Proceedings of the 2007 IEEE Symposium on
Foundations of Computational Intelligence (FOCI 2007)

Definability of Approximations for a Generalization of the Indiscernibility Relation

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***Abstract*— We discuss a generalization of the indiscernibility relation, i.e., a relation R that is not necessarily reflexive, symmetric, or transitive. On the basis of granules, defined by R, we introduce the idea of *definability*. Twelve different basic definitions of approximations are discussed. Since four of these approximations do not satisfy, in general, the inclusion property, four additional modified approximations are introduced. Fur thermore, eight other approximations are constructed by *duality*. The main objective is to study definability of approximations. We study definability of all approximations for reflexive, symmetric, or transitive relations. In particular, for reflexive relations the set of these twenty four approximations is reduced, in general, to the set of fourteen approximations.**

I. Introduction

One of the basic ideas of rough set theory is the indis cernibility relation [13], [14]. The usual assumption is that the indiscernibility relation is an equivalence relation. In this paper we will discuss a generalization of the indiscernibility relation, an arbitrary binary relation R defined on the nonemptyfinite set U called *universe*. Such relation R does not need to be reflexive, symmetric, or transitive.

Our main objective was to study the definability of approxi mations of any subset X of the universe U. This idea is based on a union of granules, defined by R, that are also known as R *successor* or R *predecessor sets* or as *neighborhoods*.

In this paper twelve definitions of approximations are dis cussed. Since four of these approximations do not satisfy, in general, the inclusion property, four modified approximations are introduced. Additionally, using *duality*, we define eight ex tra approximations. Among these twenty four approximations, nine are introduced for the first time.

Such generalizations of the indiscernibility relation have immediate application to data mining (machine learning) from incomplete data sets. In such applications the binary relation R, called the *characteristic relation* and describing such data, is reflexive. For reflexive relations the system of twelve approximations plus the system of four additional approxi mations is reduced to eight approximations, while the system of eight approximations defined by duality is reduced to six. Thus, only fourteen different approximations are possible. Note that some of these fourteen approximations are not useful for data mining from incomplete data [2], [3], [4], [5], [6], [7].

II. Basic Definitions

First we will introduce the basic granules (or neighbor hoods), defined by a relation R. In this paper R is a gen eralization of the indiscernibility relation. The relation R, in general, does not need to be reflexive, symmetric, or transitive, while the indiscernibility relation is an equivalence relation. Such granules are called here R *successor* and R *predecessor* sets.

Let U be a finite nonempty set, called a *universe*, let R be a binary relation on U, and let x be a member of U. The R *successor* set of x, denoted by $R_s(x)$, is defined as follows

$$R_s(x) = \{y \mid xRy\}.$$

The R *predecessor* set of x, denoted by $R_p(x)$, is defined as follows

$$R_p(x) = \{y \mid yRx\}.$$

R *successor* and R *predecessor* sets are used to form larger sets that are called R *successor* and R *predecessor definable*.

Let X be a subset of U. A set X is R *successor definable* if and only if $X = \emptyset$ or there exists a subset Y of U such that

$$X = \cup \{R_s(y) \mid y \in Y\}.$$

A set X is R *predecessor definable* if and only if $X = \emptyset$ or there exists a subset Y of U such that

$$X = \cup \{R_p(y) \mid y \in Y\}.$$

Let X be a subset of U. The R *singleton successor lower approximation* of X, denoted by $\underline{appr}_s^{singleton}(X)$, is defined as follows

$$\{x \in U \mid R_s(x) \subseteq X\}.$$

The singleton successor lower approximations were studied in many papers, see, e.g., [1], [2], [8], [9], [10], [11], [12], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25].

The R *singleton predecessor lower approximation* of X, denoted by $\underline{appr}_p^{singleton}(X)$, is defined as follows

$$\{x \in U \mid R_p(x) \subseteq X\}.$$

The singleton predecessor lower approximations were stud ied in [17].

The R *singleton successor upper approximation* of X, de noted by $\overline{appr}_s^{singleton}(X)$, is defined as follows

$$\{x \in U \mid R_s(x) \cap X \neq \emptyset\}.$$

The singleton successor upper approximations, like single ton successor lower approximations, were also studied in many papers, e.g., [1], [2], [8], [9], [17], [18], [19], [20], [21], [22], [23], [24], [25].

The R *singleton predecessor upper approximation* of X, denoted by $\overline{appr}_p^{singleton}(X)$, is defined as follows

$$\{x \in U \mid R_p(x) \cap X \neq \emptyset\}.$$

The singleton predecessor upper approximations were intro duced in [17]. Note that singleton approximations were also discussed in [15].

The R *subset successor lower approximation* of X, denoted by $\underline{appr}_s^{subset}(X)$, is defined as follows

$$\cup \{R_s(x) \mid x \in U, R_s(x) \subseteq X\}.$$

The subset successor lower approximations were introduced in [1], [2].

The R *subset predecessor lower approximation* of X, de noted by $\underline{appr}_p^{subset}(X)$, is defined as follows

$$\cup \{R_p(x) \mid x \in U, R_p(x) \subseteq X\}.$$

The subset predecessor lower approximations were studied in [17].

