \overline{\sigma}(y) \| \leq K|x - y|$$

$|.|$ is the euclidian norm attached to the canonical scalar product on R^N and $\| \ . \ \|$ is the usual matricial norm i.e. $\forall A \in \mathbb{R}^{N\times m}, \| \ A \ \| = \left(\mathrm{tr} AA^T\right)^{\frac{1}{2}} = \left[\sum_{i,j=1}^N |A_{i,j}|^2\right]^{\frac{1}{2}}$. The initial position point x_0 is supposed to be fixed.
Let's have $\mathcal{X}$ be the set of adapted and continuous processes $(x_t)_{t\geq 0}$ that can be written :

$$\forall t \geq 0, \qquad x_t = x(0) + \int_0^t b(s)\, ds + \sum_{i=1}^m \int_0^t \sigma_i(s)\, dW_s^i$$

where the vector fields $(b, \sigma_1, ..., \sigma_m)$ verify the minimale conditions that give sense to the previous stochastic differential equation, that is :

$$\forall t \geq 0, \qquad E\left[\int_0^t |b(s)|^2\, ds + \sum_{i=1}^m \int_0^t |\sigma_i(s)|^2\, ds\right] < \infty$$

The problem is : " Can we find Lagrangians of the form $L(x, \dot{x}, \Sigma)$, where $(x, \dot{x}, \Sigma) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{N\times m}$, and a strictly positive discount factor ρ so that the observed process $(x_t^*)_{t\geq 0}$, unique solution to (5), is the optimal process for the following problem $(\mathcal{P})$? "

$$\inf_{x(.)\in\mathcal{X}} E\left[\int_0^{+\infty} e^{-\rho t} L(x_t, b(t), \sigma_1(t), ..., \sigma_m(t)) dt \mid x(0) = x_0\right]$$

If that's possible we will say that the observed dynamics $(x_t^*)_{t\geq 0}$ can be globally rationalizable.
We find a similar result as the one of Boldrin and Montrucchio [4], that is, for a discount factor high enough, any admissible markovian Itô process is rational :

Theorem 3.1: Under hypotheses **(H1.1)** *and* **(H1.2)***, if we take ρ high enough, then the Itô process $(x_t^*)_{t\geq 0}$ is globaly rationalizable in the sense of Brock, i.e. there exists a Lagrangian $L(.,.,.)$ defined on the whole space and strictly convex in the last two variables and a real $\rho > 0$ so that $\forall x \in \mathcal{X}$:*

$$\liminf_{T\to+\infty} \left(E\left[\int_0^T e^{-\rho t} L(x_t, b_t, \sigma_t) dt\right] - E\left[\int_0^T e^{-\rho t} L(x_t^*, b(x_t^*), \sigma(x_t^*)) dt\right]\right) \geq 0$$

We can even choose (L, ρ) so that :

$$-\infty < E\left[\int_0^{+\infty} e^{-\rho t} L(x_t^*, b(x_t^*), \sigma(x_t^*)) dt\right] \leq \liminf_{T\to+\infty} E\left[\int_0^T e^{-\rho t} L(x_t, b_t, \sigma_t) dt\right]$$

B. Particular Lagrangian forms

Let's first see if we always can have a semi-linear quadratic lagrangian that rationalize a given markovian process of the form (5)

$$L_{L-Q}(x, b, \sigma) = q(x) + \frac{1}{2}\, b^T S(x) b + \mathrm{tr}(\sigma Q(x))$$

where $q(.), S(.), Q(.)$ are respectively functions of the vectorial variable $x \in \mathbb{R}^N$ with values in $\mathbb{R}, S_N^+(\mathbb{R}), \mathbb{R}^{m\times N}$, $S_N^+(\mathbb{R})$ representing the set of semi-definite symetric matrices of size N. The set $S_N^{++}(\mathbb{R})$ will be defined as the classical subset of invertible matrices of $S_N^+(\mathbb{R})$.
We first show in this part that even if there are no necessary conditions, if we look for sufficiently smooth value functions with some convexity property, the conditions imposed on the dynamics of the supposed optimal path restrict hardly the set of admissible dynamics. Here is a partial result that shows this phenomenon

Proposition 3.2: In general, we do not always find a convex twice continuously function on the whole space that is the *value function for a semi-linear quadratic type lagrangian with $(\overline{b}(x),\overline{\sigma}(x))_{x\in\mathbb{R}^N}$ as the unique optimal solution of the stochastic variational problem.*

The proof of this result rely on some classical linear algebra methods; we take for that a particular drift function $\overline{b}(x)=x$ for which we assume that it is possible to find a suitable convex value function sufficiently smooth. We use the HJB equation associated to the optimal problem and the maximal Liouville principle for convex functions to derive a contradiction.
The two following last results are about the proof of existence of a Lagrangian rationalizing the diffusion (5) in the case of a constant volatility and a linear drift. Our approach is contructive for both the one-dimensional and the multidimensional.
The following first result gives existence in dimension one of quadratic lagrangian and a preference rate for the future for which the observed diffusion is the unique solution to the stochastic calculus of variations problem with this Lagrangian and this preference rate :

Proposition 3.3: Let's take the following diffusion

$$\begin{aligned} dx_t^* &= \mu x_t^* \, dt + \sigma \, dW_t \qquad (6) \\ x_0^* &= x \end{aligned}$$

where $\mu > 0$ and σ is a m-dimensional vector different than the null vector. Then we will find two reals $q < 0$, $r > 0$ and a future preference rate $\rho > 0$ so that the process (6) is the unique optimal solution of :

$$\inf_{x(.)\,\in\,\mathcal{X}} E_x\left[\int_0^{+\infty} e^{-\rho t}\left(q\, x_t^2 + r\, b_t^2\right)dt\right]$$

