Flocking of Multi-Agent Systems with a Virtual Leader

Hong Shi, Long Wang, Tianguang Chu, Feng Fu
Center for Systems and Control
College of Engineering, Peking University
Beijing 100871, China

Minjie Xu
School of Electrical Engineering
Beijing Jiaotong University
Beijing 100044, China

Abstract— This paper considers the flocking problem of a group of autonomous agents moving in the space with a virtual leader. We investigate the dynamic properties of the group for the case where the state of the virtual leader may be time-varying and the topology of the neighboring relations between agents is dynamic. To track such a leader, we introduce a set of switching control laws that enable the entire group to generate the desired stable flocking motion. The control law acting on each agent relies on the state information of its neighboring agents and the external reference signal (or "virtual leader"). Then we prove that, if the acceleration input of the virtual leader is known, then each agent can follow the virtual leader, and moreover, the convergence rate of the center of mass (CoM) can be estimated; if the acceleration input is unknown, then the velocities of all agents asymptotically approach the velocity of the CoM, and thus the flocking motion can be obtained, however in this case, the final velocity of the group may not be equal to the desired velocity. Numerical simulations are worked out to further illustrate our theoretical results.

I. INTRODUCTION

Flocking is ubiquitous in nature, e.g., flocking of birds, schooling of fish, and swarming of bacteria, and it is a form of collective behavior of multiple interacting agents. In recent years, there has been an increasing research interest in the distributed control/coordination of the motion of multiple dynamic agents/robots and the control design of multi-agent systems. There has been considerable effort in modeling and exploring the collective dynamics, and trying to understand how a group of autonomous creatures or man-made mobile autonomous agents/robots can cluster in formations without centralized coordination and control [1]–[16]. Many results have been obtained with local rules applied to each agent in a considered multi-agent system.

Stimulated by the simulation results in [8], Tanner et al. [4] considered a group of mobile agents moving in the plane with double-integrator dynamics. They introduced a set of control laws that enable the group to generate stable flocking motion, but these control laws cannot regulate the final speed and heading of the group. Due to the fact that in some cases, the regulation of agents has certain purposes such as achieving desired common speed and heading, or arriving at a desired destination, the cooperation/coordination of multiple mobile agents with some virtual leaders is an interesting and

important topic. There have been some papers dealing with this issue in the literature. For example, Leonard and Fiorelli [7] viewed reference points as virtual leaders for manipulating the geometry of an autonomous vehicle group and directing the motion of the group; Olfati-Saber [1] used virtual leaders to accomplish obstacle avoidance.

In this paper, we consider the flocking problem of multiple mobile autonomous agents moving in an n-dimensional Euclidean space with point mass dynamics. By viewing the external reference signal as a virtual leader, we show that all agents eventually move ahead at a common velocity and maintain constant distances between them. We propose a set of control laws and develop a Lyapunov-based approach to analyze the problem. With the proposed control laws and the selected Lyapunov functions, the tracking problem can be solved if the acceleration input of the virtual leader is known by all agents; and the problem may not be solved if the acceleration input is unknown, but in this case all agents still eventually move at a common velocity, i.e., the flocking motion can be achieved.

This paper is organized as follows. In Section II, we formulate the problem to be investigated. We analyze the system stability and the motion of the center of mass (CoM) by using some control laws in Section III. Some numerical simulations are presented in Section IV. Finally, we briefly summarize our results in Section V.

II. PROBLEM FORMULATION

In this paper, we consider a group of N ($N \geq 2$) agents moving in an n-dimensional Euclidean space; each has point mass dynamics described by

$$\begin{aligned}
\dot{x}^i &= v^i, \\
m_i \dot{v}^i &= u^i, \quad i = 1, \dots, N,
\end{aligned} \tag{1}$$

where $x^i \in \mathbf{R}^n$ is the position vector of agent i, $v^i \in \mathbf{R}^n$ is its velocity vector, $m_i > 0$ is its mass, and $u^i \in \mathbf{R}^n$ is the control input acting on agent i.

Our objective is to make the entire group move at a desired velocity and maintain constant distances between the agents. The desired velocity is supposed to be a time-varying and smooth function, which means that the state of the virtual leader keeps changing. In order to achieve our control objective, we try to regulate agent velocities to the desired velocity, reduce the velocity differences between neighboring agents, and at the same time, regulate their distances such

that their potentials become minima. Hence, we choose the control law u^i for agent i to be

$$u^i = \alpha^i + \beta^i + \gamma^i, \qquad (2)$$

where α^i is used to regulate the potentials among agents, β^i is used to regulate the velocity of agent i to the weighted average of the velocities of its neighbors, and γ^i is used to regulate the velocity of agent i to the desired velocity (all to be designed later). α^i is derived from the social potential field which is described by an artificial social potential function, V^i, a function of the relative distances between agent i and its flockmates. Freedom from collisions and cohesion in the group can be guaranteed by this term. β^i reflects the alignment or velocity matching with neighbors among agents. γ^i is designed to regulate the velocity of agent i based on the external signal (the desired velocity).

Definition 1: [4] (*Neighboring Graph*) The neighboring graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, is an undirected graph consisting of a set of vertices, $\mathcal{V} = \{n_1, \ldots, n_N\}$, indexed by the agents in the group, and a set of edges, $\mathcal{E} = \{(n_i, n_j) \in \mathcal{V} \times \mathcal{V} \mid n_j \sim n_i\}$, containing unordered pairs of vertices that represent the neighboring relations.

The neighboring graph is used to describe the sensor information flow in the group. In $\mathcal{G}$, an edge (n_i, n_j) means that agent i can sense agent j, and it will regulate its state based on the position and velocity of agent j. In this paper, we mainly consider the dynamic and symmetric neighboring relations between agents. Let $\mathcal{I} = \{1, \ldots, N\}$. We write $\mathcal{N}_i \triangleq \{j \mid \|x^{ij}\| \leq R\} \subseteq \mathcal{I} \backslash \{i\}$ for the set which contains all neighbors of agent i, where $x^{ij} = x^i - x^j$ denotes the relative position vector between agents i and j; $R > 0$ is a constant and can be viewed as the sensing radius of the sensors. Here we assume that the sensors of all agents have the same sensing range. During the course of motion, the relative distances between agents vary with time, so the neighbors of each agent are not fixed, which generates the switching neighboring graph. In the discussion to follow, we assume that the neighboring graph $\mathcal{G}$ remains connected, which ensures that the group will not be divided into several isolated subgroups. In order to depict the potential between the agents, we present the following definition.