The R *subset successor upper approximation* of X, denoted by $\overline{appr}_s^{subset}(X)$, is defined as follows

$$\cup \{R_s(x) \mid x \in U, R_s(x) \cap X \neq \emptyset\}.$$

The subset successor upper approximations were introduced in [1], [2].

The R *subset predecessor upper approximation* of X, de noted by $\overline{appr}_p^{subset}(X)$, is defined as follows

$$\cup \{R_p(x) \mid x \in U, R_p(x) \cap X \neq \emptyset\}.$$

The subset predecessor upper approximations were studied in [17].

The R *concept successor lower approximation* of X, de noted by $\underline{appr}_s^{concept}(X)$, is defined as follows

$$\cup \{R_s(x) \mid x \in X, R_s(x) \subseteq X\}.$$

The concept successor lower approximations were intro duced in [1], [2].

The R *concept predecessor lower approximation* of X, denoted by $\underline{appr}_p^{concept}(X)$, is defined as follows

$$\cup \{R_p(x) \mid x \in X, R_p(x) \subseteq X\}.$$

The concept predecessor lower approximations are intro duced, for the first time, in this paper.

The R *concept successor upper approximation* of X, de noted by $\overline{appr}_s^{concept}(X)$, is defined as follows

$$\cup \{R_s(x) \mid x \in X, R_s(x) \cap X \neq \emptyset\}$$

The concept successor upper approximations were studied in [1], [2], [12].

The R *concept predecessor upper approximation* of X, denoted by $\overline{appr}_p^{concept}(X)$, is defined as follows

$$\cup \{R_p(x) \mid x \in X, R_p(x) \cap X \neq \emptyset\}$$

The concept predecessor upper approximations were studied in [17].

III. Dual Approximations

Let X be a subset of U. A *complement* of X, i.e., the set $U - X$, will be denoted by $\neg X$. As it was shown in [23], singleton approximations are *dual*, i.e.,

$$\underline{appr}_s^{singleton}(X) = \neg(\overline{appr}_s^{singleton}(\neg X)),$$

$$\overline{appr}_s^{singleton}(X) = \neg(\underline{appr}_s^{singleton}(\neg X)),$$

$$\underline{appr}_p^{singleton}(X) = \neg(\overline{appr}_p^{singleton}(\neg X)),$$

$$\overline{appr}_p^{singleton}(X) = \neg(\underline{appr}_p^{singleton}(\neg X)).$$

Additionally, as it was shown in [23], subset approximations are not dual. Moreover, concept approximations are not dual as well. Consider the example from Section VIIA and set X = {1, 2, 4, 8}. Then

$$\underline{appr}_s^{concept}(X) = \{1, 2, 4, 8\} \neq \neg(\overline{appr}_s^{concept}(\neg X)) = \{1, 2\}.$$

Replacing R with R^{-1} in the example from Section VII A shows that Rconcept predecessor approximations are not dual, either.

Two additional approximations were defined in [23]. The first approximation denoted by $\overline{appr}_s^{dualsubset}(X)$ was defined by

$$\neg(\underline{appr}_s^{subset}(\neg X)),$$

while the second one denoted by $\underline{appr}_s^{dualsubset}(X)$ was defined by

$$\neg(\overline{appr}_s^{subset}(\neg X)).$$

Obviously, there are additional possible approximations, if we replace subscript s by p, i.e.,

$$\neg(\underline{appr}_p^{subset}(\neg X))$$

and

$$\neg(\overline{appr}_p^{subset}(\neg X)),$$

or if we replace in the last four definitions "subset" by "con cept". Four former approximations of these eight approxima tions are called R *dual subset approximations* of X and latter four approximations are called R *dual concept approximations* of X, (*successor* or *predecessor*, *lower* or *upper*, respectively).

IV. Rough Inclusion

Duality of lower and upper approximations of arbitrary subset X of the universe U is a basic property of rough ap proximations defined for the indiscernibility relation originally formulated by Z. Pawlak [13], [14]. Rough inclusion is another important property. The following property is called a *rough inclusion*:

$$\underline{appr}_a^b(X) \subseteq X \subseteq \overline{appr}_a^b(X)$$

or

$$\neg(\overline{appr}_a^b(\neg X)) \subseteq X \subseteq \neg(\underline{appr}_a^b(\neg X)),$$

where $a \in \{s, p\}$ and $b \in \{subset, concept\}$.

For subset and concept approximations, any pair of the same type of lower and upper approximations, as well as any pair of associated dual lower and upper approximations satisfies the property of rough inclusion for arbitrary relation R. However, for not reflexive relation R and singleton approximations the following situations may happen, dose not matter if R is symmetric or transitive:

$$\sim (\underline{appr}_a^{singleton}(X) \subseteq X),$$

$$\sim (X \subseteq \overline{appr}_a^{singleton}(X)),$$

$$\overline{appr}_a^{singleton}(X) \subsetneq X \subsetneq \underline{appr}_a^{singleton}(X),$$

where $a \in \{s, p\}$. The following example shows such three situations for a symmetric and transitive relation R.

