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A NEW LOOK AT FREQUENCY RESOLUTION IN POWER SPECTRAL DENSITY ESTIMATION

Sudeshna Pal, Soosan Beheshti

Electrical and Computer Engineering Department, Ryerson University, Toronto, Canada
spal@ee.ryerson.ca soosan@ee.ryerson.ca

ABSTRACT

The problem of frequency resolution in non-parametric power spectrum density estimation (PSDE) for noisy signals is considered. In this setting, finite length of data as well as the additive noise, both contribute to a decreased frequency resolution. The existing PSDE approaches offer different forms of averaging and windowing of the available data only to improve statistical properties of the estimates, however, at the expense of reducing frequency resolution. In these approaches, the additive noise and the finiteness of data which are the causes of the original loss of the frequency resolution are not treated separately. In this paper, we suggest a new approach which takes advantages of these two different causes of the problem. Therefore, the new nonparametric approach to spectrum estimation tackles the problem of resolution in two steps. First, the method optimally reduces noise interference with the signal via minimum noiseless description length (MNDL). The new power spectrum estimation MNDL-Periodogram (P^{MNDL}) of the denoised signal is then computed via conventional indirect periodogram to improve frequency resolution.

1. INTRODUCTION

Spectrum estimation plays an important role in signal detection and tracking. In many applications, much interest lies in narrow-band signal detection which maybe recorded in very noisy environment. Therefore signal detection and frequency estimation become nontrivial problems that require robust, high-resolution spectrum estimation techniques [1].

Conventional PSDE approaches are used for both modeling PSD of stochastic signals and PSD estimation of noisy signals. We separate these two problems. In this paper, we consider the PSDE problem for noisy signals and discuss a different approach to this problem. In application, we are always dealing with finite-length data that already constitute in decreased frequency resolution of any signals. Further windowing of the signal-data, as done by the existing modified versions of the periodogram for improved spectrum estimation, consequently result in further deteriorated frequency resolution of the signal. The additive noise embedded in noisy signals is not considered distinctly in the existing PSDE methods which impacts frequency resolution of these noisy signals as well.

This paper considers nonparametric (or classical) methods for spectrum estimation that are the most often-used and explored techniques [3] in spectral analysis. It is known that these classical methods emphasize on obtaining a consistent estimate of the power spectrum through some averaging or smoothing operations performed directly on the periodogram or on the autocorrelation of the noisy data. Although variance of the modified periodogram estimates is decreased, the effects of these operations are performed at the expense of reducing the frequency resolution. Section 2 provides a closer look at these effects on frequency resolution for the existing nonparametric methods.

In this paper since frequency resolution of noisy signals is the main focus, the problem of PSDE becomes two-fold. First, it is necessary to denoise the signal from its interfering background and then, to compute its power spectrum estimation such that frequency resolution is not decreased due to further windowing of data as is the case in existing methods. The proposed method chooses a new spectrum from the noisy spectrum via minimum noiseless description length (MNDL) and fixed thresholding, similar to [4]. Section 3 provides the calculation approach which follows the same fundamentals used for signal denoising in [4] and for optimum order selection in [5]. The denoised spectrum is then used to find the new PSDE of the signal via MNDL-Periodogram.

Section 4, illustrates frequency resolution reduction in classical averaging or modified periodograms for estimating power spectrum density. In comparison, the novel approach for power spectrum estimation is proposed to maintain frequency resolution close to the original spectrum, which is also demonstrated in the simulation results.

2. PROBLEM STATEMENT

Additive noise as well as the broadening of the spectrum being estimated due to windowing are particularly a problem when we wish to resolve signals with closely spaced frequency components.

In this paper, we consider noisy signal of the following form

$$y[n] = \bar{y}[n] + w[n] \tag{1}$$

for an available finite N length sample of Y, $\bar{y}[1], \bar{y}[2], \cdots,$ $\bar{y}[N]$, where $w[n]$ is white Gaussian noise with variance σ_w^2. Note $\bar{y}(n)$ is the available noiseless data.