Definition 2: [4] (Potential Function) Potential V^{ij} is a continuous, nonnegative function of the distance $\|x^{ij}\|$ between agents i and j, such that $V^{ij}(\|x^{ij}\|) \to \infty$ as $\|x^{ij}\| \to 0$, V^{ij} attains its unique minimum when agents i and j are located at a desired distance, V^{ij} is differentiable on $(0, R) \cup (R, \infty)$, and V^{ij} is a constant $V^{ij}(R)$ for $\|x^{ij}\| > R$.

Function V^{ij} can be nonsmooth at $\|x^{ij}\| = R$. By the definition of V^{ij}, the total potential of agent i can be expressed as

$$V^i = \sum_{j \notin \mathcal{N}_i, j \neq i} V^{ij}(R) + \sum_{j \in \mathcal{N}_i} V^{ij}(\|x^{ij}\|).$$

Certainly, in reality, according to different cases, we can define different interaction potential functions such as the functions considered in [1], [7], and [14]–[16].

III. MAIN RESULTS

In this section, we investigate the stability properties of the system described in (1). We will present explicit control input in (2) for the terms α^i, β^i, and γ^i. We will employ matrix analysis, algebraic graph theory and nonsmooth analysis as basic tools for our discussion, and some concepts and results are available in [17]–[21].

During the course of motion, each agent regulates its position and velocity based on the external signal and the state information of its neighbors. However, it is known that, in reality, because of the influence of some external factors, the reference signal is not always detected by all agents in the group. In this paper, we will consider the case where the signal is sent continuously and at any time, there exists at least one agent in the group who can detect it.

We take the control law u^i for agent i to be

$$u^i = -\sum_{j \in \mathcal{N}_i} \nabla_{x^i} V^{ij} - \sum_{j \in \mathcal{N}_i} w_{ij}\left(v^i - v^j\right) \\ -h_s^i m_i \left(v^i - v^0(t)\right) + g_i m_i a_0(t), \qquad (3)$$

where $v^0(t) \in \mathbf{R}^n$ is the desired velocity and $\dot{v}^0(t) = a_0(t)$; $h_s^i \geq 0$ represents the intensity of influence of the reference signal on the motion of agent i; $g_i = 1$ if agent i knows the acceleration input $a_0(t)$ and is 0 otherwise; $w_{ij} \geq 0$, $w_{ij} = w_{ji}$, and $w_{ii} = 0$, $i, j = 1, \ldots, N$, represent the interaction coefficients. Assume that $h_s^i = h_i$ if agent i can detect the reference signal, where $h_i > 0$ is a constant, and is 0 otherwise. $w_{ij} = c_{ij}$ is fixed if agent j is a neighbor of agent i, where $c_{ij} > 0$ ($\forall i \neq j$) is a constant, and is 0 otherwise. Here we always assume that $c_{ij} = c_{ji}$, which means that the interaction between agents is reciprocal. We write $W_\sigma = [w_{ij}]_\sigma \in \mathbf{R}^{N \times N}$ for the interaction coefficient matrix (or coupling matrix), where σ is a switching signal and is a piecewise constant function $\sigma(t) : [0, \infty) \to \mathcal{P}$, $\mathcal{P}$ is a finite index set where the number of the indices is equal to the number of the connected graph $\mathcal{G}_\sigma$ in the group. The switching signal σ relies on the distances between agents. Thus, W_σ is always symmetric, and by the assumption of the connectivity of the neighboring graph $\mathcal{G}_\sigma$, W_σ is always irreducible.

A. Not all agents can obtain the desired velocity $v^0(t)$, but they all know the acceleration input $a_0(t)$, i.e., $g_i = 1$ for all $i \in \mathcal{I}$.

Lemma 1: [4] Function V^{ij} is regular everywhere in its domain. Moreover, the generalized gradient of V^{ij} at R and the (partial) generalized gradient of V^{ij} with respect to x^i at R are empty sets.

Theorem 1: Taking the control law in (3), all agent velocities in the group described in (1) asymptotically approach the desired velocity, avoidance of collisions between the agents is ensured, and the group final configuration minimizes all agent potentials.

This theorem becomes clearly true after Theorem 2 is proved, so we proceed to present Theorem 2.

We define the following error vectors:

$$e_p^i = x^i - \int_{t_0}^t v^0(\tau)d\tau \quad \text{and} \quad e_v^i = v^i - v^0(t),$$

where t is the time variable and t_0 is the initial time. Then e_v^i represents the velocity difference vector between the actual velocity and the desired velocity of agent i. It is easy to see that $\dot{e}_p^i = e_v^i$ and $\dot{e}_v^i = \dot{v}^i - a_0(t)$. Hence, the error dynamics is given by

$$\begin{aligned} \dot{e}_p^i &= e_v^i, \\ m_i \dot{e}_v^i &= u^i - m_i a_0(t), \quad i = 1, \ldots, N. \end{aligned} \tag{4}$$

Note that, in fact, we choose a moving reference frame and take the position of the virtual leader as the origin. By the definition of V^{ij}, it follows that $V^{ij}(\|x^{ij}\|) = V^{ij}(\|e_p^{ij}\|) \triangleq \widetilde{V}^{ij}$, where $e_p^{ij} = e_p^i - e_p^j$, and hence $\widetilde{V}^i = V^i$ and $\nabla_{e_p^i} \widetilde{V}^{ij} = \nabla_{x^i} V^{ij}$. Thus the control input for agent i in the error system (4) has the following form:

$$u^i = -\sum_{j \in \mathcal{N}_i} \nabla_{e_p^i} \widetilde{V}^{ij} - \sum_{j \in \mathcal{N}_i} w_{ij} \left(e_v^i - e_v^j \right) - h_s^i m_i e_v^i + m_i a_0(t). \tag{5}$$

Theorem 2: Taking the control law in (5), all agent velocities in the system described in (4) asymptotically approach zero, avoidance of collisions between the agents is ensured, and the group final configuration minimizes all agent potentials.

