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Efficient Scheduling Focusing on the Duality of MPL Representation

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***Abstract*—A max-plus linear (MPL) representation for describing the behavior of a repetitious execution system with a MIMO-FIFO structure is proposed. A conventional MPL form is required to recalculate the representation matrices by each job when applied to systems whose processing times differ by each job. Approximately twice the calculation volume is required to obtain the earliest and latest times using conventional MPL compared with our proposed MPL representation. This work assigns the state variables to events other than conventional ones, reduces the number of independent system parameters in representation matrices, and improves the form to schedule efficiently, even when applied to systems whose processing times differ by each job. The derived equations are similar to the dual system in modern control theory, which means that the calculation load for scheduling can be reduced remarkably comparing with the conventional method.**

I. Introduction

THIS research considers a scheduling method for repetitious execution systems with MIMO (Multiple Inputs and Multiple Outputs) -FIFO structure. This kind of system can also be understood as an extended class of project scheduling problems based on PERT [1], which is improved for MIMO type and repetitive systems. This class has the following features and constraints.

- Parallel execution: Multiple processes can work simultaneously.
- Synchronization: The subsequent processes cannot start until a certain process has been completed.
- No-concurrency: While a certain resource is in use, the subsequent job cannot start.

The behavior of systems with the above features can also be described with a subclass of Petri net called the TEG (Timed Event Graph) [2] in which all places have single upstream and downstream transitions. Moreover, as an independent variable it is favorable to adopt the number of event occurrences, called the event counter, rather than 'time'.

It is also known that the behavior of TEGs can be expressed in MPL (Max-Plus Linear) form as a set of linear equations in max-plus algebra [3], [4]. The times of event changes, especially the earliest starting times or the output times, can be calculated by combinations of 'max' and '+' operations. Thus, many studies on scheduling problems utilizing the MPL representation are performed [5], [6]. On the other hand, 'min' and '-' operators are used to calculate the latest starting times in internal processes. This formulation is similar to one for earliest starting times except the difference of operators [4]. In project management, production scheduling, or transportation systems, it is important to grasp the location of bottlenecks or floats in the respective processes. Hence, calculation of the earliest and the latest times is required.

However, if we use the MPL description from previous research, when the execution times of processes vary by each job, all representation matrices should be recalculated accordingly. When calculating the latest times, a comparable calculation load is added. This indicates that the simplicity and the similarity of the state equations are not utilized effectively and it is not advantageous in terms of calculation load.

This paper derives a simpler state-space representation in MPL form with fewer numbers of independent system parameters by improving the assignment of the state variables. The representation matrices are decomposed into two parts; one dependent on the system structure and the other on the execution times of processes. This method leads to a reduction of the calculation load even when applied to systems whose processing times vary by each job. In addition, a similarity between the two MPL representations of forward and backward types is examined. The former calculates the earliest times and the latter for the latest times, and they are similar to the relationship of dual systems in modern control theory [7]. With the help of this relationship, the calculation volume for obtaining the representation matrices is remarkable reduced.

Note that in the following discussions, assembly production systems are mainly used, however the principal ideas can also be applied to batch processing systems [8], transportation systems [9] [10], and project scheduling [11], etc.

II. Conventional MPL Representation

This section reviews briefly the basic operation rules of

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max-plus algebra and explains a method for deriving the conventional MPL state-space representation utilizing an example of a simple production system.

A. Basic Operators

Max-plus algebra is an algebraic system in which the max and the + operations are defined as addition and multiplication, respectively. Denoting the real field by $\boldsymbol{R}$, if $x, y \in \mathfrak{D}$, $\mathfrak{D} = \boldsymbol{R} \cup \{-\infty\}$, then

$$x \oplus y = \max(x, y)\,,\ x \otimes y = x + y$$

where the symbol for multiplication $\otimes$ is suppressed when no confusion is likely to arise. Denoting the unit elements for these operators by $\varepsilon\,(= -\infty)$ and $e\,(=0)$ then the following two operators are defined additionally.

$$x \wedge y = \min(x, y)\,,\ x \setminus y = -x + y$$

Operators for multiple numbers are as follows. If $m \le n$,

$$\bigoplus_{k=m}^{n} x_k = \max(x_m, x_{m+1}, \cdots, x_n)\quad,\quad \bigwedge_{k=m}^{n} x_k = \min(x_m, x_{m+1}, \cdots, x_n)$$

Statements for matrices are as follows. For $\boldsymbol{X} \in \mathfrak{D}^{m\times n}$, $[\boldsymbol{X}]_{ij}$ represents the (i, j)-th element of $\boldsymbol{X}$ and $\boldsymbol{X}^T$ is the transpose matrix. For $\boldsymbol{X} \in \mathfrak{D}^{m\times l}$, $\boldsymbol{Y} \in \mathfrak{D}^{l\times p}$,

$$[\boldsymbol{X}\otimes\boldsymbol{Y}]_{ij} = \bigoplus_{k=1}^{l}\left([\boldsymbol{X}]_{ik} \otimes [\boldsymbol{Y}]_{kj}\right) = \max_{k=1,\cdots,l}([\boldsymbol{X}]_{ik} + [\boldsymbol{Y}]_{kj})$$

$$[\boldsymbol{X}\odot\boldsymbol{Y}]_{ij} = \bigwedge_{k=1}^{l}\left([\boldsymbol{X}]_{ik} \setminus [\boldsymbol{Y}]_{kj}\right) = \min_{k=1,\cdots,l}(-[\boldsymbol{X}]_{ik} + [\boldsymbol{Y}]_{kj})$$