We obtain noisy spectrum $Y(e^{j\omega})$ via FFT such that

$$Y(e^{j\omega}) = \frac{1}{N} \sum_{n=0}^{N} y[n] e^{-j\omega n} \tag{2}$$

where $Y(e^{j\omega})$ is the Fourier Transform of noisy signal $y[n]$.

The new PSDE problem is to first find the best denoised spectrum of $Y(e^{j\omega})$ and use it to estimate PSD of the denoised signal such that frequency resolution is not affected by any additional windowing.

2.1. Fundamentals of Existing PSDE Methods

Existing modifications to the periodogram that have been proposed to improve only the statistical properties of the spectrum estimate. The effects of these modifications on frequency resolution of spectrum are explained as follows: The Bartlett method, also known as averaging periodogram, allows data to be subdivided into smaller segments prior to computing the periodogram. The effect of reducing the length of data into shorter segments results in a window whose spectral width has been increased by a certain factor. Consequently, the frequency resolution is reduced by the same factor. Similarly, the Welch method, known as modified periodogram, allows data segments to not only overlap but also applies a window for variance reduction. Resolution in this case is not only window dependent but also suffers from the same effects as Bartlett due to data segmentation. In the Blackman-Tukey method the autocorrelation estimate is windowed first prior to spectrum estimation computation. The effect of windowing the autocorrelation is to smooth the periodogram estimate and thus decreasing the variance in the estimate as a result. However, this is done at the expense of reducing the resolution since a smaller number of estimates are used to form the estimate of the power spectrum [2].

We realize that the periodogram is only modified to improve its statistical properties at the cost of deteriorating frequency resolution, and therefore propose the new method in the following section to overcome the problem of noise and windowing data via optimally denoising the signal first to improve the spectrums frequency resolution.

3. MNDL-PERIODOGRAM: A NEW PSDE APPROACH

In this approach, a new PSD estimate is obtained in the following two steps:

3.1. Spectrum Denoising

Consider the noisy signal $y[n]$ in (1). To evaluate the noisy spectrum, the FFT error for each n-point is an important unavailable factor

$$e[e^{j\omega_o n}] = Y(e^{j\omega_o n}) - \bar{Y}(e^{j\omega_o n}) \tag{3}$$

where $\bar{Y}(e^{j\omega_o n})$ is the noiseless spectrum and $\omega_o = \frac{2\pi}{N+1}$. After obtaining the N-point FFT of the signal, we sort the absolute value of this FFT and denote the sorted versions by $Y^s[n]$ and its associated denoised coefficients by $\bar{Y}^s[n]$.
The tail of the sorted spectrum is more affected by the noise than the FFT points with the highest absolute values. Therefore, for the denoising step, the goal is to choose the optimum number of these sorted noisy spectrum. For each value m, $0 \leq m \leq N$ the chosen noiseless FFT is

$$Y_m[n] = \begin{cases} Y^s[n] & \text{if } 0 \leq n \leq m, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

that represents the choice of the first m estimates and sets the rest of the FFT values to zero.
and the new error criterion becomes

$$e_m Y[n] = \bar{Y}^s[n] - Y_m[n]. \tag{5}$$

Note that n in this error and n in (3) possibly represent two different frequencies. Our goal is to obtain the optimum value of m which results in the minimum spectrum mean-square estimation error (SMSE). This criterion's estimate is provided with a new adaptive method based on the observed noisy spectrum such that

$$\text{SMSE}[m] = E(||\bar{Y}^s[n] - Y_m[n]||^2) \tag{6}$$

$$= \sum_{n=0}^{m-1} E(||e_m Y[n]||^2) + \Delta[m] \tag{7}$$

where

$$\Delta[m] = \sum_{n=m}^{N-1} \|\bar{Y}^s[n]\|^2. \tag{8}$$

SMSE is similar to the MNDL criterion introduced in [4]where the objective is to minimize the error or noise between noisy and noiseless data or spectrum in this case.
Note that as m grows, the $e_m Y[n]$ dependent part of SMSE grows, while $\Delta[m]$ decreases. This always leads to an optimum m for which the SMSE is minimized. This desired criterion is not available, however, from the observed data, an

estimate of the following available spectrum error (ASE) is available

$$\mathrm{ASE}[m] = \sum_{n=0}^{N-1} \mathrm{E}(||Y^s[n] - Y_m[n]||^2) \qquad (9)$$

Here, we present a novel approach which uses the available spectrum error to provide an estimate of the desired criterion SMSE$[m]$.