Proof: Consider the following positive semi-definite function:

$$J = \frac{1}{2} \sum_{i=1}^N \left(\widetilde{V}^i + m_i e_v^{iT} e_v^i \right).$$

Function J is continuous but may be nonsmooth whenever $\|e_p^{ij}\| = R$ for some $(i,j) \in \mathcal{I} \times \mathcal{I}$. Define the level set of J in the space of agent velocities and relative distances in the error system: $\Omega = \left\{ \left(e_v^i, e_p^{ij} \right) \mid J \leq c, \ c > 0 \right\}$. Though the neighboring relations vary with time, under the assumption of the connectivity of $\mathcal{G}_\sigma$, the set Ω is compact. This is because the set $\{e_v^i, e_p^{ij}\}$ with $J \leq c$ is closed by continuity. Moreover, boundedness can be proved by the connectivity of $\mathcal{G}_\sigma$. More specifically, because $\mathcal{G}_\sigma$ is always connected, there must be a path connecting any two agents i and j in the group and its length does not exceed $N-1$, and on the other hand, the distance between two interconnected agents is not more than R, hence, we have $\|e_p^{ij}\| \leq (N-1)R$. By similar analysis, $e_v^{iT} e_v^i \leq 2c/m_i$; thus $\|e_v^i\| \leq \sqrt{2c/m_i}$. Note that the restriction of J in Ω ensures collision avoidance and the differentiability of $\|e_p^{ij}\|$, $\forall i,j \in \mathcal{I}$.

By the definition of $\widetilde{V}^{ij}$, $\widetilde{V}^{ij}$ is continuous and locally Lipschitz. From Lemma 1, $\widetilde{V}^{ij}$ is regular everywhere in its domain and then $\widetilde{V}^i$ is regular everywhere, hence, J is regular as a sum of regular functions [21]. Then, we have

$$\partial J \subset \left[\sum_{j=2}^N \left(\partial_{e_p^1} \widetilde{V}^{1j} \right)^T, \ldots, \sum_{j=1}^{N-1} \left(\partial_{e_p^N} \widetilde{V}^{Nj} \right)^T, m_1 e_v^{1T}, \ldots, m_N e_v^{NT} \right]^T.$$

Hence, the generalized time derivative of J is

$$\dot{\tilde{J}} \subset \sum_{i=1}^N \left(\bigcap_{\xi_i} \xi_i^T e_v^i \right)$$

$$- e_v^T K \left[(L_\sigma \otimes I_n) e_v + \begin{pmatrix} \vdots \\ \nabla_{e_p^i} \widetilde{V}^i \\ \vdots \end{pmatrix} + (H_s \otimes I_n) e_v \right],$$

where $\xi_i \in \sum_{j=1, j \neq i}^N \partial_{e_p^i} \widetilde{V}^{ij}$; $e_v = \left(e_v^{1T}, \ldots, e_v^{NT} \right)^T$ is the stack vector of all agent velocity vectors in the error system; $L_\sigma = [l_{ij}]_\sigma \in \mathbf{R}^{N \times N}$ with

$$l_{ij} = \begin{cases} -w_{ij}, & i \neq j, \\ \sum_{k=1, k \neq i}^N w_{ik}, & i = j; \end{cases}$$

$\otimes$ stands for the Kronecker product; I_n is the identity matrix of order n; $\nabla_{e_p^i} \widetilde{V}^i = \sum_{j \in \mathcal{N}_i} \nabla_{e_p^i} \widetilde{V}^{ij}$; and $H_s = \operatorname{diag} \left(h_s^1 m_1, \ldots, h_s^N m_N \right)$. Due to the switching topology of the neighboring relations, L_σ and $\nabla_{e_p^i} \widetilde{V}^i$ will correspondingly change. By Lemma 1, we get

$$\dot{\tilde{J}} \subset -\overline{co} \left\{ e_v^T \left[(L_\sigma + H_s) \otimes I_n \right] e_v \right\}.$$

It is easy to see that L_σ is symmetric and has the properties that each row sum is equal to 0, the diagonal entries are positive, and all the other entries are nonpositive. On the other hand, H_s is a diagonal matrix with nonnegative entries and there exists at least one diagonal entry which is positive. Furthermore, since the neighboring graph $\mathcal{G}_\sigma$ is connected, $L_\sigma + H_s$ is irreducible. Hence, matrix $L_\sigma + H_s$ is irreducibly diagonally dominant. By Corollary 6.2.27 in [18], it follows that matrix $L_\sigma + H_s$ is positive definite. Thus $-\overline{co} \left\{ e_v^T \left[(L_\sigma + H_s) \otimes I_n \right] e_v \right\}$ is an interval of the form $[l, 0]$ with $l < 0$, and 0 is contained in it only when $e_v^1 = \cdots = e_v^N = \mathbf{0}$. The rest of the analysis is similar to Theorem 2 in [6], and thus is omitted. ∎

Remark 1: Note that, if all agents know the desired velocity, i.e., $h_s^i = h_i > 0$ for all $i \in \mathcal{I}$, then they still eventually move at the desired velocity even when the neighboring graph is not connected.