Example. Let $U = \{1, 2, 3, 4, 5\}$, $X = \{1, 2\}$, $R = \{(1, 1), (3, 3), (4, 4), (3, 4), (4, 3)\}$. Then for $a \in \{s, p\}$ $\underline{appr}_a^{singleton}(X)$ = $\{1, 2, 5\}$, $\overline{appr}_a^{singleton}(X)$ = $\{1\}$, so that

$$\sim (\underline{appr}_a^{singleton}(X) \subseteq X),$$

$$\sim (X \subseteq \overline{appr}_a^{singleton}(X)),$$

$$\overline{appr}_a^{singleton}(X) \subsetneq X \subsetneq \underline{appr}_s^{singleton}(X).$$

Thus, for singleton approximations—in general—the in clusion property does not hold. To avoid this situation, the following modification of corresponding definitions may be introduced:

The R *modified singleton successor lower approximation* of X, denoted by $\underline{appr}_s^{modsingleton}(X)$, is defined as follows

$$\{x \in U \mid R_s(x) \subseteq X \wedge R_s(x) \neq \emptyset\}.$$

The R *modified singleton predecessor lower approximation* of X, denoted by $\underline{appr}_p^{modsingleton}(X)$, is defined as follows

$$\{x \in U \mid R_p(x) \subseteq X \wedge R_p(x) \neq \emptyset\}.$$

The R *modified singleton successor upper approximation* of X, denoted by $\overline{appr}_s^{modsingleton}(X)$, is defined as follows

$$\{x \in U \mid R_s(x) \cap X \neq \emptyset \vee R_s(x) = \emptyset\}.$$

The R *modified singleton predecessor upper approximation* of X, denoted by $\overline{appr}_p^{modsingleton}(X)$, is defined as follows

$$\{x \in U \mid R_p(x) \cap X \neq \emptyset \vee R_p(x) = \emptyset\}.$$

Pairs of corresponding modified singleton approximations are dual. To prove that property, let arbitrary element $x \in U$ be considered. $x \in \underline{appr}_s^{modsingleton}(X)$ if and only if $R_s(x) \neq \emptyset$ and $R_s(x) \subseteq X$. The latter inclusion is equivalent to $R_s(x) \cap (\neg X) = \emptyset$ and together with $R_s(x) \neq \emptyset$ give $x \notin \overline{appr}_s^{modsingleton}(\neg X)$.

V. Coalescence of Rough Approximations

Directly from respective definitions, if R is reflexive,

$$\underline{appr}_a^{subset}(X) = \underline{appr}_a^{concept}(X),$$

for $a \in \{s, p\}$, i.e., for any $X \subseteq U$, Rsubset successor lower approximation of X is equal to the Rconcept successor lower approximation of X and the Rsubset predecessor lower approximation of X is equal to the Rconcept predecessor lower approximation of X. Again, if R is reflexive,

$$\underline{appr}_a^{singleton}(X) = \underline{appr}_a^{modsingleton}(X),$$

and

$$\overline{appr}_a^{singleton}(X) = \overline{appr}_a^{modsingleton}(X),$$

for $a \in \{s, p\}$, i.e., for any $X \subseteq U$, corresponding Rsingleton approximations and Rmodified singleton approximations are equal to each other.

Additionally, it was proven in [17] that if R is reflexive,

$$\{x \in U \mid R_s(x) \cap X \neq \emptyset\} = \cup \{R_p(x) \mid x \in X, R_p(x) \cap X \neq \emptyset\}.$$

i.e., for any $X \subseteq U$, the Rsingleton successor upper approx imation of X is equal to the Rconcept predecessor upper approximation of X.

By analogy with the previous result, if R is reflexive,

$$\{x \in U \mid R_p(x) \cap X \neq \emptyset\} = \cup \{R_s(x) \mid x \in X, R_s(x) \cap X \neq \emptyset\},$$

i.e., for any $X \subseteq U$, the Rsingleton predecessor upper approximation of X is equal to the Rconcept successor upper approximation of X.

For reflexive relation R corresponding Rsubset lower and Rconcept lower approximations coalesce. Namely

$$\underline{appr}_s^{subset}(X) = \underline{appr}_s^{concept}(X),$$

and

$$\underline{appr}_p^{subset}(X) = \underline{appr}_p^{concept}(X).$$

From those equalities and duality of Rdual subset approx imations and Rdual concept approximations

$$\overline{appr}_s^{dualsubset}(X) = \overline{appr}_s^{dualconcept}(X),$$
$$\overline{appr}_p^{dualsubset}(X) = \overline{appr}_p^{dualconcept}(X).$$

VI. Definability

In general, for any subset X of U and a binary relation R on U, the Rsingleton approximations of X (successor or predecessor, lower or upper) and Rmodified singleton approximations of X are neither Rsuccessor definable nor Rpredecessor definable. Similarly, approximations defined as dual sets to subset approximations or to concept approxi mations are neither Rsuccessor definable nor Rpredecessor definable. Subset successor approximations and concept suc cessor approximations are only Rsuccessor definable while subset predecessor approximations and concept predecessor approximations are only Rpredecessor definable. On the other hand, all presented in the paper approximations are both R successor definable and Rpredecessor definable when R is an equivalence relation relation. All necessary examples and proofs are presented in the approaching subsections. In the sequel we investigate which of three properties of relation R: reflexivity, symmetry, transitivity, are necessary or sufficient to ensure definability of introduced approximations. For arbitrary subset X of a universe U and arbitrary relation R on U an R successor lower approximation of X is Rsuccessor definable if and only if corresponding Rpredecessor lower approxi mation is Rpredecessor definable. Moreover an Rsuccessor lower approximation of X is Rpredecessor definable if and only if corresponding Rpredecessor lower approximation is Rsuccessor definable. Obviously, the same situation arises for any kind of upper approximation considered in the paper. It follows from the fact that a binary relation R is reflexive, symmetric or transitive if and only if relation R^{-1} is reflex ive, symmetric or transitive, respectively. Thus situations of Rsuccessor and Rpredecessor definability of Rsuccessor approximations will be considered only in the approaching subsections.