Using (6) for an unbiased estimator ($E(e_m Y[n]) = 0$), then $ASE[m]$ in (9) becomes

$$\mathrm{ASE}[m] = \sum_{n=m}^{N-1} E(||e_m Y[n]||^2) + \Delta[m] \qquad (10)$$

We now show that an estimate of $e_m Y[n]^2$ dependent parts in (8) and (10) can be provided based of the observed data. Therefore, an estimate of unavailable $\Delta[m]$ can be calculated from (8) and substituted back in (6).

The estimate of $e_m Y[n]^2$ dependent parts of SMSE and ASE in (8,10) are calculated as

$$\varepsilon_{1u}[m] = \sum_{|n|=m}^{N-1} (N - |n|)\sigma_w^2, \;\; \varepsilon_{2u}[m] = \sum_{|n|=0}^{m} |n|\sigma_w^2 \qquad (11)$$

Furthermore, variance of the additive noise $w[n]$ is also calculated and sorted in descending order, keeping high noise variance at the front and low variance towards the tail such that

$$\sigma_{ws}^2[n] = \frac{1}{N}\sum_{i=1}^{S}(||w_i[n] - \bar{w}_i[n]||^2) \qquad (12)$$

where S is the number of additive noise samples and $\bar{w}_i[n]$ represents the mean of that i^{th} noise sample.
Sorted estimates of the dependent parts in (11) now become

$$\varepsilon_{1s}[m] = \frac{1}{N}\sum_{|n|=m}^{N} \sigma_{ws}^2, \;\; \varepsilon_{2s}[m] = \sum_{|n|=0}^{m} |n|\sigma_{ws}^2 \qquad (13)$$

Finally, the estimate of $\Delta[m]$ and SMSE$[m]$ are

$$\hat{\Delta}[m] = \widehat{ASE}[m] - \varepsilon_{1x}[m], \qquad (14)$$

$$\widehat{SMSE}[m] = \hat{\Delta}[m] + \varepsilon_{2x}[m] \qquad (15)$$

where $x = u$ for unsorted estimates and $x = s$ for the sorted estimates.

The optimum window length for obtaining acceptable noiseless spectrum in our case is

$$m^* = \arg\min_m \widehat{SMSE}[m] \qquad (16)$$

Minimization of the estimated desired criterion above provides optimum denoising of noisy signal for our case.

3.2. PSDE of Denoised Spectrum

We obtain PSD estimate of the denoised spectrum via conventional indirect periodogram. That is, we first obtain the denoised time-domain signal $y^{MNDL}[n]$ via m-point ifft of the denoised spectrum $Y^{MNDL}(e^{j\omega_o})$ by choosing m^* frequency components. Computation of the new MNDL-Periodogram ($P^{MNDL}(e^{j\omega_o})$) is as follows:

$$P_{yy}^{MNDL}(e^{j\omega_o}) = \sum_{m=-(N-1)}^{N-1} r_{yy}^{MNDL}[m]e^{-j\omega_o m} \qquad (17)$$

where $r_{yy}^{MNDL}[m]$ is the autocorrelation of denoised time-domain signal $y^{MDNL}[n]$.

4. SIMULATION RESULTS

We work with two sets of noisy sinusoidal signals of length $N = 1024$ and are generated with (1) as follows:

$$y_1[n] = 5sin[0.4\pi n] + w[n] \qquad (18)$$

$$y_2[n] = 5sin[0.4\pi n] + 5sin[0.41\pi n] + w[n] \qquad (19)$$

where $w[n]$ is the added white Gaussian noise with unit variance ($\sigma_w^2 = 1$).