Remark 2: If the coupling matrix W_σ is asymmetric, we can regulate the control law acting on each agent to generate the desired flocking motion. The main analysis is as follows:

Define the position neighboring graph $\mathcal{G}_\sigma$ and the velocity neighboring graph $\mathcal{D}_\sigma$ as in [5] and assume that $\mathcal{G}_\sigma$ and $\mathcal{D}_\sigma$ are always strongly connected. From [16], we obtain that, if $\mathcal{D}_\sigma$ is strongly connected, then its Laplacian matrix L_σ is irreducible and for each L_σ, there is only one left eigenvector $\xi_\sigma = [\xi_1, \ldots, \xi_N]_\sigma^T \in \mathbf{R}^N$ such that $0 < \xi_i < 1$ for all $i \in \mathcal{I}$, $\xi_\sigma^T L_\sigma = 0$, and $\sum_{i=1}^N \xi_i = 1$. Then, we modify the control law u^i to

$$\begin{aligned} u^i &= -\sum_{j \in \mathcal{N}_i} \nabla_{x^i} V^{ij} - \sum_{j \in \mathcal{N}_i^*} \xi_i w_{ij} \left(v^i - v^j \right) \\ &\quad - h_s^i m_i \left(v^i - v^0(t) \right) + g_i m_i a_0(t), \end{aligned}$$

where $\mathcal{N}_i^* \triangleq \{j \mid w_{ij} > 0\}$. By a similar analysis, we get

$$\dot{\tilde{J}} \subset -\frac{1}{2}\overline{co}\left\{e_v^T \left[(\Lambda_\sigma L_\sigma + L_\sigma^T \Lambda_\sigma + 2H_s) \otimes I_n\right] e_v\right\},$$

where $\Lambda_\sigma = \mathrm{diag}(\xi_1,\ldots,\xi_N)_\sigma \in \mathbf{R}^{N\times N}$. It is easy to see that $\Lambda_\sigma L_\sigma + L_\sigma^T \Lambda_\sigma$ is symmetric and has the properties that each row sum is equal to 0, the diagonal entries are positive, and all the other entries are nonpositive. The rest analysis is similar to Theorem 2, and thus is omitted.

In what follows, we will analyze the motion of system (1) in the case where $h_s^i = h_0$ for all $i \in \mathcal{I}$, where $h_0 > 0$ is a constant. This means that the intensities of influence of the external signal on all agents are equal. Hence, the control law in (3) has the following form:

$$u^i = -\sum_{j\in\mathcal{N}_i} \nabla_{x^i} V^{ij} - \sum_{j\in\mathcal{N}_i} w_{ij}\left(v^i - v^j\right)$$
$$-h_0 m_i\left(v^i - v^0(t)\right) + m_i a_0(t). \tag{6}$$

Certainly, in this case, we also have the conclusions in Theorems 1 and 2. Here we will choose another moving reference frame to study the flocking problem.

The position vector of the CoM in system (1) is defined as

$$x^* = \frac{\sum_{i=1}^N m_i x^i}{\sum_{i=1}^N m_i}.$$

Thus, the velocity vector of the CoM is

$$v^* = \frac{\sum_{i=1}^N m_i v^i}{\sum_{i=1}^N m_i}.$$

On using control law (6) and by the symmetry of W_σ and the symmetry of function V^{ij} with respect to x^{ij}, we get

$$\dot{v}^* = -h_0 v^* + h_0 v^0(t) + a_0(t). \tag{7}$$

By solving (7), we get

$$v^*(t) = v^0(t) + \left(v^*(t_0) - v^0(t_0)\right) e^{-h_0(t-t_0)}.$$

Thus, it follows that, if $v^*(t_0) = v^0(t_0)$, then the velocity of the CoM equals $v^0(t)$ for all time; if $v^*(t_0) \neq v^0(t_0)$, then the velocity of the CoM exponentially converges to the desired velocity $v^0(t)$ with a time constant of h_0 s. Moreover, since $\dot{x}^* = v^*$, we have

$$x^*(t) = x^*(t_0) + \int_{t_0}^t v^0(\tau)d\tau + \frac{v^*(t_0)-v^0(t_0)}{h_0}\left[1 - e^{-h_0(t-t_0)}\right].$$

We define the error vectors:

$$e_p^* = x^* - \int_{t_0}^t v^0(\tau)d\tau \quad \text{and} \quad e_v^* = v^* - v^0(t).$$

Then e_p^* represents the position difference vector between the CoM and the virtual leader, whereas e_v^* represents the velocity difference vector between them. By the calculation above, it is easy to see that

$$\lim_{t\to\infty} e_p^* = x^*(t_0) + \frac{v^*(t_0) - v^0(t_0)}{h_0}.$$

Thus, it follows that, if $v^*(t_0) = v^0(t_0)$, then the difference of the positions between the CoM and the virtual leader equals $x^*(t_0)$ for all time; if $v^*(t_0) \neq v^0(t_0)$, then the difference exponentially approaches the constant vector $x^*(t_0) + \frac{v^*(t_0)-v^0(t_0)}{h_0}$ with a time constant of h_0 s.

Therefore, from the analysis above, we have the following theorem.

Theorem 3: Taking the control law in (6), if the initial velocity of the CoM is equal to the desired initial velocity, then the velocity of the CoM equals the desired velocity for all time and the position difference between the CoM and the virtual leader always equals $x^*(t_0)$; otherwise the velocity of the CoM will exponentially converge to the desired velocity with a time constant of h_0 s and the position difference between the CoM and the virtual leader will exponentially approach the constant vector $x^*(t_0) + \frac{v^*(t_0)-v^0(t_0)}{h_0}$.

In this case, we can also choose the moving reference frame proposed in [1] to analyze the stability of system (1), and take the position of the CoM of the group as the origin. We define the error vectors:

$$\varepsilon_p^i = x^i - x^* \quad \text{and} \quad \varepsilon_v^i = v^i - v^*.$$

Then ε_v^i represents the velocity difference vector between agent i and the CoM. It is easy to see that $\dot{\varepsilon}_p^i = \varepsilon_v^i$ and $\dot{\varepsilon}_v^i = \dot{v}^i - \dot{v}^*$. Hence, the error dynamics is given by

$$\dot{\varepsilon}_p^i = \varepsilon_v^i,$$
$$m_i \dot{\varepsilon}_v^i = u^i - m_i \dot{v}^*, \quad i = 1,\ldots,N. \tag{8}$$