A. Definability of Singleton Approximations

If a relation R is reflexive and transitive, then $\underline{appr}_s^{singleton}(X)$ is Rsuccessor definable and $\underline{appr}_p^{singleton}(X)$ is Rpredecessor definable.

It was shown in [23] that for a reflexive and transitive relation R

$$\underline{appr}_s^{singleton}(X) = \underline{appr}_s^{subset}(X)$$

and

$$\underline{appr}_p^{singleton}(X) = \underline{appr}_p^{subset}(X)$$

Since $\underline{appr}_s^{subset}(X)$ is Rsuccessor definable, then $\underline{appr}_s^{singleton}(X)$ is also Rsuccessor definable. If R is not reflexive and transitive simultaneously, then $\underline{appr}_s^{singleton}(X)$ may not be Rsuccessor definable. To show that, consider the examples from Sections VIIC and VIIE. In these examples the relation R is respectively reflexive and symmetric but not transitive or R is symmetric and transitive but not reflexive. In the former case $\underline{appr}_s^{singleton}(X) = \{2, 8, 9\}$ is not R successor definable because the element 2 occurs always with elements 1 or 3 and none of them belongs to $\{2, 8, 9\}$. In the latter case $\underline{appr}_s^{singleton}(X) = \{1\}$ is not Rsuccessor definable because the element 1 does not occur in any R successor set.

If a relation R is an equivalence relation then $\underline{appr}_s^{singleton}(X)$ is Rpredecessor definable. The proof of this fact follows the equalities

$$\underline{appr}_s^{singleton}(X) = \underline{appr}_s^{subset}(X)$$

for reflexive and transitive relation R [23] and

$$\underline{appr}_s^{subset}(X) = \underline{appr}_p^{subset}(X)$$

for symmetric relation R. If a relation R is not an equivalence relation, then $\underline{appr}_s^{singleton}(X)$ may not be Rpredecessor definable. Indeed, if R is symmetric and not an equivalence relation, the same examples as for Rsuccessor definability of $\underline{appr}_s^{singleton}(X)$ are appropriate to show that any set X may be not Rpredecessor definable. If R is reflexive and transitive but not an equivalence relation, then in the example from Section VIID $\underline{appr}_s^{singleton}(X) = \{1, 6\}$ is not R predecessor definable, because the element 1 occurs only in one Rpredecessor set $R_p(1) = \{1, 2\}$ and $2 \notin \{1, 6\}$.

If relation R is symmetric, then $\overline{appr}_s^{singleton}(X)$ is R successor definable. To prove this fact we will show that for any $x \in \overline{appr}_s^{singleton}(X)$ x must belong to an Rsuccessor set included in $\overline{appr}_s^{singleton}(X)$. Since $x \in \overline{appr}_s^{singleton}(X)$, there exists $y \in X$ such that $y \in R_s(x)$. From symmetry of R, $x \in R_s(y)$. If $R_s(y) = \{x\}$, then obviously $R_s(y) \subseteq \overline{appr}_s^{singleton}(X)$. If $R_s(y) \neq \{x\}$, then there exists $z \neq x$, such that $z \in R_s(y)$. For any such z from symmetry of R $y \in R_s(z)$, and in consequence $z \in \overline{appr}_s^{singleton}(X)$. So, also in this case $R_s(y) \subseteq \overline{appr}_s^{singleton}(X)$.

To show that for reflexive and transitive relation R a set $\overline{appr}_s^{singleton}(X)$ may not be Rsuccessor definable, it is enough to consider the example from Section VIID. For the set X from the example $\overline{appr}_s^{singleton}(X) = \{1, 2, 5, 6, 7\}$, but the element 2 occurs in Rsuccessor sets together with the element 3 which does not belong to $\overline{appr}_s^{singleton}(X)$.

If a relation R is reflexive or symmetric, then $\overline{appr}_s^{singleton}(X)$ is Rpredecessor definable. For

$\underline{appr}_s^{singleton}(X)$ and symmetric relation we have just proved property which directly implies this one. Moreover, it was shown in Section V that for a reflexive relation R

$$\overline{appr}_s^{singleton}(X) = \overline{appr}_p^{concept}(X).$$

For definability of $overline appr_p^{concept}(X)$ see Sec tion VID. If R is neither reflexive nor symmetric, then $\overline{appr}_s^{singleton}(X)$ may be not Rpredecessor definable, as the example from Section VIIB shows. $\overline{appr}_s^{singleton}(X)$ = {1, 3, 5} and the element 1 occurs in the Rpredecessor sets together with the element $2 \notin \overline{appr}_s^{singleton}(X)$.