Suppose the priorities of the operators $\otimes$, $\setminus$, and $\odot$ are higher than $\oplus$ and $\wedge$, they would then hold the following properties. If $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{D}^{l\times m}$, $\boldsymbol{Z} \in \mathfrak{D}^{n\times l}$, $\boldsymbol{v}, \boldsymbol{w} \in \mathfrak{D}^{m}$,

$$(\boldsymbol{X} \oplus \boldsymbol{Y})\odot\boldsymbol{v} = (\boldsymbol{X}\odot\boldsymbol{v}) \wedge (\boldsymbol{Y}\odot\boldsymbol{v}) \tag{1}$$

$$\boldsymbol{X}\odot(\boldsymbol{v} \wedge \boldsymbol{w}) = (\boldsymbol{X}\odot\boldsymbol{v}) \wedge (\boldsymbol{X}\odot\boldsymbol{w}) \tag{2}$$

$$\boldsymbol{Y}^T \odot(\boldsymbol{Z}^T \odot\boldsymbol{v}) = (\boldsymbol{Z}\boldsymbol{Y})^T \odot\boldsymbol{v} \tag{3}$$

Unit matrices are denoted as follows.

$\boldsymbol{\varepsilon}_{mn}$: All elements are ε and its size is $m \times n$
$\boldsymbol{e}_{m}$: Only diagonal elements are e, all off-diagonal elements are ε, and its size is $m \times m$

The two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathfrak{D}^{m}$, if

$$[\boldsymbol{v}]_i \le [\boldsymbol{w}]_i \ \text{ for all } i\ (1 \le i \le m)$$

is satisfied are denoted simply by $\boldsymbol{v} \le \boldsymbol{w}$.

B. State-Space Representation in MPL form

In conventional studies for describing the behavior of systems in MPL form, the state variables are assigned to the starting times for manufacturing processes. Consider a simple production system shown in Fig. 1 as an illustrative example. Processes 1 and 2 receive materials from external inputs 1 and 2, respectively, manufactures them and sends the resulting parts to process 3. Process 3 manufactures the received part and sends the finished product to the external output. Suppose the same processing operations are repeated multiple times and the following constraints are imposed to the system.

- Jobs cannot be started until the previous job is finished.
- Processes 1 and 2 cannot begin until they have received the corresponding materials from inputs 1 and 2.
- Process 3 cannot start processing until it has received the manufactured part from processes 1 and 2.
- The manufacturing starts as soon as all the above conditions are satisfied.

For the k-th job, the starting times for manufacturing in processes 1-3 are denoted by $x_i(k)\ \ (1 \le i \le 3)$ and those for the manufacturing times by $d_i(k)\,(\ge 0)$. Moreover, the output time of the finished product is represented by $y(k)$. We then find the following relations between the relevant variables where the multiplication symbol $\otimes$ is suppressed

$$\begin{aligned} x_1(k) &= \max\{x_1(k-1) + d_1(k-1), u_1(k)\} \\ &= d_1(k-1)\,x_1(k-1) \oplus u_1(k) \end{aligned} \tag{4}$$

$$\begin{aligned} x_2(k) &= \max\{x_2(k-1) + d_2(k-1), u_2(k)\} \\ &= d_2(k-1)\,x_2(k-1) \oplus u_2(k) \end{aligned} \tag{5}$$

$$\begin{aligned} x_3(k) &= \max\{x_3(k-1) + d_3(k-1), x_1(k) + d_1(k), \\ &\qquad x_2(k) + d_2(k)\} \\ &= d_3(k-1)\,x_3(k-1) \oplus d_1(k)\,x_1(k) \\ &\quad \oplus d_2(k)\,x_2(k) \end{aligned} \tag{6}$$

$$y(k) = x_3(k) + d_3(k) = d_3(k)\,x_3(k) \tag{7}$$

and k is the event counter. By substituting (4), (5) for (6), (4)-(7) can be summarized in the matrix forms shown below which are similar to the state-space representation in modern control theory [2].

$$\boldsymbol{x}_E(k) = \boldsymbol{A}\boldsymbol{x}(k-1) \oplus \boldsymbol{B}\boldsymbol{u}(k) \tag{8}$$

$$\boldsymbol{y}_E(k) = \boldsymbol{C}\boldsymbol{x}(k) \tag{9}$$

where

$$\boldsymbol{A} = \begin{bmatrix} d_1(k-1) & \varepsilon & \varepsilon \\ \varepsilon & d_2(k-1) & \varepsilon \\ d_1(k-1)d_1(k) & d_2(k-1)d_2(k) & d_3(k-1) \end{bmatrix}, \tag{10}$$

$$\boldsymbol{B} = \begin{bmatrix} e & \varepsilon \\ \varepsilon & e \\ d_1(k) & d_2(k) \end{bmatrix},\ \boldsymbol{C} = \begin{bmatrix} \varepsilon \\ \varepsilon \\ d_3(k) \end{bmatrix}^T \tag{11}$$

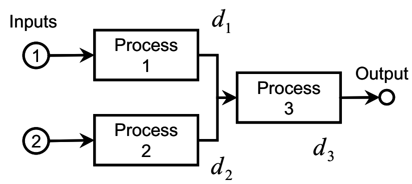

Fig. 1. A simple manufacturing system with two-inputs and one-output.

The subscripts E in (8) and (9) represent the earliest times. The representation matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ include the system parameters. Matrix $\boldsymbol{A}$ includes the parameters for the event counters $k-1$ and k. This induces a disadvantage on the calculation load. For example, the number of unknown parameters grows large when adjusting the system parameters for obtaining desirable outputs [11].

Moreover, the representation matrices (10) and (11) depend on both the precedence constraints and the manufacturing times in the respective processes. Thus, it is difficult to grasp the effects on system matrices caused solely by changes of the system parameters where the manufacturing times vary by each job. Reference [11] utilized the idea of an incidence matrix and decomposed the system matrices to the precedence constraints, locations of inputs and outputs. However, since the state variables are assigned in the same way as conventional methods, the matrices also include the system parameters for the event counters $k-1$ and k.