By the definition of V^{ij}, it follows that $V^{ij}(\|x^{ij}\|) = V^{ij}(\|\varepsilon_p^{ij}\|) \triangleq \overline{V}^{ij}$, where $\varepsilon_p^{ij} = \varepsilon_p^i - \varepsilon_p^j$, and hence $\overline{V}^i = V^i$ and $\nabla_{\varepsilon_p^i} \overline{V}^{ij} = \nabla_{x^i} V^{ij}$. Thus the control input u^i for agent i in the error system (8) has the following form:

$$u^i = -\sum_{j\in\mathcal{N}_i} \nabla_{\varepsilon_p^i} \overline{V}^{ij} - \sum_{j\in\mathcal{N}_i} w_{ij}\left(\varepsilon_v^i - \varepsilon_v^j\right)$$
$$-h_0 m_i \varepsilon_v^i - h_0 m_i e_v^* + m_i a_0(t).$$

We consider the error system (8) and choose the following Lyapunov function:

$$J = \frac{1}{2}\sum_{i=1}^N \left(\overline{V}^i + m_i \varepsilon_v^{iT} \varepsilon_v^i\right). \tag{9}$$

By a similar calculation, we get

$$\dot{\tilde{J}} \subset -\overline{co}\left\{\varepsilon_v^T \left[(L_\sigma + H_0) \otimes I_n\right] \varepsilon_v\right\},$$

where $H_0 = \mathrm{diag}(h_0 m_1,\ldots,h_0 m_N) \in \mathbf{R}^{N\times N}$ and $\varepsilon_v = \left(\varepsilon_v^{1T},\ldots,\varepsilon_v^{NT}\right)^T$. Using the analysis method in Theorem 2, we obtain that the velocities of all agents asymptotically approach the velocity of the CoM, avoidance of collisions between the agents is ensured, and the group final configuration minimizes all agent potentials. Furthermore, from Theorem 3, we conclude that the velocities of all agents in group (1) asymptotically approach the desired velocity.

Remark 3: One issue to be mentioned here is that, when the intensities of influence of the external signal on the

motions of all agents are not equal, it is difficult to estimate the motion of the CoM and analyze the stability properties of system (1) by using the second moving reference frame.

Remark 4: By the analysis above, it is easy to see that, when all agents know the desired velocity $v^0(t)$, the desired flocking motion can still be obtained though all agents do not regulate their velocities according to their neighboring agents, i.e., we can omit the term β^i in (2).

B. All agents can obtain the desired velocity, but they all do not know the acceleration input $a_0(t)$, i.e., $g_i = 0$ for all $i \in \mathcal{I}$. Here we still assume that the coefficients $h_s^i = h_0 > 0$ for all $i \in \mathcal{I}$.

In this case, the control law acting on agent i is

$$u^i = -\sum_{j \in \mathcal{N}_i} \nabla_{x^i} V^{ij} - \sum_{j \in \mathcal{N}_i} w_{ij} \left(v^i - v^j \right) \\ -h_0 m_i \left(v^i - v^0(t) \right), \qquad (10)$$

where $v^0(t)$, h_0, and w_{ij} are defined as before.

Theorem 4: Taking the control law in (10), all agent velocities in the group described in (1) become asymptotically the same, avoidance of collisions between the agents is ensured, and the group final configuration minimizes all agent potentials.

This theorem becomes true after Theorem 5 is proved. First, on using control law (10), we get

$$\dot{v}^* = -h_0 v^* + h_0 v^0(t). \qquad (11)$$

We consider the error dynamics (8). The control input u^i for agent i in the error system has the following form:

$$u^i = -\sum_{j \in \mathcal{N}_i} \nabla_{\varepsilon_p^i} \overline{V}^{ij} - \sum_{j \in \mathcal{N}_i} w_{ij} \left(\varepsilon_v^i - \varepsilon_v^j \right) \\ -h_0 m_i \varepsilon_v^i - h_0 m_i e_v^*. \qquad (12)$$

Theorem 5: Taking the control law in (12), all agent velocities in the error system (8) asymptotically approach zero, avoidance of collisions between the agents is ensured, and the group final configuration minimizes all agent potentials.

Choosing the Lyapunov function J defined as in (9) and calculating the generalized time derivative of J along the solution of the error system (8), we have $\dot{\tilde{J}} \subset -\overline{co} \left\{ \varepsilon_v^T \left[(L_\sigma + H_0) \otimes I_n \right] \varepsilon_v \right\}$. Following the analysis method in Theorem 2, we can obtain the proof of Theorem 5. Due to space limitation, we omit the detailed proof.

Theorem 5 implies that all agent velocities in group (1) asymptotically approach the velocity of the CoM by using control law (10). But in what follows, we will show that in this case the final velocity of the group may not asymptotically approach the desired velocity. In fact, for some cases, all agents can track the external signal, but for others, they cannot. We will present two simple examples, which is enough to illustrate the problem. By solving (11), we have

$$v^*(t) = v^*(t_0)e^{-h_0(t-t_0)} + h_0 \int_{t_0}^t e^{-h_0(t-\tau)} v^0(\tau) d\tau,$$

and moreover, we obtain that

$$x^*(t) = x^*(t_0) + \frac{v^*(t_0)}{h_0} \left[1 - e^{-h_0(t-t_0)} \right] \\ + h_0 \int_{t_0}^t \int_{t_0}^s e^{-h_0(s-\tau)} v^0(\tau) d\tau ds.$$

Example 1: Suppose the desired velocity $v^0(t)$ be a constant vector v_0, then we get

$$v^*(t) = v_0 + \left(v^*(t_0) - v_0 \right) e^{-h_0(t-t_0)}.$$

It is obvious that the velocity of the CoM equals the desired velocity for all time or it will exponentially converge to it with a time constant of h_0 s. Furthermore, by Theorem 5, we obtain that the velocities of all agents asymptotically approach the desired velocity. Moreover, we have

$$x^*(t) = x^*(t_0) + v_0(t - t_0) + \frac{v^*(t_0) - v_0}{h_0} \left[1 - e^{-h_0(t-t_0)} \right],$$

hence,

$$\lim_{t \to \infty} e_p^* = x^*(t_0) + \frac{v^*(t_0) - v_0}{h_0}.$$

This implies that the position difference between the CoM and the virtual leader will asymptotically approach a constant vector. By the analysis above, we know that, when the desired velocity is a constant vector, the desired stable flocking motion can be obtained by using control law (10). More information can be found in [5]–[6].