We obtain noisy spectrum ($Y_1(e^{j\omega_o n})$ and $Y_2(e^{j\omega_o n})$) and corresponding noiseless spectrum ($\bar{Y}_1(e^{j\omega_o n})$ and $\bar{Y}_2(e^{j\omega_o n})$) for both $y_1[n]$ and $y_2[n]$ noisy signals above. Note that here the noiseless spectrums, which are obtained by simply taking the spectrum of pure sine waves, is computed only as a comparison criterion to check against the estimated noiseless spectrums in the simulations. Fig. 1 shows the power spectrum for $y_1[n]$ noisy signal $Y_1(e^{j\omega_o n})$ and its corresponding noiseless spectrum $\bar{Y}_1(e^{j\omega_o n})$.

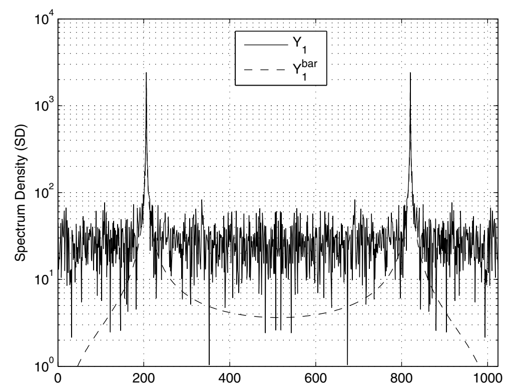

Fig. 1. Noisy $Y_1(e^{j\omega_o n})$ and Noiseless $\bar{Y}_1(e^{j\omega_o n})$ PSD for $y_1[n]$

4.1. Optimum Spectrum Denoising

In Fig. 2 we see the estimated $\widehat{\text{SMSE}}$ criterion for $y_1[n]$ and observe that the optimum minimum occurs at $m^* = 61$ for sorted noise variance and $m^* = 437$ for unsorted noise, while the desired SMSE in (7) picks m^* at 18.

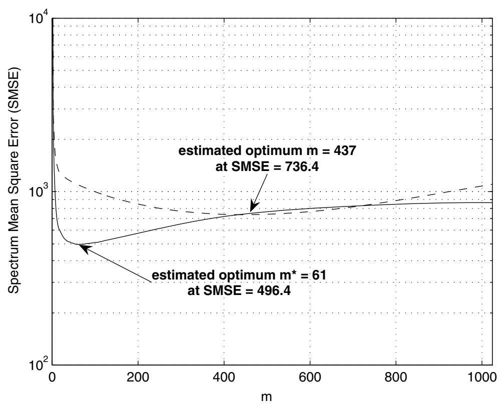

Fig. 2. $\widehat{\text{SMSE}}$ with optimum $m^* = 61$ for sorted noise (solid line) and $m^* = 437$ for unsorted noise (dashed line).

Similarly, for $y_2[n]$ the optimum minimum occurs at $m^* = 105$ for sorted noise variance while the desired SMSE in 6 picks m^* at 26.

However in Fig. 3 and 4 we note that the denoised power spectrum $Y_1^{MNDL}(e^{j\omega_o n})$ with $m^* = 18$ and $Y_1^{MNDL}(e^{j\omega_o n})$ with $m^* = 61$ are successful estimated denoised spectrums with respect to the pure $\bar{Y}_1(e^{j\omega_o n})$ noiseless spectrum which is not obtained in the existing spectral estimation methods. As will be observed in following figures, the denoising of the spectrum prior to spectral estimation greatly enhances the frequency resolution.

The thresholding in the above figures is compared with the well-known hard thresholding approach, Donoho and Johnstone thresholding $\sigma_w\sqrt{2\log N} = 2373$ and MDL thresholding $\sigma_w\sqrt{\log N} = 1678$. However, these thresholds are much worse than the estimated threshold obtained by $\widehat{\text{SMSE}}$ as they pick up most of the noise. Therefore, optimum m from (16) provides the best denoised spectrum.