Hence, by assigning the state variables to other events, the system matrices can avoid dependence on multiple event counters and a simpler MPL state-space representation is derived.

III. Forward MPL Representation

We derive a MPL state-space representation whose system matrices are decomposed to the precedence constraints and the manufacturing times in the respective processes. The assumptions and constraints imposed to the system are as follows.

- The number of processes, external inputs, and external outputs are n, p, and q, respectively.
- Each job goes through all of the processes and each individual process is used only once per job.
- The subsequent job cannot start while the previous job is in manufacturing.
- All tasks in a preceding process must finish before the following process can begin manufacturing.
- Processes that have external outputs cannot start manufacturing until all corresponding materials are received.

A. State Equation

The state variable $[\boldsymbol{x}^-(k)]_i$ is assigned to the starting time of manufacturing in the i-th $(1 \le i \le n)$ process and $[\boldsymbol{x}^+(k)]_i$ to the completion time. The next relation holds.

$$[\boldsymbol{x}^-(k)]_i + d_i(k) = [\boldsymbol{x}^+(k)]_i \tag{12}$$

Fig. 2 depicts the relationship between the process i and its preceding processes and external inputs. Suppose $\mathcal{P}_i$ and $\mathcal{R}_i$ represents the number sets of attached external inputs and preceding processes, respectively. The input variable $[\boldsymbol{u}(k)]_j$ is assigned to the feeding time of material. The process i can begin after all of the following three conditions are satisfied; the manufacturing of the previous job is completed, all of the required parts are received from preceding processes, and all of the required materials are received from external inputs. Thus, the earliest starting time of manufacturing for the k-th job in process i can be formulated as follows.

$$[\boldsymbol{x}_E^-(k)]_i = [\boldsymbol{x}^+(k-1)]_i \oplus \bigoplus_{j\in\mathcal{R}_i}[\boldsymbol{x}_E^+(k)]_j \oplus \bigoplus_{j\in\mathcal{P}_i}[\boldsymbol{u}(k)]_j \tag{13}$$

From (12) and (13), the next relation is as follows.

$$\begin{aligned}[\boldsymbol{x}_E^+(k)]_i &= ([\boldsymbol{x}^+(k-1)]_i + d_i(k)) \\ &\oplus \bigoplus_{j\in\mathcal{R}_i}([\boldsymbol{x}_E^+(k)]_j + d_i(k)) \oplus \bigoplus_{j\in\mathcal{P}_i}([\boldsymbol{u}(k)]_j + d_i(k))\end{aligned}$$

Here we introduce matrices $\boldsymbol{P}_k \in \mathcal{D}^{n\times n}$, $\boldsymbol{F}_0 \in \mathcal{D}^{n\times n}$, and $\boldsymbol{B}_0 \in \mathcal{D}^{n\times p}$ which are defined in the following way.

$$\boldsymbol{P}_k = \mathrm{diag}[d_1(k), d_2(k), \cdots, d_n(k)]$$

$$[\boldsymbol{F}_0]_{ij} = \begin{cases} e & : \mathcal{R}_i \ne \{\phi\}\text{, if the } i\text{-th process has a preceding process } j. \\ \varepsilon & : \mathcal{R}_i = \{\phi\}\text{, if the process does not have any preceding processes.}\end{cases}$$

$$[\boldsymbol{B}_0]_{ij} = \begin{cases} e & : \mathcal{P}_i \ne \{\phi\}\text{, if the } i\text{-th process has an external input } j. \\ \varepsilon & : \mathcal{P}_i = \{\phi\}\text{, if the process does not have any external inputs.}\end{cases}$$

Accordingly, $[\boldsymbol{x}_E^+(k)]_i$ can be transformed as follows.

$$\begin{aligned}[\boldsymbol{x}_E^+(k)]_i &= \bigoplus_{j=1}^{n}([\boldsymbol{P}_k]_{ij} \otimes [\boldsymbol{x}^+(k-1)]_j) \\ &\oplus \bigoplus_{j=1}^{n}([\boldsymbol{F}_0]_{ij} \otimes [\boldsymbol{x}_E^+(k)]_j + d_i(k)) \\ &\oplus \bigoplus_{j=1}^{p}([\boldsymbol{B}_0]_{ij} \otimes [\boldsymbol{u}(k)]_j + d_i(k)) \\ &= [\boldsymbol{P}_k\boldsymbol{x}^+(k-1)]_i \\ &\quad \oplus ([\boldsymbol{F}_0\boldsymbol{x}_E^+(k)]_i + d_i(k)) \oplus ([\boldsymbol{B}_0\boldsymbol{u}(k)]_i + d_i(k)) \\ &= [\boldsymbol{P}_k\boldsymbol{x}^+(k-1)]_i \oplus [\boldsymbol{P}_k\boldsymbol{F}_0\boldsymbol{x}_E^+(k)]_i \oplus [\boldsymbol{P}_k\boldsymbol{B}_0\boldsymbol{u}(k)]\end{aligned}$$

Since this is followed for all $i\,(1 \le i \le n)$, the next relation is satisfied.

$$\boldsymbol{x}_E^+(k) = \boldsymbol{P}_k\boldsymbol{F}_0\boldsymbol{x}_E^+(k) \oplus \boldsymbol{P}_k[\boldsymbol{x}^+(k-1) \oplus \boldsymbol{B}_0\boldsymbol{u}(k)] \tag{14}$$

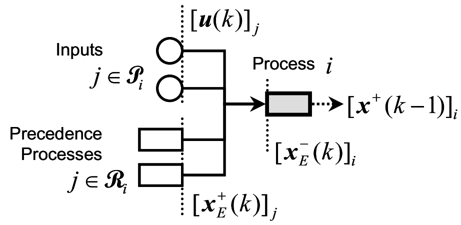

Fig. 2. External inputs and preceding processes attached to the i-th process.