Example 2: Suppose $n = 1$ and $v^0(t) = \alpha t$, where α is a positive constant, then we get

$$v^*(t) = \alpha t + (v^*(t_0) - \alpha t_0)e^{-h_0(t-t_0)} - \frac{\alpha}{h_0} \left[1 - e^{-h_0(t-t_0)} \right].$$

It is easy to see that $\lim_{t \to \infty} e_v^* = -\frac{\alpha}{h_0}$. Moreover, we have

$$x^*(t) = x^*(t_0) + \frac{\alpha}{2} \left(t^2 - t_0^2 \right) - \frac{\alpha}{h_0}(t - t_0) \\ + \left(\frac{v^*(t_0) - \alpha t_0}{h_0} + \frac{\alpha}{h_0^2} \right) \left[1 - e^{-h_0(t-t_0)} \right],$$

thus $\lim_{t \to \infty} e_p^* = -\infty$. This implies that the velocity of the CoM cannot asymptotically approach the desired velocity, i.e., the CoM cannot track the external signal. Hence, in this case, the desired stable flocking motion cannot be obtained by using the control law in (10).

In what follows, we will demonstrate that in this case the desired flocking motion still may not be achieved even when the position information of the virtual leader is considered in the design of the control law. The initial position of the virtual leader is still chosen as the origin, then its position vector is $x^0(t) = \int_{t_0}^t v^0(\tau) d\tau$. We modify the control law u^i in (10) to

$$u^i = -\sum_{j \in \mathcal{N}_i} \nabla_{x^i} V^{ij} - \sum_{j \in \mathcal{N}_i} w_{ij} \left(v^i - v^j \right) \\ -h_0 m_i \left(v^i - v^0(t) \right) - r_0 m_i \left(x^i - x^0(t) \right), (13)$$

where $r_0 > 0$ is a constant. By a similar calculation, we get

$$\dot{v}^* = -h_0 v^* + h_0 v^0(t) - r_0 x^* + r_0 x^0(t).$$

We consider the error system (8) and choose the following Lyapunov function:

$$J = \frac{1}{2} \sum_{i=1}^{N} \left(\overline{V}^i + m_i \varepsilon_v^{iT} \varepsilon_v^i + r_0 m_i \varepsilon_p^{iT} \varepsilon_p^i \right).$$

Then we have $\dot{\tilde{J}} \subset -\overline{co} \left\{ \varepsilon_v^T \left[(L_\sigma + H_0) \otimes I_n \right] \varepsilon_v \right\}$. The rest of the analysis is similar, and thus is omitted. Hence, on using the control law in (13), the velocities of all agents in group (1) still asymptotically approach the velocity of the CoM.

Next, we analyze the motion of the CoM. By the calculation above, we have

$$\begin{bmatrix} \dot{x}^* \\ \dot{v}^* \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -r_0 I_n & -h_0 I_n \end{bmatrix} \begin{bmatrix} x^* \\ v^* \end{bmatrix} + \begin{bmatrix} 0 \\ h_0 v^0 + r_0 x^0 \end{bmatrix},$$
(14)

where $v^0 \triangleq v^0(t)$ and $x^0 \triangleq x^0(t)$. Let

$$A = \begin{bmatrix} 0 & I_n \\ -r_0 I_n & -h_0 I_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -r_0 & -h_0 \end{bmatrix} \otimes I_n.$$

In the following, we will present an example to illustrate the fact that the velocity of the CoM may not asymptotically approach the desired value $v^0(t)$ by using control law (13).

Example 3: Suppose $t_0 = 0$, $n = 1$, and $v^0(t) = e^{\alpha h_0 t}$, then $x^0(t) = \frac{1}{\alpha h_0}(e^{\alpha h_0 t} - 1)$, where α is a positive constant. Thus the eigenvalues of matrix A is

$$\lambda_1 = \frac{-h_0 + \sqrt{h_0^2 - 4r_0}}{2} \quad \text{and} \quad \lambda_2 = \frac{-h_0 - \sqrt{h_0^2 - 4r_0}}{2},$$

and they all are negative or have the negative real parts.

(i) If $h_0^2 - 4r_0 \neq 0$, then $\lambda_1 \neq \lambda_2$ and the eigenvectors associated with them are $(1, \lambda_1)^T$ and $(1, \lambda_2)^T$, respectively. Let

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix},$$

then,

$$P^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}.$$

Thus $A = P\text{diag}(\lambda_1, \lambda_2)P^{-1}$. By solving (14), we obtain

$$\begin{bmatrix} x^* \\ v^* \end{bmatrix} = \begin{bmatrix} \frac{\alpha h_0^2 + r_0}{\alpha h_0 [(\alpha^2 + \alpha)h_0^2 + r_0]} e^{\alpha h_0 t} - \frac{1}{\alpha h_0} + (*)_1 \\ \frac{\alpha h_0^2 + r_0}{(\alpha^2 + \alpha)h_0^2 + r_0} e^{\alpha h_0 t} + (*)_2 \end{bmatrix},$$

where $\lim_{t \to \infty}(*)_1 = 0$ and $\lim_{t \to \infty}(*)_2 = 0$. Hence, we get

$$\lim_{t \to \infty} e_v^* = \lim_{t \to \infty} \left[\frac{-\alpha^2 h_0^2}{(\alpha^2 + \alpha)h_0^2 + r_0} e^{\alpha h_0 t} + (*)_2 \right] = -\infty,$$

and

$$\lim_{t \to \infty} e_p^* = \lim_{t \to \infty} \left[\frac{-\alpha h_0}{(\alpha^2 + \alpha)h_0^2 + r_0} e^{\alpha h_0 t} + (*)_1 \right] = -\infty.$$

(ii) If $h_0^2 - 4r_0 = 0$, then $\lambda_1 = \lambda_2 = -\frac{h_0}{2}$ and the eigenvectors associated with them are $(1, \lambda_1)^T$ and $(1, 1 + \lambda_1)^T$. Let