B. Definability of Modified Singleton Approximations

If R is symmetric and transitive or R is reflexive and tran sitive, then set $\underline{appr}_s^{modsingleton}(X)$ is Rsuccessor definable.

Let us start the proof with the assumption that $x \in \underline{appr}_s^{modsingleton}(X)$. It means $R_s(x) \neq \emptyset$ and $R_s(x) \subseteq \underline{appr}_s^{modsingleton}(X)$. From symmetry and transitivity of R follows that $x \in R_s(x)$. Since x is an arbitrary ele ment of set $\underline{appr}_s^{modsingleton}(X)$, we have $x \in R_s(x)$ for all $x \in \underline{appr}_s^{modsingleton}(X)$ and as consequence $\underline{appr}_s^{modsingleton}(X)$ X is Rsuccessor definable. For a reflex ive relation R, the Rmodified singleton approximations are equal to corresponding Rsingleton approximations, and it has been shown that a Rsingleton successor lower approximation is definable if R is reflexive and transitive. The examples from Sections VIIB and VIIC shows that if a relation R is only transitive or it is reflexive and symmetric, then the set $\underline{appr}_s^{modsingleton}(X)$ may be not Rsuccessor definable. For the transitive relation R $\underline{appr}_s^{modsingleton}(X) = \{5\}$. This set is not Rsuccessor definable, since the element 5 does not occur in the Rsuccessor sets. For a reflexive and symmetric relation R from the example from Section VIIC $\underline{appr}_s^{modsingleton}(X)$ = {2, 8, 9}, and the element 2 occurs always in the Rsuccessor sets together with the elements 1 or 3, that do not belong to set {2, 8, 9}.

If a relation R is symmetric and transitive, then for any X set $\underline{appr}_s^{modsingleton}(X)$ is Rpredecessor definable. Proof of this property is based on the fact that for a symmetric relation R, the Rpredecessor sets are equal to Rsuccessor sets and that sets $\underline{appr}_s^{modsingleton}(X)$ are Rsuccessor definable if R is symmetric and transitive.

The examples from Sections VIIC and VIID shows that if a relation R is not symmetric and transitive then the set $\underline{appr}_s^{modsingleton}(X)$ may be not R successor definable. In the example from Section VIIC $\underline{appr}_s^{modsingleton}(X)$ = {2, 8, 9} and 2 occurs together with 1 or 3. In the example from Section VIID $\underline{appr}_s^{modsingleton}(X)$ = {1, 6} and the element 1 occurs always with the element 2. If R is reflexive and symmetric, then for any X set $\overline{appr}_s^{modsingleton}(X)$ is R successor definable. To prove this fact let us note that for a reflexive relation R

$$\overline{appr}_s^{modsingleton}(X) = \overline{appr}_s^{singleton}(X),$$

as it was observed in Section V and that the set $\overline{appr}_s^{singleton}(X)$ is Rsuccessor definable if R is symmetric. The examples from Sections VIID and VIIE show that for not reflexive or not symmetric relation R set $\overline{appr}_s^{modsingleton}(X)$ may be not Rsuccessor definable. In the former example the element 2 belongs to $\overline{appr}_s^{modsingleton}(X)$, but it occurs in R successor sets together with the element 3 that does not belong to $\overline{appr}_s^{modsingleton}(X)$. In the latter example the element 1 belongs to $\overline{appr}_s^{modsingleton}(X)$ and this element does not occur in any Rsuccessor set.

If a relation R is reflexive, then set $\overline{appr}_s^{modsingleton}(X)$ is Rpredecessor definable for any set X. To prove that we will use the fact that reflexivity of R is a sufficient condition for sets $\overline{appr}_s^{singleton}(X)$ to be Rpredecessor defin able and that sets $\overline{appr}_s^{modsingleton}(X)$ and $\overline{appr}_s^{singleton}(X)$ are equal. On the other hand, the example from Section VIIE shows that for a not reflexive relation R, the set $\overline{appr}_s^{modsingleton}(X)$ may be not Rpredecessor definable. Indeed, $1 \in \overline{appr}_s^{modsingleton}(X)$ but $1 \notin R_p(x)$, for any $x \in U$.

C. Definability of Subset Approximations

As mentioned at the beginning of Section VI, sets $\underline{appr}_s^{subset}(X)$ and $\overline{appr}_s^{subset}(X)$ are Rsuccessor definable for any set X and relation R. It follows directly from defini tions of these approximations. Obviously, if R is symmetric, then Rsuccessor subset approximations (lower and upper) are Rpredecessor definable. On the other hand the example from Section VIID proves that for a not symmetric relation R, the Rsuccessor subset approximations (lower and upper) may be not Rpredecessor definable. Indeed, $1 \in \underline{appr}_s^{subset}(X)$, but $1 \notin \underline{appr}_p^{subset}(X)$, because the element 1 occurs in R predecessor sets with the element 2 which does not belong to $\underline{appr}_s^{subset}(X)$. The element $3 \in \overline{appr}_s^{subset}(X)$ but $3 \notin \overline{appr}_p^{subset}(X)$ because the element 4 occurs always with 3 in Rpredecessor sets.