In order to simplify (14), the following theorem is proved.

Theorem 1. *There exists an instance of l that satisfies*

$$(\boldsymbol{P}_k \boldsymbol{F}_0)^l = \boldsymbol{\varepsilon}_{nn},\ 1 \le l \le n$$

Proof. In [11], the existence of the following constant l is proved.

$$(\boldsymbol{F}_0 \boldsymbol{P}_k)^l = \boldsymbol{\varepsilon}_{nn},\ 1 \le l \le n$$

Thus,

$$(\boldsymbol{F}_0 \boldsymbol{P}_k)^l = \boldsymbol{F}_0 (\boldsymbol{P}_k \boldsymbol{F}_0)^{l-1} \boldsymbol{P}_k = \boldsymbol{\varepsilon}_{nn}$$

holds, and multiplying $\boldsymbol{P}_k$ on each term from the left side leads to the next relation.

$$(\boldsymbol{P}_k \boldsymbol{F}_0)^l \boldsymbol{P}_k = \boldsymbol{\varepsilon}_{nn} \tag{15}$$

The (i, j)-th element of (15) is

$$\bigoplus_{h=1}^{n} [(\boldsymbol{P}_k \boldsymbol{F}_0)^l]_{ih} \otimes [\boldsymbol{P}_k]_{hj} = [(\boldsymbol{P}_k \boldsymbol{F}_0)^l]_{ij} + d_j(k) = \varepsilon$$

Recalling $d_j(k) \ge 0$,

$$[(\boldsymbol{P}_k \boldsymbol{F}_0)^l]_{ij} = \varepsilon - d_j(k) = \varepsilon$$

This holds true for all i and j $(1 \le i, j \le n)$, which proves the proposition. □

By substituting the entire right-hand side of (14) for the first term, $\boldsymbol{x}_E^+(k)$, in the right-hand side, the following relation is obtained.

$$\begin{aligned} \boldsymbol{x}_E^+(k) &= (\boldsymbol{P}_k \boldsymbol{F}_0)^2 \boldsymbol{x}_E^+(k) \\ &\oplus (\boldsymbol{e}_n \oplus \boldsymbol{P}_k \boldsymbol{F}_0) \boldsymbol{P}_k [\boldsymbol{x}^+(k-1) \oplus \boldsymbol{B}_0 \boldsymbol{u}(k)] \end{aligned} \tag{16}$$

Similarly, substituting (14) for $\boldsymbol{x}_E^+(k)$ in the right-hand side of (16), repeating the same procedures, and utilizing Theorem 1, (14) can be transformed into

$$\boldsymbol{x}_E^+(k) = (\boldsymbol{P}_k \boldsymbol{F}_0)^* \boldsymbol{P}_k [\boldsymbol{x}^+(k-1) \oplus \boldsymbol{B}_0 \boldsymbol{u}(k)] \tag{17}$$

where

$$(\boldsymbol{P}_k \boldsymbol{F}_0)^* = \boldsymbol{e}_n \oplus \boldsymbol{P}_k \boldsymbol{F}_0 \oplus \cdots \oplus (\boldsymbol{P}_k \boldsymbol{F}_0)^{l-1}$$

Consider

$$\boldsymbol{x}^+(k-1) \oplus \boldsymbol{B}_0 \boldsymbol{u}(k) \tag{18}$$

in the right-hand side of (17). The first term represents the completion times of the $k-1$-th job and the second term is the feeding time of material in processes that have external inputs. Thus, (18) can be interpreted as the times for turning into the ready state. Note that the above two terms do not depend on the manufacturing times explicitly. On the other hand, since $\boldsymbol{F}_0$ means the precedence constraints and $\boldsymbol{P}_k$ represents the manufacturing times, $(\boldsymbol{P}_k \boldsymbol{F}_0)^* \boldsymbol{P}_k$ represents the shortest transportation times between two arbitrary processes. It should be noted that $(\boldsymbol{P}_k \boldsymbol{F}_0)^* \boldsymbol{P}_k$ depends only on the event counter k and parameters for the counter $k-1$ do not appear. As an example, consider the production system shown in Fig. 1 where the representation matrices are as follows.

$$\boldsymbol{P}_k = \begin{bmatrix} d_1(k) & \varepsilon & \varepsilon \\ \varepsilon & d_2(k) & \varepsilon \\ \varepsilon & \varepsilon & d_3(k) \end{bmatrix},\ \boldsymbol{F}_0 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ e & e & \varepsilon \end{bmatrix},\quad \boldsymbol{B}_0 = \begin{bmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \end{bmatrix}^T, \tag{19}$$

$$(\boldsymbol{P}_k \boldsymbol{F}_0)^* \boldsymbol{P}_k = \begin{bmatrix} d_1(k) & \varepsilon & \varepsilon \\ \varepsilon & d_2(k) & \varepsilon \\ d_1(k)d_3(k) & d_2(k)d_3(k) & d_3(k) \end{bmatrix} \tag{20}$$

We compare (8) and (17), the former is conventional and the latter is based on the proposed method. Equations (8), (10), and (11) include five independent system parameters, whereas only three parameters appear in (20). Moreover, the coefficient matrix $\boldsymbol{B}$ of the feeding times of material $\boldsymbol{u}(k)$ depends on the system parameters in (11) but does not depend on them in (19). Equations (8) and (17) have similar forms to each other, however the proposed representation is simpler and is advantageous especially when treated as unknown variables.

B. Output Equation

Consider an equation when the earliest output time is assigned to the output variable. Cases where the external inputs and outputs are directly connected are taken into account for the purpose of this general discussion. Fig. 3 depicts the definitions of the relevant variables. Suppose $\mathscr{T}_i$ and $\mathscr{V}_i$ state the number set of the attached preceding processes and the one for the external inputs, respectively.