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & 1 + \lambda_1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix},$$

then $P^{-1} = \begin{bmatrix} 1 + \lambda_1 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$ and $A = PJP^{-1}$. By solving (14), we have

$$\begin{bmatrix} x^* \\ v^* \end{bmatrix} = \begin{bmatrix} \frac{4\alpha + 1}{\alpha h_0 (2\alpha + 1)^2} e^{\alpha h_0 t} - \frac{1}{\alpha h_0} + (*)_3 \\ \frac{4\alpha + 1}{(2\alpha + 1)^2} e^{\alpha h_0 t} + (*)_4 \end{bmatrix},$$

where $\lim_{t \to \infty}(*)_3 = 0$ and $\lim_{t \to \infty}(*)_4 = 0$. Hence, we get

$$\lim_{t \to \infty} e_v^* = \lim_{t \to \infty} \left[\frac{-4\alpha^2}{(2\alpha + 1)^2} e^{\alpha h_0 t} + (*)_4 \right] = -\infty,$$

and

$$\lim_{t \to \infty} e_p^* = \lim_{t \to \infty} \left[\frac{-4\alpha}{h_0 (2\alpha + 1)^2} e^{\alpha h_0 t} + (*)_3 \right] = -\infty.$$

From the analysis above, we conclude that the CoM cannot track the virtual leader and thus the desired flocking motion cannot be achieved by using the control law in (13).

Remark 5: All the results in this paper can be analogously extended to the case where there is velocity damping in the environment. We can use the analysis method in [5]–[6] to modify the corresponding control laws.

IV. NUMERICAL SIMULATIONS

In this section, we will present some numerical simulations for system (1) in order to illustrate the theoretical results obtained in the previous sections.

These simulations are performed with ten agents, labelled with circles, moving in the plane, whose initial positions, velocities and neighboring relations are set randomly, but which satisfy: (1) all initial positions are set within a circle of radius of $R^* = 10$ m centered at the origin; (2) all initial velocities are set with arbitrary directions and magnitudes within the range of $[0, 4]$ m/s; and (3) the initial neighboring graph is connected. All agents have different masses and they are set randomly in the range of $(0,1]$ kg. Suppose the desired velocity $v^0(t) = [\sin(t), \cos(t)]^T$ and the initial time $t_0 = 0$ s. We run all simulations for 100 s and choose suitable coordinate axes to show our simulation results.

Fig. 1 depicts the curves of the desired velocity along x-axis and y-axis. Figs. 3–7 show the simulation results for the same group, and the group has the same initial state shown in Fig. 2 where the solid lines represent the neighboring relations between agents and the dotted arrows represent the initial velocities of all agents. However, different control laws are taken in the form of (3) (in Fig. 3), (6) (in Fig. 4), (10) (in Fig. 5), or (13) (in Fig. 6) with the explicit potential function

$$V^{ij}(\|x^{ij}\|) = \begin{cases} 0.05 \ln \|x^{ij}\|^2 + \frac{0.05}{\|x^{ij}\|^2}, & 0 < \|x^{ij}\| \leq R, \\ 0.05 \ln R^2 + \frac{0.05}{R^2}, & \|x^{ij}\| > R. \end{cases}$$

The agent's sensing range is chosen as $R = 4$ m. In Figs. 3–7, h_i is generated randomly such that $0 \leq h_i \leq 1$ such that there exists at least one nonzero constant. Take $r_0 = 1$ in Fig. 6. The interaction coefficient w_{ij} equals c_{ij} if agent j is a neighbor of agent i and is 0 otherwise, where the coefficient c_{ij} is generated randomly such that $0 < c_{ij} = c_{ji} < 1$ and $c_{ii} = 0$ for all $i, j = 1, \ldots, 10$. Fig. 3 presents the simulation

results for the case where not all agents know the desired velocity $v^0(t)$ but they all know the acceleration input $a_0(t)$, and it explicitly demonstrates that the desired stable flocking motion can be obtained though the neighboring graph varies with time. When all agents know the desired velocity and its acceleration input, they can still eventually move at the desired velocity though the neighboring graph is not always connected in the course of motion, as shown in Fig. 4. Figs. 5 and 6 show the simulation results in the case where all agents do not know the acceleration input. It is easy to see from them that the desired stable flocking motion cannot be achieved by using the control laws in (10) and (13). Hence, it is difficult for all agents to track a variable velocity $v^0(t)$ in the case where they do not know its acceleration input $a_0(t)$. Fig. 7 depicts the curves of the velocity errors between the agents and the CoM along x-axis and y-axis in the simulations shown in Figs. 5 and 6, respectively, and from it, it is easy to see that the flocking motion can be obtained and the velocities of all agents converge to the velocity of the CoM.
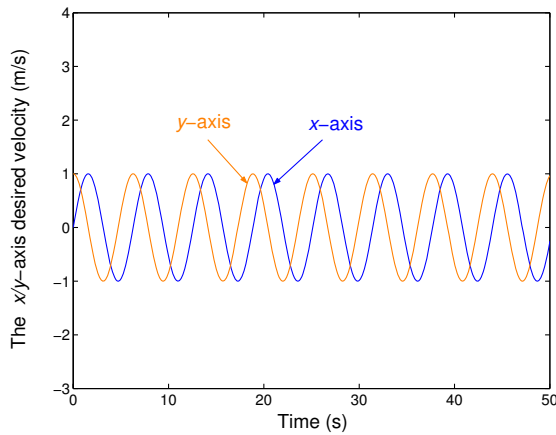

Fig. 3. (a) and (b) depict the curves of the velocity errors of all agents along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the final group configuration and all agents' velocities at $t = 100$s.