4.2. PSDE comparison: MNDL-Periodogram vs. existing methods

After denoised frequency spectrum $Y_1^{MNDL}(e^{j\omega_o n})$ is obtained via minimum m^* of $\widehat{SMSE}[m]$, we compute its power spectrum $P_1^{MNDL}(e^{j\omega_o n})$ by applying indirect periodogram method. That is, we first convert $Y_1^{MNDL}(e^{j\omega_o n})$ spectrum

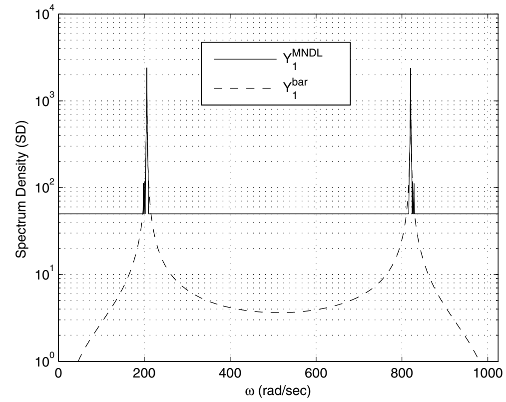

Fig. 3. Denoised spectrum Y_1^{MNDL} with optimum m = 18.

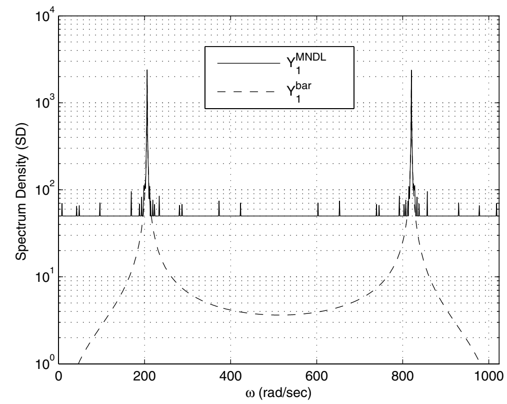

Fig. 4. Denoised spectrum Y_1^{MNDL} with optimum m = 61.

into time-domain signal. We then compute its autocorrelation and take the Fourier Transform according to (17). Fig. 5 illustrates the newly obtained $P_1^{MNDL}(e^{j\omega_o n})$ against the noiseless PSD $P_{\bar{y}_1}(e^{j\omega_o n})$.

We now demonstrate a comparison of the new MNDL-Periodogram against existing modified periodogram already mentioned in Section 2.

We observe that the Bartlett spectrum in Fig. 6 is very noisy and relative to MNDL-Periodogram's spectrum, farther away from the desired $P_1^{\bar{y}_1}(e^{j\omega_o n})$ noiseless power spectrum.

The Welch spectrum in Fig. 7 is even further away from the desired $P_1^{\bar{y}_1}(e^{j\omega_o n})$ noiseless power spectrum. However its noise level is better than the Bartlett spectrum and therefore may provide better frequency resolution.

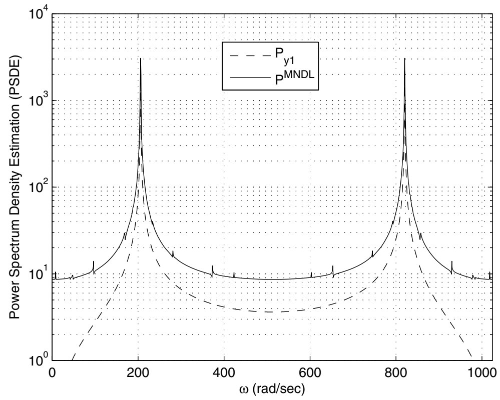

Fig. 5. PSDE of y_1 via MNDL-Periodogram (P_1^{MNDL}) against PSD of noiseless signal ($P_{\bar{y}_1}(f)$)

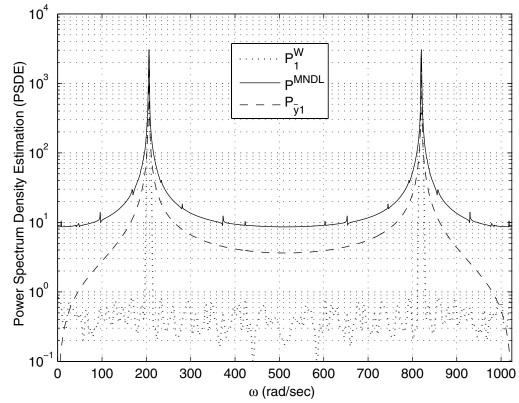

Fig. 7. PSDE of y_1 via Welch (P_1^W) and MNDL-Periodogram (P_1^{MNDL}) against PSD of noiseless signal ($P_1^{\bar{y}_1}$)