The earliest output time can be formulated in the following manner.

$$[\boldsymbol{y}_E(k)]_i = \bigoplus_{j\in\mathcal{T}_i}[\boldsymbol{x}^+(k)]_j \oplus \bigoplus_{j\in\mathcal{V}_i}[\boldsymbol{u}(k)]_j$$

We introduce matrices $\boldsymbol{C}_0 \in \mathcal{D}^{q\times n}$ and $\boldsymbol{D}_0 \in \mathcal{D}^{q\times p}$ that are defined as follows.

$$[\boldsymbol{C}_0]_{ij} = \begin{cases} e & : \mathcal{T}_i \neq \{\phi\}\text{, if the } i\text{-th output has a preceding process } j. \\ \varepsilon & : \mathcal{T}_i = \{\phi\}\text{, if the output does not have any preceding processes.} \end{cases}$$

$$[\boldsymbol{D}_0]_{ij} = \begin{cases} e & : \mathcal{V}_i \neq \{\phi\}\text{, if the } i\text{-th output has an external input } j. \\ \varepsilon & : \mathcal{V}_i = \{\phi\}\text{, if the output does not have any external inputs.} \end{cases}$$

By utilizing these matrices, the output variable $[\boldsymbol{y}_E(k)]_i$ can be stated as follows.

$$\begin{aligned}&[\boldsymbol{y}_E(k)]_i \\ &= \bigoplus_{j=1}^{n}([\boldsymbol{C}_0]_{ij} \otimes [\boldsymbol{x}^+(k)]_j) \oplus \bigoplus_{j=1}^{p}([\boldsymbol{D}_0]_{ij} \otimes [\boldsymbol{u}(k)]_j) \\ &= [\boldsymbol{C}_0\boldsymbol{x}^+(k)]_i \oplus [\boldsymbol{D}_0\boldsymbol{u}(k)]_i\end{aligned}$$

This holds true for all $i\ (1 \le i \le q)$ and the summarized form is shown below.

$$\boldsymbol{y}_E(k) = \boldsymbol{C}_0\boldsymbol{x}^+(k) \oplus \boldsymbol{D}_0\boldsymbol{u}(k) \tag{21}$$

For the production system depicted in Fig. 1, the representation matrices are as follows.

$$\boldsymbol{C}_0 = [\varepsilon\ \varepsilon\ e],\ \boldsymbol{D}_0 = [\varepsilon\ \varepsilon]$$

The coefficient matrix $\boldsymbol{C}$ of the state vector $\boldsymbol{x}(k)$ in (9) includes the system parameter for the output equation. On the other hand, since it is not included in (21), it can be said that the proposed MPL representation is simpler than the corresponding conventional one.

The following two lemmas are proved in order to represent (21) in a different form.

Lemma 1. *The state vector* $\boldsymbol{x}^+(k)$ *that represents the completion of manufacturing times satisfies the following inequality.*

$$\boldsymbol{P}_k\boldsymbol{F}_0\boldsymbol{x}^+(k) \le \boldsymbol{x}^+(k) \tag{22}$$

Proof. Consider the i-th $(1 \le i \le n)$ element of $\boldsymbol{F}_0\boldsymbol{x}^+(k)$. It can be expanded as

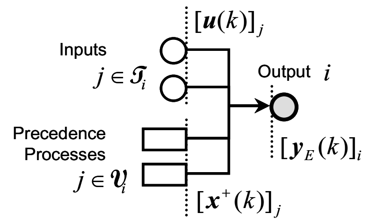

Fig. 3. External inputs and preceding processes attached to the i-th external output

$$\begin{aligned}[\boldsymbol{F}_0\boldsymbol{x}^+(k)]_i &= \bigoplus_{j=1}^{n}([\boldsymbol{F}_0]_{ij} + [\boldsymbol{x}^+(k)]_j) \\ &= \bigoplus_{j\in\mathcal{R}_i}[\boldsymbol{x}^+(k)]_j \le [\boldsymbol{x}^-(k)]_i = [\boldsymbol{x}^+(k)]_i - d_i(k)\end{aligned}$$

Hence, the next inequality holds.

$$\begin{aligned}[\boldsymbol{P}_k\boldsymbol{F}_0\boldsymbol{x}^+(k)]_i &= \bigoplus_{j=1}^{n}([\boldsymbol{P}_k]_{ij} + [\boldsymbol{F}_0\boldsymbol{x}^+(k)]_j) \\ &= d_i(k) + [\boldsymbol{F}_0\boldsymbol{x}^+(k)]_i \le [\boldsymbol{x}^+(k)]_i\end{aligned}$$

Since this relation is followed for all $i\ (1 \le i \le n)$, the proposition is proved. □

Lemma 2. *The state vector* $\boldsymbol{x}^+(k)$ *holds the next relation.*

$$\boldsymbol{x}^+(k) = (\boldsymbol{P}_k\boldsymbol{F}_0)^*\boldsymbol{x}^+(k) \tag{23}$$

Proof. Utilizing the result of Lemma 1 repeatedly,

$$(\boldsymbol{P}_k\boldsymbol{F}_0)^2\boldsymbol{x}^+(k) \le \boldsymbol{P}_k\boldsymbol{F}_0\boldsymbol{x}^+(k) \le \boldsymbol{x}^+(k) \tag{24}$$

Similarly, the following inequalities are satisfied.

$$\begin{aligned}(\boldsymbol{P}_k\boldsymbol{F}_0)^3\boldsymbol{x}^+(k) &\le \boldsymbol{x}^+(k) \\ &\vdots \\ (\boldsymbol{P}_k\boldsymbol{F}_0)^{l-1}\boldsymbol{x}^+(k) &\le \boldsymbol{x}^+(k)\end{aligned} \tag{25}$$

Utilize (22), (24), (25) and the next trivial relation.

$$\boldsymbol{x}^+(k) = \boldsymbol{x}^+(k)$$

Operating $\oplus$ on both sides induces (23). □

Utilizing Lemma 2, $\boldsymbol{y}_E(k)$ in (21) can also be expressed as follows.

$$\boldsymbol{y}_E(k) = \boldsymbol{C}_0(\boldsymbol{P}_k\boldsymbol{F}_0)^*\boldsymbol{x}^+(k) \oplus \boldsymbol{D}_0\boldsymbol{u}(k) \tag{26}$$

The coefficient matrix of the state vector $\boldsymbol{x}^+(k)$ includes $\boldsymbol{P}_k$ which depends on the manufacturing times. Thus, it is not advantageous in reducing the calculation load. However, it is closely relevant to another MPL representation for the latest starting times examined in the next section.