The second and last proposition generalizes the previous to a squared matrix for the drift; we see that strong conditions appears for the matrix A :

Proposition 3.4: Consider the following diffusion

$$\begin{aligned} dx_t^* &= A x_t^* \, dt + \sigma \, dW_t \qquad (7) \\ x_0^* &= x \end{aligned}$$

where $A \in \mathbb{R}^{N\times N}$ and $\sigma \in \mathbb{R}^{N\times m}\backslash\{0\}$. Suppose A verifies the following hypothesis :

$$\exists (U,V) \in S_N^{++}(\mathbb{R}) \times S_N^{+}(\mathbb{R}) \qquad / \qquad A = UV$$

Then, there exist two matrices $Q \in S_N(\mathbb{R})$, $R \in S_N^{++}(\mathbb{R})$ and a future preference rate $\rho > 0$ so that the process (7) is the unique solution of the optimization problem :

$$\inf_{x(.)\,\in\,\mathcal{X}} E_x\left[\int_0^{+\infty} e^{-\rho t}\left(x_t^T Q\, x_t + b_t^T R\, b_t\right)dt\right]$$

IV. Conclusion

This paper was devoted to the presentation of a generalization of the classical variational calculus to the stochastic case, by adding a diffusion part to the deterministic part. The two horizons were both treated. We first gave a general existence result for the problem of stochastic calculus of variations for a very precise class of diffusions and for relatively strong hypotheses on the Lagrangian. Future work could consist in seeing if we could weaken those hypothesis without losing existence of an optimal solution. In finite horizon, we found a necessary optimality condition : under certain regularity hypotheses on the Lagrangian, a system of $m+1$ coupled equations must be verified with a transversality terminale condition; this system of equations naturally extends the classical case because we find the usual Douglas equation by eliminating the stochastic part of the diffusion. For the infinite horizon, we used a weak notion of optimality so as to avoid too strong integrability hypotheses on the Lagrangian. We showed in case of optimality of a diffusion process in infinite horizon, we must necessarily have a transversality condition at infinity. We also saw that there are some serious economic litterature which used stochastic growth path to extend neoclassical models of economic growth.
We also proved that any Itô process that we perfectly know its dynamical characteristics (drift and volatility) can result from a stochastic optimization program. We gave a general rationalizability result. Finally, we saw that we could specify some particular forms of the Lagrangian for certain diffusions of linear type. Future work could consist in looking for particular diffusions that would have semi-linear-quadratic Lagrangians as inverse problem solution.

Acknowledgment

The author would like to thank to Ivar Ekeland for suggesting this problem, and to Jos Alexandre Scheinkman and Nizar Touzi for their valuable comments.

References

[1] H. Basta, "Strategic Use of Information and Inverse Problems", PhD Thesis, CEREMADE, University of Paris-Dauphine, 2003.
[2] J. M. Bismut, "Conjugate Convex Functions in Optimal Stochastic Control", *Journal of Mathematical Analysis and Applications*, **44**, 384-404, 1973.
[3] J. M. Bismut, "An Introductory Approach to Duality in Optimal Stochastic Control", *Siam Review*, **20**, 62-78, 1978.
[4] M. Boldrin and L. Montrucchio, "On the Indeterminacy of Capital Accumulation Paths", *Journal of Economic Theory*, **40**, 26-39, 1986.
[5] F. Bourguignon, "A Particular Class of Continuous Time Stochastic Growth Models", *Journal of Economic Theory*, **9**, 141-158, 1974.
[6] W. A. Brock, "On Existence of Weakly Maximal Programmes in a Multi-Sector Economy", *Review of Economic Studies*, **37**, 275-280, 1970.
[7] J. Douglas, "Solution of the Inverse Problem of the Calculus of Variations" , *Transactions of the American Mathematical Society*, **50**, 71-128, 1941.
[8] L. W. McKenzie, "Optimal Economic Growth, Turnpike Theorems and Comparative Dynamics", in *Handbook of Mathematical Economics: Volume III* (K. Arrow and M. Intriligator, eds), North-Holland, Amsterdam, 1986.
[9] R. Merton, "An Asymptotic Theory of Growth Under Uncertainty", *Review of Economic Studies*, **42**, 375-393, 1975.
[10] T. Mitra and G. Sorger, "Rationalizing Policy Functions by Dynamic Optimization", *Econometrica*, **67**, 375-392, 1999.

[11] L. Montrucchio, "Lipschitz Continuous Policy Functions for Strongly Concave Optimization Problems", *Journal of Mathematical Economics*, **16**, 259-273, 1987.
[12] L. Montrucchio, "Dynamic Complexity of Optimal Paths and Discount Factors for Strongly Concave Problems", *Journal of Optimization Theory and Applications*, **80**, 385-406, 1994.
[13] R. M. Solow, "A Contribution to the Theory of Economic Growth", *the Quarterly Journal of Economics*, **70**, 65-94, 1956.
[14] N. L.Stockey and R. E.Lucas Jr., "Recursive Methods in Economics", Harvard University Press, Cambridge, 1989.
[15] J. Yong and X. Y. Zhou, "Stochastic Controls : Hamiltonian Systems and HJB Equations", New York: Springer Verlag, 1999.