D. Definability of Concept Approximations

Definability of concept approximations requires the same properties of a relation R as in the case of subset approx imations. Namely, directly from definitions, $\underline{appr}_s^{concept}(X)$ and $\overline{appr}_s^{concept}(X)$ are Rsuccessor definable for any set X and relation R. If R is symmetric, then Rsuccessor concept lower approximation and Rconcept successor upper approxi mation are Rpredecessor definable because of equality of R successor and Rpredecessor sets. The example from Section VIID confirms the necessity of symmetry of a relation R. The element $1 \in \underline{appr}_s^{concept}(X)$ and $1 \in \overline{appr}_s^{concept}(X)$ but it occurs in Rpredecessor sets together with the element 2 which neither belongs to $\underline{appr}_s^{concept}(X)$ nor to $\overline{appr}_s^{concept}(X)$.

E. Definability of Dual Subset Approximations

If a relation R is an equivalence relation then the sets $\underline{appr}_s^{dualsubset}(X)$ and $\overline{appr}_s^{dualsubset}(X)$ are Rsuccessor definable. Indeed, in this case a relation R partitions uni verse U. Directly from definitions it follows that the sets $\underline{appr}_s^{dualsubset}(X)$ and $\overline{appr}_s^{dualsubset}(X)$ are complementary to sets $\overline{appr}_s^{subset}(\neg X)$ and $\underline{appr}_s^{subset}(\neg X)$ that are R successor definable. On the other hand one may check that for a not an equivalence relation relation R those sets may be not Rsuccessor definable. The example from Section VII C shows that $\underline{appr}_s^{dualsubset}(X)$ is not Rsuccessor definable because $\underline{appr}_s^{dualsubset}(X) = \{9\}$, and 9 occurs always with 8. In the same example set $\overline{appr}_s^{dualsubset}(X)$ is not R successor definable either, because $7 \in \overline{appr}_s^{dualsubset}(X)$ and 7 occurs always together with 6 which does not be long to $\overline{appr}_s^{dualsubset}(X)$. The example from Section VII E shows that for a symmetric and transitive relation R $1 \in \underline{appr}_s^{dualsubset}(X)$ and $1 \in \overline{appr}_s^{dualsubset}(X)$, but the element does not occur in any Rsuccessor set, thus sets $\underline{appr}_s^{dualsubset}(X)$ and $\overline{appr}_s^{dualsubset}(X)$ are not R successor definable. The example from Section VIID may be used to check that for a reflexive and transitive relation R dual successor approximations may be not Rsuccessor definable. The set $\underline{appr}_s^{dualsubset}(X) = \{7\}$ is not Rsuccessor definable because the element 7 occurs in Rsuccessor sets together with the element 6. Set $\overline{appr}_s^{dualsubset}(X) = \{1, 2, 5, 6, 7\}$ is not Rsuccessor definable because the element 2 occurs in Rsuccessor sets together with the element 3 which does not belong to set $\overline{appr}_s^{dualsubset}(X)$.

A sufficient condition of the Rpredecessor definability of dual subset successor approximations (lower and upper) is reflexivity and transitivity of a relation R. At the beginning of the proof let us notice that because of reflexivity of R, every element $x \in U$ occurs in at least one Rpredecessor set. Let us assume there is an element $x \in \underline{appr}_s^{dualsubset}(X)$ such that it always occurs in Rpredecessor sets together with elements from outside of set $\underline{appr}_s^{dualsubset}(X)$. Such assumption implies there exists element $y \neq x$, such that $y \notin \underline{appr}_s^{dualsubset}(X)$ and $x \in R_p(y)$. If $y \notin X$ then $x \notin \underline{appr}_s^{dualsubset}(X)$ and that contradiction ends the proof. If $y \in X$ then there exists $z \notin X$, such that $z \in R_p(y)$. But in such a situation transitivity of R implies $z \in R_p(x)$. It means that $x \in \overline{appr}_s^{subset}(\neg X)$, so x cannot be an ele ment of $\underline{appr}_s^{dualsubset}(X)$. This observation ends the proof. Let us start the proof for a Rdual subset successor upper approximation in the same way as above, i.e., with obser vation of membership of every element x in at least one R predecessor set and with assuming the existence of an element $x \in \overline{appr}_s^{dualsubset}(X)$ which always occurs in Rpredecessor sets with elements from outside of $\overline{appr}_s^{dualsubset}(X)$. Such assumptions imply that $R_p(x) \neq x$, i.e., there exists an element y, such that $y \in R_p(x)$ and $y \notin \overline{appr}_s^{dualsubset}(X)$. Thus $y \in \underline{appr}_s^{subset}(\neg X)$. But $y \in R_p(x)$, and that means that $x \in R_s(y)$ and now it is easy to see that y cannot belong to $\underline{appr}_s^{subset}(\neg X)$. This observation leads to the following: if $y \in R_s(z)$ for any $z \in (\neg X)$ than $R_s(z) \nsubseteq \underline{appr}_s^{subset}(\neg X)$. Such situation arises since $x \in R_s(y)$, $y \in R_s(z)$ and since transitivity of R.