IV. Backward MPL Representation

This section derives a MPL representation for obtaining the latest starting times in the respective processes. It is performed by setting the finishing times of products to the input variables and feeding times of materials to the output variables. The target system is identical to the previous section and the same symbols are used.

A. State Equation

The latest starting times for manufacturing processes are considered by regarding the input variable $[\boldsymbol{u}(k)]_j$ $(1 \le j \le q)$ as the output time to the external output. Fig. 4 shows the succeeding processes and the external outputs of the process i. $\mathcal{Q}_i$ is the number set of external outputs attached and $\boldsymbol{S}_i$ represents the number set of the succeeding processes. The completion time of manufacturing in the i-th process is equal or earlier than the minimization of the following three times; the starting time of the next job, the times for manufacturing in succeeding processes, and the output times to the external outputs. Hence, the latest completion time in the i-th process can be formulated as follows.

$$[\boldsymbol{x}_L^+(k)]_i = [\boldsymbol{x}^-(k+1)]_i \wedge \bigwedge_{j\in \boldsymbol{S}_i} [\boldsymbol{x}_L^-(k)]_j \wedge \bigwedge_{j\in \mathcal{Q}_i} [\boldsymbol{u}(k)]_j \qquad (27)$$

Using (12) and (27), the next relation holds.

$$\begin{aligned}[\boldsymbol{x}_L^-(k)]_i &= ([\boldsymbol{x}^+(k+1)]_i - d_i(k)) \\ &\wedge \bigwedge_{j\in \boldsymbol{S}_i} ([\boldsymbol{x}_L^-(k)]_j - d_i(k)) \wedge \bigwedge_{j\in \mathcal{Q}_i} ([\boldsymbol{u}(k)]_j - d_i(k))\end{aligned}$$

Moreover, utilizing matrices $\boldsymbol{P}_k$, $\boldsymbol{F}_0$ and $\boldsymbol{B}_0$ introduced in the previous section transforms this in the following way.

$$\begin{aligned}[\boldsymbol{x}_L^-(k)]_i &= \bigwedge_{j=1}^{n} ([\boldsymbol{P}_k^T]_{ij} \setminus [\boldsymbol{x}^-(k+1)]_j) \\ &\quad \wedge \bigwedge_{j=1}^{n} ([\boldsymbol{F}_0^T]_{ij} \setminus [\boldsymbol{x}_L^-(k)]_j - d_i(k)) \\ &\quad \wedge \bigwedge_{j=1}^{q} ([\boldsymbol{C}_0^T]_{ij} \setminus [\boldsymbol{u}(k)]_j - d_i(k)) \\ &= [\boldsymbol{P}_k^T \odot \boldsymbol{x}^-(k+1)]_i \wedge ([\boldsymbol{F}_0^T \odot \boldsymbol{x}_L^-(k)]_i - d_i(k)) \\ &\quad \wedge ([\boldsymbol{C}_0^T \odot \boldsymbol{u}(k)]_i - d_i(k)) \\ &= [\boldsymbol{P}_k^T \odot \boldsymbol{x}^-(k+1)]_i \wedge [\boldsymbol{P}_k^T \odot [\boldsymbol{F}_0^T \odot \boldsymbol{x}_L^-(k)]]_i \\ &\quad \wedge [\boldsymbol{P}_k^T \odot [\boldsymbol{C}_0^T \odot \boldsymbol{u}(k)]]_i \\ &= [(\boldsymbol{F}_0 \boldsymbol{P}_k)^T \odot \boldsymbol{x}_L^-(k)]_i \\ &\quad \wedge [\boldsymbol{P}_k^T \odot [\boldsymbol{x}^-(k+1) \wedge \boldsymbol{C}_0^T \odot \boldsymbol{u}(k)\end{aligned}$$

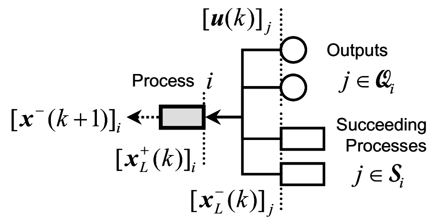

Fig. 4. External outputs and succeeding processes attached to the i-th process

Since this holds true for all $i\,(1 \le i \le n)$, the next relation is satisfied.

$$\begin{aligned}\boldsymbol{x}_L^-(k) &= (\boldsymbol{F}_0 \boldsymbol{P}_k)^T \odot \boldsymbol{x}_L^-(k) \\ &\quad \wedge \boldsymbol{P}_k^T \odot [\boldsymbol{x}^-(k+1) \wedge \boldsymbol{C}_0^T \odot \boldsymbol{u}(k)]\end{aligned} \qquad (28)$$

Substituting the right-hand side of (28) for the first term $\boldsymbol{x}_L^-(k)$ leads to the following equality.

$$\begin{aligned}\boldsymbol{x}_L^-(k) &= [(\boldsymbol{F}_0 \boldsymbol{P}_k)^2]^T \odot \boldsymbol{x}_L^-(k) \\ &\quad \wedge [(\boldsymbol{e}_n \oplus \boldsymbol{P}_k \boldsymbol{F}_0) \boldsymbol{P}_k]^T \odot [\boldsymbol{x}^-(k+1) \wedge \boldsymbol{C}_0^T \odot \boldsymbol{u}(k)]\end{aligned} \qquad (29)$$