Fig. 4. (a) and (b) depict the curves of the velocity errors of all agents along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the final group configuration and all agents' velocities at $t = 100$s.
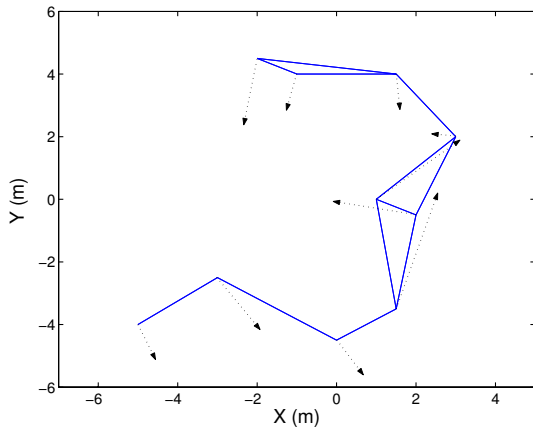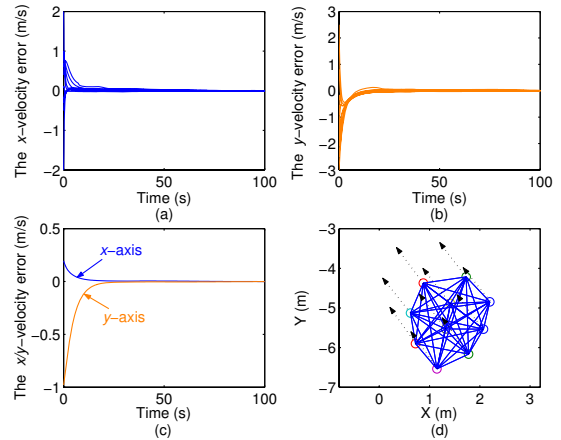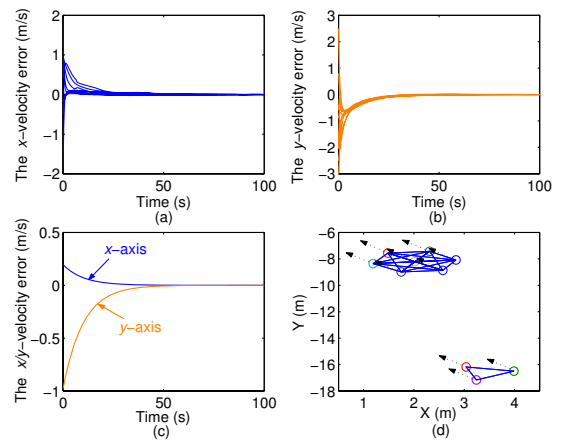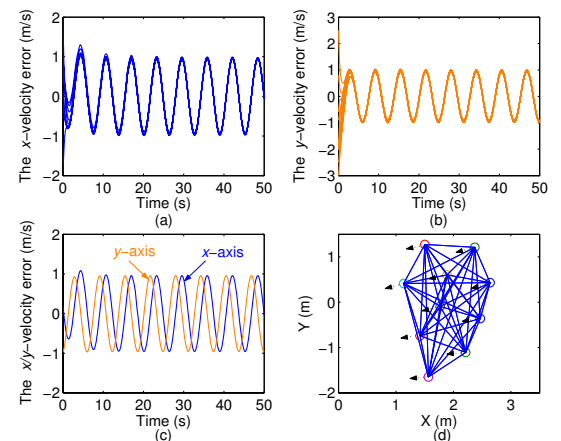

Fig. 1. The desired velocity curves

Fig. 2. The initial state of the group

Fig. 5. (a) and (b) depict the curves of the velocity errors of all agents along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the final group configuration and all agents' velocities at $t = 100$s.
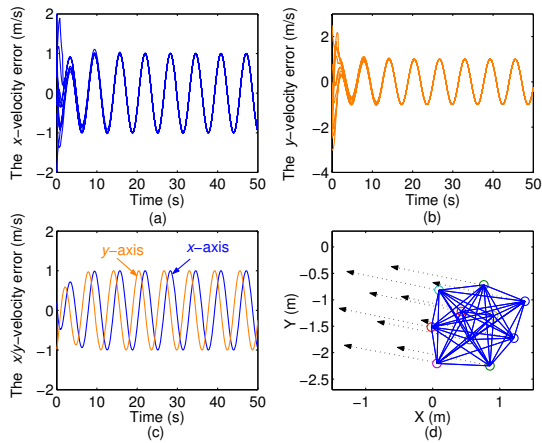
Fig. 6.   (a) and (b) depict the curves of the velocity errors of all agents along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the final group configuration and all agents' velocities at $t = 100$s.
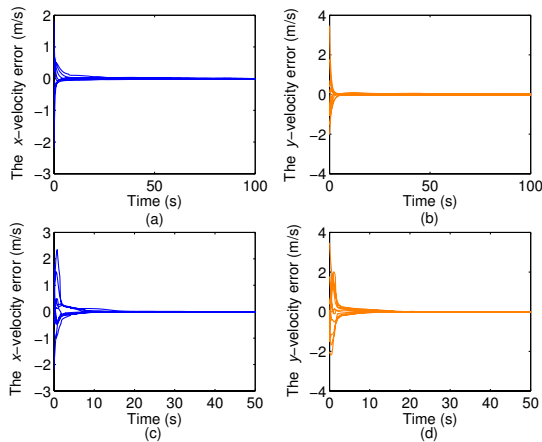

Fig. 7.   (a)–(b) and (c)–(d) depict the curves of the velocity errors between the agents and the CoM along x-axis and y-axis by using control laws (10) and (13), respectively.

V. CONCLUSIONS

This paper studied the flocking problem of a group of agents moving in an n-dimensional Euclidean space with a dynamic virtual leader. To solve the problem, we proposed a set of switching control laws, and the control law acting on each agent relies on the state information of its neighbors and the external signal. We proved that, in the case where the acceleration input of the virtual leader is known, all agents can follow the virtual leader, freedom from collisions between the agents is ensured, the final tight formation minimizes all agents potentials, and moreover, the velocity of the CoM equals the desired velocity for all time or it will exponentially converge to the desired velocity; in the case where the acceleration input is unknown, the velocities of all agents asymptotically approach the velocity of the CoM, however in this case, the final velocity of the group may not be equal to the desired value. Numerical simulation agrees very well with the theoretical analysis.

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