Symmetry of a relation R causes that the same examples from Sections VIIC and VIIE that were used to illustrate the case of Rsuccessor definability of Rdual subset successor approximations shows that R dual subset successor approxi mations (lower or upper) may be not Rpredecessor definable if R is not reflexive and transitive, simultaneously.

F. Definability of Dual Concept Approximations

If a relation R is an equivalence relation, then $\underline{appr}_s^{dualconcept}(X)$ and $\overline{appr}_s^{dualconcept}(X)$ are Rsuccessor definable.

Proof of this property is analogous to the respective proof for the Rdual subset approximations case. We use the fact that an equivalence relation partitions universe and Rdual concept approximations are complementary sets to respective Rconcept approximations. If R is not an equivalence relation then Rdual concept approximations (lower or upper) may be not Rsuccessor definable. To show that let us consider three examples for Rdual concept lower approximation. For the first one from Section VIIC set $\underline{appr}_s^{dualconcept}(X) = \{2, 8, 9\}$ is not Rsuccessor definable because the ele ment 2 occurs in Rsuccessor sets together with the ele ments 1 or 3 and none of them belong to set $\{2, 8, 9\}$. The second example comes from Section VIID. Now, set $\underline{appr}_s^{dualconcept}(X) = \{7\}$ is not Rsuccessor definable be cause there is not Rsuccessor set $\{7\}$. For this two cases relation R is reflexive, so $\underline{appr}_s^{concept}(X) = \underline{appr}_s^{subset}(X)$ and thus $\overline{appr}_s^{dualconcept}(X) = \overline{appr}_s^{dualsubset}(X)$. We proved that for a not equivalence relation R set $\overline{appr}_s^{dualsubset}(X)$ may be not Rsuccessor definable. The last example comes from Section VIIE. Now, the element $1 \in \underline{appr}_s^{dualconcept}(X)$ and the element does not belong to any Rsuccessor set. The same element makes set $\overline{appr}_s^{dualconcept}(X)$ Rsuccessor undefinable.

If a relation R is reflexive and transitive then the sets $\underline{appr}_s^{dualconcept}(X)$ and $\overline{appr}_s^{dualconcept}(X)$ are R predecessor definable.

The proof of that fact for set $\overline{appr}_s^{dualconcept}(X)$ is based on the equality of $\underline{appr}_s^{concept}(X) = \underline{appr}_s^{subset}(X)$ for a reflexive relation R and on Rpredecessor definability of $\overline{appr}_s^{subset}(X)$ if R is reflexive and transitive. The proof for the set $\underline{appr}_s^{dualconcept}(X)$ is similar to the part of respective proof for $\underline{appr}_s^{dualsubset}(X)$ in which the element $y \in X$ is assumed (c.f. Section VIE),

G. Summary

Table I summarizes all described results for the Rsuccessor and Rpredecessor definability. The following notation is used: r denotes reflexivity, s denotes symmetry, t denotes transitivity and "any" denotes lack of constrains on a relation R that are needed to guarantee given kind of definability of an arbitrary subset of the universe.

VII. Illustrative Examples

A. Reflexive Relations

Example. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $R = \{(1, 1), (1, 8), (2, 2), (2, 8), (3, 3), (4, 4), (4, 8), (5, 4), (5, 5), (5, 8), (6, 6), (6, 8), (7, 7), (8, 2), (8, 4), (8, 6), (8, 8)\}$. $R_s(x)$ and $R_p(x)$, for $x \in U$, are the following sets

TABLE I
SUMMARY RESULTS OF ROUGH APPROXIMATIONS DEFINABILITY

Approximation	Rsuccessor def.	Rpredecessor def.
$\underline{appr}_s^{singleton}(X)$	$r \wedge t$	$r \wedge s \wedge t$
$\underline{appr}_p^{singleton}(X)$	$r \wedge s \wedge t$	$r \wedge t$
$\overline{appr}_s^{singleton}(X)$	s	$r \vee s$
$\overline{appr}_p^{singleton}(X)$	$r \vee s$	s
$\underline{appr}_s^{modsingleton}(X)$	$r \wedge t \vee s \wedge t$	$s \wedge t$
$\underline{appr}_p^{modsingleton}(X)$	$s \wedge t$	$r \wedge t \vee s \wedge t$
$\overline{appr}_s^{modsingleton}(X)$	$r \wedge s$	r
$\overline{appr}_p^{modsingleton}(X)$	r	$r \wedge s$
$\underline{appr}_s^{subset}(X)$	any	s
$\underline{appr}_p^{subset}(X)$	s	any
$\overline{appr}_s^{subset}(X)$	any	s
$\overline{appr}_p^{subset}(X)$	s	any
$\underline{appr}_s^{dualsubset}(X)$	$r \wedge s \wedge t$	$r \wedge t$
$\underline{appr}_p^{dualsubset}(X)$	$r \wedge t$	$r \wedge s \wedge t$
$\overline{appr}_s^{dualsubset}(X)$	$r \wedge s \wedge t$	$r \wedge t$
$\overline{appr}_p^{dualsubset}(X)$	$r \wedge t$	$r \wedge s \wedge t$
$\underline{appr}_s^{concept}(X)$	any	s
$\underline{appr}_p^{concept}(X)$	s	any
$\overline{appr}_s^{concept}(X)$	any	s
$\overline{appr}_p^{concept}(X)$	s	any
$\underline{appr}_s^{dualconcept}(X)$	$r \wedge s \wedge t$	$r \wedge t$
$\underline{appr}_p^{dualconcept}(X)$	$r \wedge t$	$r \wedge s \wedge t$
$\overline{appr}_s^{dualconcept}(X)$	$r \wedge s \wedge t$	$r \wedge t$
$\overline{appr}_p^{dualconcept}(X)$	$r \wedge t$	$r \wedge s \wedge t$