Substitute (28) for the first term $\boldsymbol{x}_L^-(k)$ in the right-hand side of (29) again. Repeat this procedure and utilize the next theorem proved in [11].

$$(\boldsymbol{F}_0 \boldsymbol{P}_k)^l = \boldsymbol{\varepsilon}_{nn}, \; 1 \le {}^{\exists} l \le n$$

Finally, (29) can be simplified in the following way.

$$\boldsymbol{x}_L^-(k) = [(\boldsymbol{P}_k \boldsymbol{F}_0)^* \boldsymbol{P}_k]^T \odot [\boldsymbol{x}^-(k+1) \wedge \boldsymbol{C}_0^T \odot \boldsymbol{u}(k)] \qquad (30)$$

B. Output Equation

We consider an output equation when the latest feeding times are assigned to output variables. For general discussion, systems in which the external inputs and the external outputs are connected directly are also taken into consideration. Fig. 5 presents the definitions of the relevant variables and collections. $\mathcal{W}_i$ and $\mathcal{U}_i$ represent the number sets of succeeding processes which are attached to the i-th external input and the external outputs, respectively. Accordingly, the latest feeding time can be stated as follows.

$$[\boldsymbol{y}_L(k)]_i = \bigwedge_{j\in \mathcal{W}_i} [\boldsymbol{x}^-(k)]_j \wedge \bigwedge_{j\in \mathcal{U}_i} [\boldsymbol{u}(k)]_j \qquad (31)$$

Utilizing the representation matrices $\boldsymbol{B}_0$ and $\boldsymbol{D}_0$ introduced in the previous section, (31) can be transformed into

$$[\boldsymbol{y}_L(k)]_i = \bigwedge_{j=1}^{n} [\boldsymbol{B}_0^T]_{ij} \setminus [\boldsymbol{x}^-(k)]_j \wedge \bigwedge_{j=1}^{q} [\boldsymbol{D}_0^T]_{ij} \setminus [\boldsymbol{u}(k)]_j$$
$$= [\boldsymbol{B}_0^T \odot \boldsymbol{x}^-(k)]_i \wedge [\boldsymbol{D}_0^T \odot \boldsymbol{u}(k)]_i$$

Since this relation holds true for all $i\ (1 \le i \le p)$, the next equation holds.

$$\boldsymbol{y}_L(k) = \boldsymbol{B}_0^T \odot \boldsymbol{x}^-(k) \wedge \boldsymbol{D}_0^T \odot \boldsymbol{u}(k) \quad (32)$$

In order to represent (32) in another form, the following two lemmas are proved.

Lemma 3. *The state vector* $\boldsymbol{x}^-(k)$ *that represents the starting times for manufacturing satisfies the next inequality.*

$$(\boldsymbol{F}_0 \boldsymbol{P}_k)^T \odot \boldsymbol{x}^-(k) \ge \boldsymbol{x}^-(k) \quad (33)$$

Proof. For the i-th $(1 \le i \le n)$ element of $\boldsymbol{F}_0^T \odot \boldsymbol{x}^-(k)$, the following relation is followed.

$$[\boldsymbol{F}_0^T \odot \boldsymbol{x}^-(k)]_i = \bigwedge_{j=1}^{n} ([\boldsymbol{F}_0^T]_{ij} \setminus [\boldsymbol{x}^-(k)]_j)$$
$$= \bigwedge_{j \in S_i} [\boldsymbol{x}^-(k)]_j \ge [\boldsymbol{x}^+(k)]_i = [\boldsymbol{x}^-(k)]_i + d_i(k)$$

Thus,

$$[(\boldsymbol{F}_0 \boldsymbol{P}_k)^T \odot \boldsymbol{x}^-(k)]_i = \bigwedge_{j=1}^{n} ([\boldsymbol{P}_k^T]_{ij} \setminus [\boldsymbol{F}_0^T \odot \boldsymbol{x}^-(k)]_j)$$
$$= -d_i(k) + [\boldsymbol{F}_0^T \odot \boldsymbol{x}^-(k)]_i \ge [\boldsymbol{x}^-(k)]_i$$

Since this is applicable for all $i\ (1 \le i \le n)$, the proposition is proved. □

Lemma 4. *The state vector* $\boldsymbol{x}^-(k)$ *follows the next equality.*

$$\boldsymbol{x}^-(k) = \boldsymbol{F}_k^{*T} \odot \boldsymbol{x}^-(k) \quad (34)$$

where

$$(\boldsymbol{F}_0 \boldsymbol{P}_k)^* = \boldsymbol{e}_n \oplus \boldsymbol{F}_0 \boldsymbol{P}_k \oplus \cdots \oplus (\boldsymbol{F}_0 \boldsymbol{P}_k)^{l-1}$$

Proof. Utilizing the result of Lemma 3 repeatedly,

$$[(\boldsymbol{F}_0 \boldsymbol{P}_k)^2]^T \odot \boldsymbol{x}^-(k) \ge (\boldsymbol{F}_0 \boldsymbol{P}_k)^T \odot \boldsymbol{x}^-(k) \ge \boldsymbol{x}^-(k) \quad (35)$$

holds true. In a similar way, the following inequalities are satisfied.