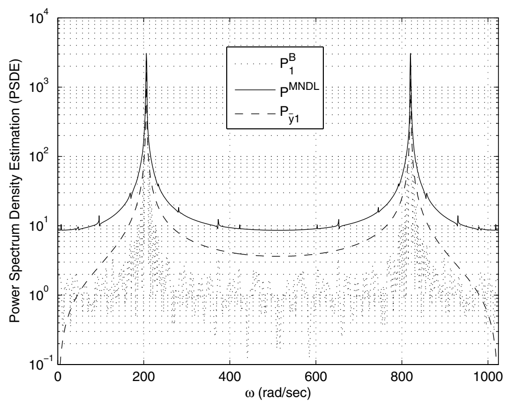

Fig. 6. PSDE of y_1 via Bartlett (P_1^B) and MNDL-Periodogram (P_1^{MNDL}) against PSD of noiseless signal ($P_1^{\bar{y}_1}$)

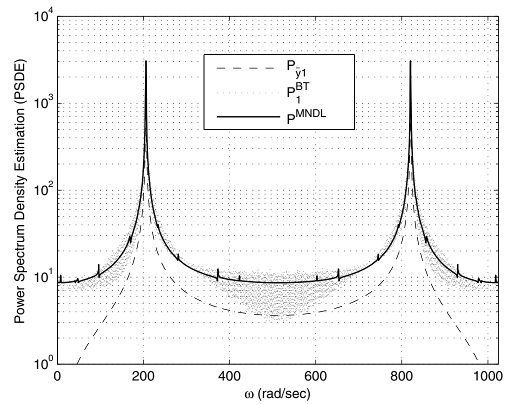

Fig. 8. PSDE of y_1 via Blackman-Tukey (P_1^{BT}) and MNDL-Periodogram (P_1^{MNDL}) against PSD of noiseless signal ($P_1^{\bar{y}_1}$)

We clearly observe that power spectrum of $P_1^B(e^{j\omega_o n})$, $P_1^W(e^{j\omega_o n})$, and $P_1^{BT}(e^{j\omega_o n})$ are definitely broadened/ corrupted in comparison to $P_1^{\bar{y}_1}(e^{j\omega_o n})$. This is due to the existing noise and the windowing of the data or autocorrelation function of the noisy signal as already mentioned in Section 2. However, the spectrum of $P_1^{MNDL}(e^{j\omega_o n})$ has no modified effects and therefore has similar frequency resolution with respect to $P_1^{\bar{y}_1}(e^{j\omega_o n})$.

We demonstrate the improvement of frequency resolution over the existing methods further by computing the aforementioned PSDE of y_2, which has two closely spaced spectra in the signal.

Fig. 9 illustrates the PSDE of y_2 performed on y_1 in the previous figure. The two closely spaced spectra are encircled and zoomed-in versions of the two peaks are shown in figures below.

We take a closer look at the two closely spaced spectra in the following zoomed in figures displaying the Bartlett, Welch and Blackman-Tukey methods, each against MNDL-Periodogram and available noiseless PSD $P_2^{\bar{y}_2}$.

In Fig. 10 the distance between the peak and the dip is very well distinguished in the MNDL-Periodogram spectrum whereas the same resolution is not obtained with the Bartlett approach.

Furthermore, the Welch method in Fig. 11 displays even

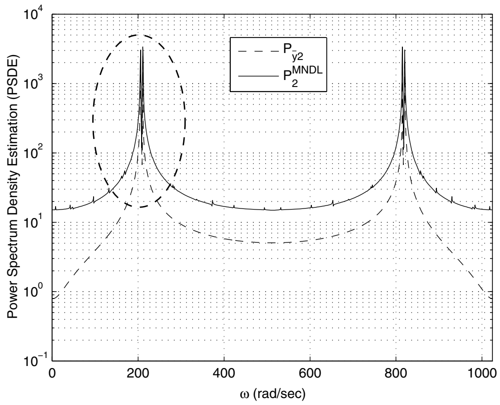

Fig. 9. PSDE of y_2 via MNDL-Periodogram(P_2^{MNDL}) against PSD of noiseless signal ($P_2^{\bar{y}_2}$)