$R_s(1) = \{1, 8\}$,
$R_s(2) = \{2, 8\}$,
$R_s(3) = \{3\}$,
$R_s(4) = \{4, 8\}$,
$R_s(5) = \{4, 5, 8\}$,
$R_s(6) = \{6, 8\}$,
$R_s(7) = \{7\}$, and
$R_s(8) = \{2, 4, 6, 8\}$.

Moreover,
$R_p(1) = \{1\}$,
$R_p(2) = \{2, 8\}$,
$R_p(3) = \{3\}$,
$R_p(4) = \{4, 5, 8\}$,
$R_p(5) = \{5\}$,
$R_p(6) = \{6, 8\}$,
$R_p(7) = \{7\}$, and
$R_p(8) = \{1, 2, 4, 5, 6, 8\}$.

B. Transitive Relations

Let U = {1, 2, 3, 4, 5, 6}, R = {(1, 2), (1, 4), (2, 4), (3, 2), (3, 4), (5, 6)} and X = {1, 2, 3, 6}. $R_s(x)$ and $R_p(x)$, for $x \in U$, are the following sets
$R_s(1) = \{2, 4\}$,
$R_s(2) = \{4\}$,
$R_s(3) = \{2, 3\}$,
$R_s(4) = R_s(6) = \emptyset$,
$R_s(5) = \{6\}$,

Moreover
$R_p(1) = R_p(3) = R_p(5) = \emptyset$,
$R_p(2) = \{1, 3\}$,
$R_p(4) = \{1, 2, 3\}$,
$R_p(6) = \{5\}$.

C. Reflexive and Symmetric Relations

Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, R = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (3, 8), (4, 1), (4, 3), (4, 4), (5, 5), (5, 6), (6, 5), (6, 6), (6, 7), (7, 7), (8, 3), (8, 8), (8, 9), (9, 8), (9, 9)} and X = {1, 2, 3, 7, 8, 9}. $R_s(x)$ and $R_p(x)$, for $x \in U$, are the following sets
$R_s(1) = R_p(1) = \{1, 2, 4\}$,
$R_s(2) = R_p(2) = \{1, 2, 3\}$,
$R_s(3) = R_p(3) = \{2, 3, 4, 8\}$,
$R_s(4) = R_p(4) = \{1, 3, 4\}$
$R_s(5) = R_p(5) = \{5, 6\}$,
$R_s(6) = R_p(6) = \{5, 6, 7\}$,
$R_s(7) = R_p(7) = \{6, 7\}$,
$R_s(8) = R_p(8) = \{3, 8, 9\}$,
$R_s(9) = R_p(9) = \{8, 9\}$.

D. Reflexive and Transitive Relations

Let U = {1, 2, 3, 4, 5, 6, 7 }, R = {(1, 1), (2, 1), (2, 2), (2, 3), (3, 3), (4, 3), (4, 4), (5, 5), (5, 6), (6, 6), (7, 6), (7, 7)} and X = {1, 6, 7}. $R_s(x)$ and $R_p(x)$, for $x \in U$, are the following sets
$R_s(1) = \{1\}$,
$R_s(2) = \{1, 2, 3\}$,
$R_s(3) = \{3\}$
$R_s(4) = \{3, 4\}$
$R_s(5) = \{5, 6\}$,
$R_s(6) = \{6\}$,
$R_s(7) = \{6, 7\}$.

Moreover
$R_p(1) = \{1, 2\}$,
$R_p(2) = \{2\}$,
$R_p(3) = \{2, 3, 4\}$
$R_s(4) = \{4\}$,
$R_s(5) = \{5\}$,
$R_s(6) = \{5, 6, 7\}$,
$R_s(7) = \{7\}$.

E. Symmetric and Transitive Relations

Let U = {1, 2, 3}, R = {(2, 2), (2, 3), (3, 2), (3, 3)} and X = { 2}. $R_s(x)$ and $R_p(x)$, for $x \in U$, are the following sets
$R_s(1) = R_p(1) = \emptyset$,
$R_s(2) = R_p(2) = R_s(3) = R_p(3) = \{2, 3\}$.

VIII. CONCLUSIONS

In this paper we studied twenty four approximations defined for any binary relation R on universe U, where R is not necessarily reflexive, symmetric or transitive. Our main focus was on definability of a subset X of U. We checked which approximations of X are, in general, definable. When relation R is reflexive, some of these approximations coalesce. As a result, in general, only fourteen different approximations are possible for reflexive relations. Similar results are presented for relations that are combinations of reflexive, symmetric, or transitive relations.

ACKNOWLEDGMENT

This research has been partially supported by the Ministry of Scientific Research and Information Technology of the Republic of Poland, grant 3 T11C 005 28.

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