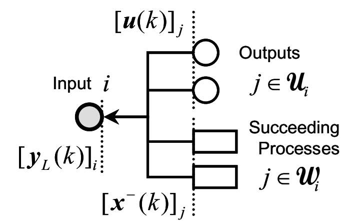

Fig. 5. External outputs and succeeding processes attached to the i-th external input

$$\begin{aligned} &[(\boldsymbol{F}_0 \boldsymbol{P}_k)^3]^T \odot \boldsymbol{x}^-(k) \ge \boldsymbol{x}^-(k) \\ &\qquad \vdots \\ &[(\boldsymbol{F}_0 \boldsymbol{P}_k)^{l-1}]^T \odot \boldsymbol{x}^-(k) \ge \boldsymbol{x}^-(k) \end{aligned} \quad (36)$$

Utilize (33), (35), (36), and the next trivial equation.

$$\boldsymbol{x}^-(k) = \boldsymbol{x}^-(k)$$

Operating $\wedge$ on both sides proves (34). □

By using Lemma 4, $\boldsymbol{y}_L(k)$ in (32) can also be expressed as follows.

$$\boldsymbol{y}_L(k) = [(\boldsymbol{F}_0 \boldsymbol{P}_k)^* \boldsymbol{B}_0]^T \odot \boldsymbol{x}^-(k) \wedge \boldsymbol{D}_0^T \odot \boldsymbol{u}(k) \quad (37)$$

This form includes the manufacturing times in $\boldsymbol{P}_k$ which is the coefficient matrix of the state vector $\boldsymbol{x}^-(k)$. Therefore, it is not advantageous in terms of the calculation load. However, it is closely related to the state equation and the output equation that is used for obtaining the earliest completion times derived in the previous section.

From the above discussions, the state equations and the output equations have two forms for the forward and backward type.

V. Numerical Example

A simple numerical example is presented here. We again consider the simple manufacturing system depicted in Fig. 1. Assume the number of jobs is one, and the manufacturing times are given as

$$[d_1(1), d_2(1), d_3(1)] = (1, 2, 3)$$

The system matrices appeared in (20) and (22) can be expressed as

$$(\boldsymbol{P}_1 \boldsymbol{F}_0)^* = \begin{bmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ 3 & 3 & e \end{bmatrix},\ (\boldsymbol{P}_1 \boldsymbol{F}_0)^* \boldsymbol{P}_1 = \begin{bmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 2 & \varepsilon \\ 4 & 5 & 3 \end{bmatrix}$$

First, the forward type state equation is considered.

Suppose the system is empty in the initial stage, and all required materials are ready at time $t = 0$, which means

$$\boldsymbol{x}^{+}(0) = [\varepsilon \quad \varepsilon \quad \varepsilon]^{T},\ \boldsymbol{u}(1) = [0\ \ 0]^{T}$$

Then, the earliest completion times can be obtained in the following way.

$$\boldsymbol{x}_{E}^{+}(1) = (\boldsymbol{P}_{1}\boldsymbol{F}_{0})^{*}\boldsymbol{P}_{1}[\boldsymbol{x}^{+}(0) \oplus \boldsymbol{B}_{0}\boldsymbol{u}(1)] = [1 \quad 2 \quad 5]^{T}$$

Moreover, the correctness of Lemma 2 is examined. For $\boldsymbol{x}_{E}^{+}(1)$ above, the next relationship can be confirmed in a straightforward way.

$$(\boldsymbol{P}_{1}\boldsymbol{F}_{0})^{*}\boldsymbol{x}_{E}^{+}(1) = [1 \quad 2 \quad 5]^{T} = \boldsymbol{x}_{E}^{+}(1)$$

Additionally, consider a case where the manufacturing has delayed for 2 unit times in process 1. The state variables are then updated to

$$\boldsymbol{x}^{+}(1) = [3 \quad 2 \quad 7]^{T}$$

By operating $(\boldsymbol{P}_{1}\boldsymbol{F}_{0})^{*} \otimes$ from left side, we can also confirm that

$$(\boldsymbol{P}_{1}\boldsymbol{F}_{0})^{*}\boldsymbol{x}^{+}(1) = [3 \quad 2 \quad 7]^{T} = \boldsymbol{x}^{+}(1)$$

is followed and Lemma 2 holds true even when the sate vector does not hold the earliest times.

Subsequently, the backward type state equation is examined. Suppose the delivery time for $k = 1$ is 7, and there is not any additional schedule for $k \geq 2$. Thus,

$$\boldsymbol{u}(1) = [7]^{T},\ \boldsymbol{x}^{-}(2) = [-\varepsilon \quad -\varepsilon \quad -\varepsilon]^{T}$$

Accordingly, the latest starting times can be calculated in the next manner.

$$\begin{aligned}\boldsymbol{x}_{L}^{-}(1) &= [(\boldsymbol{P}_{1}\boldsymbol{F}_{0})^{*}\boldsymbol{P}]^{T} \odot [\boldsymbol{x}_{L}^{-}(2) \wedge \boldsymbol{C}_{0}^{T} \odot \boldsymbol{u}(1)] \\ &= [3 \quad 2 \quad 4]^{T}\end{aligned}$$

As these inspections show, the forward and backward type equations for the online scheduling can be performed using the same system matrix $(\boldsymbol{P}_{1}\boldsymbol{F}_{0})^{*}\boldsymbol{P}_{1}$, which is accomplished by improving the assignment of the state variables.

VI. Conclusion

This paper derived a MPL state-space representation that describes the behavior of repetitious execution systems with a MIMO-FIFO structure. A disadvantage of the conventional MPL form is that all representation matrices should be recalculated when the system parameters are changed since the two elements for describing precedence constraints and the execution times are not separated.

In this research, the locations of inputs and outputs and precedence constraints are given by parameter matrices whose every element are logical numbers. The parameter matrices are independent of the system parameters. For the state variables, we assigned the earliest starting times to the completion times of manufacturing. In addition, we assigned the latest times to the starting times of manufacturing. This revised the MPL form whose representation matrices are dependent on only the system parameters of the corresponding job and reduced the number of independent system parameters.

The discussions clarified several fundamental properties for describing the corresponding systems and it is also expected that the calculation can be simplified and its load reduced.

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