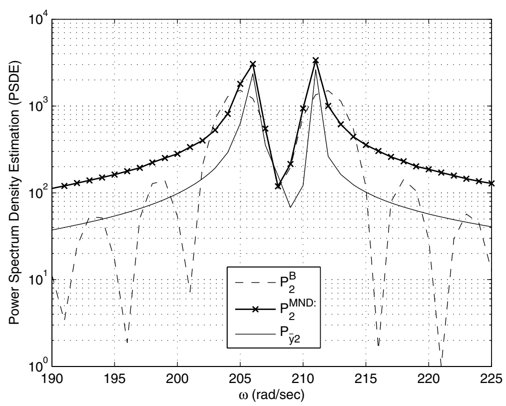

Fig. 10. Bartlett spectrum with zoomed in two closely spaced spectra

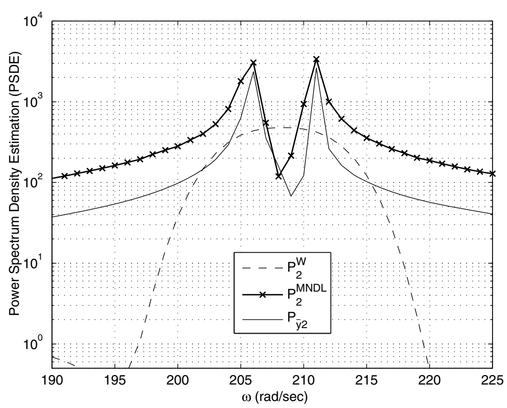

Fig. 11. Welch spectrum with zoomed in two closely spaced spectra

worse results where the frequency resolution is completely lost due to same data segmentation as in Bartlett and in addition to a hamming windowing of the segments.

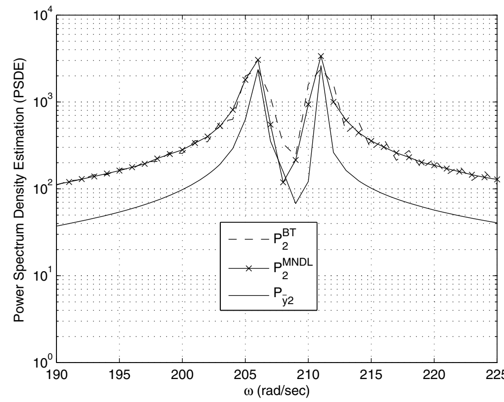

Fig. 12. Blackman-Tukey spectrum with zoomed in two closely spaced spectra

Although the Blackman-Tukey method in Fig. 12 obtains closer resolution as MNDL-Periodogram, however its dip between the two closely spaced spectra is not as distinct as the later.

We observe that the separation of the two spectra are best distinguished in $P_2^{MNDL}(e^{j\omega_o n})$ estimation with a longer dip and closer to the expected noiseless power spectrum $P_2^{\bar{y}_2}(e^{j\omega_o n})$ than the existing PSDE shown by $P_2^{B}(e^{j\omega_o n})$, $P_2^{W}(e^{j\omega_o n})$ or $P_2^{BT}(e^{j\omega_o n})$. Clearly this novel approach can be used for better signal detection and spectrum estimation, especially in noisy environments.

5. CONCLUSION

In this paper, we acknowledged the imposed problem of all existing modifications on the periodogram that have been proposed to improve only the statistical properties of the spectrum estimate at the cost of frequency resolution. We addressed the PSDE problem with presence of additive noise. In this case, windowing the available data in the existing nonparametric methods clearly demonstrates the decreased resolution in PSD estimates of noisy signals. This decrease, which is due to both the finiteness of the data and the presence of the additive noise, is improved by the MNDL-Periodogram. The approach first denoises the data and then estimates the PSD. For the first step minimizing the estimated spectrum mean square error (SMSE) provided the best denoised spectrum in comparison to other thresholding criteria. Next a simple periodogram approach is used for PSD estimation. The novel approach maintains frequency resolution close to the original noiseless spectrum as no additional windowing of signal is applied and, as a result, outperforms the existing approaches.

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