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Nonequilibrium phase transitions in stochastic systems with and without time delay: controlling various attractors with noise

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ABSTRACT

Statistical behavior of ensembles of nonlinearly coupled elements driven by external noise is studied on the basis of nonlinear Fokker-Planck equations. The models incorporate two kinds of noise, the Langevin noise and the colored noise introduced in the coupling strength, and time delays. Various types of nonequilibrium phase transitions including Hopf bifurcations and transitions between limit cycle and chaos are shown to occur as the noise level is changed.

I. INTRODUCTION

In recent years nonlinear phenomena involving noise have been attracting much attention in wide areas of science and technology because of their ubiquity and potential applicability [1, 2]. The concept of noise, indeed,plays an important role not only in statistical physics and its related fields of physical science but also in wider research areas of natural as well as social sciences. In statistical mechanics, noise is often introduced into dynamical equations for physical systems in contact with a heat bath to study irreversible phenomena such as relaxation processes to equilibrium states. One then is led to deal with a stochastic process such as the one described by a stochastic differential equation or Langevin equation, where mostly Markovian approximations are considered for the sake of convenience.

Noise playing the role of temperature is usually viewed as deteriorating the degree of coherence or order that is generated by a certain type of interactions between elements in physical systems. A typical example is thermodynamic phase transition phenomena such as ferromagnetic-paramagnetic transitions, where the spontaneous magnetization that occurs as a result of ferromagnetic couplings between spins or a certain nonlinear elements decreases to vanish, as temperature is increased beyond the Curie temperature.

A favorable aspect of effects of noise arising from its counterintuitive influences, however, is known to manifest itself in nonlinear phenomena such as phase synchronization of chaotic systems [6] and stochastic resonance [2, 4]. When a coupled system of two chaotic oscillators is subjected to noise, a certain amount of noise can assist the generating of phase synchronization of chaos [6].

On the other hand, phenomena of stochastic resonance are observed in such a way that noise can enhance the response of a nonlinear system to a weak time-periodic signal under certain conditions [2, 3]. The stochastic resonance, which shows wide applicability in engineering as well as biological sciences, has often been studied on stochastic systems exhibiting bistability. The degree of stochastic resonance is known to be enhanced in coupled bistable systems [5]. The mechanism underlying such enhancement can be attributed to the occurrence of a phase transition [2, 7].

Then studying effects of noise on coupled nonlinear systems, especially on those exhibiting phase transitions is of interest in various kinds of engineering applications. As is mentioned above, a typical type of coupled nonlinear systems exhibiting phase transition phenomena is that of stochastic bistable elements with ferromagnetic couplings, where the concept of spontaneous symmetry breaking plays an important role.

To capture essential features of the spontaneous symmetry breaking transitions, one conveniently uses the so-called mean field model that incorporates all-to-all couplings, which simplifies the matter considerably:

$$\frac{dx_i}{dt} = x_i - x_i^3 + \frac{\epsilon}{N}\sum_{j=1}^{N}(x_j - x_i) + f_i(t) \qquad i = 1, \cdots N \tag{1}$$

where N represents the total number of bistable elements, $\epsilon > 0$ the mean field coupling strength, and the Langevin forces satisfy $\langle f_i(t) f_j(t)\rangle = 2D\delta(t-t')\delta_{ij}$ $(D > 0)$.

The N-body Fokker-Planck equation corresponding to the set of Langevin equation (1) is a linear equation in the probability density. It exhibits an ergodic property of the system in accordance with the well known

H-theorem that ensures convergence to a unique equilibrium density [1].

Taking the limit $N \to \infty$ first on the N-body master equation drastically changes the ergodic properties of the system to enable one to write down a single-body nonlinear Fokker-Planck equation (hereafter referred to NFPE) that is nonlinear in the probability density [7–9]:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[\left((1-\epsilon)\,x - x^3 + \epsilon\int xpdx\right)p\right] + D\frac{\partial^2 p}{\partial x^2} \tag{2}$$

The occurrence of phase transitions, which is a result of the competition between cooperative action due to the interactions and dissipative action of thermal noise, can be well described in terms of such NFPE. It emerges as a direct consequence of applying the law of large numbers based on the scheme of the mean field coupling. Unlike the linear Fokker-Planck equation the NFPE is no longer expected to exhibit ergodicity and hence gives rise to the occurrence of bifurcations of solutions.

According to the types of interactions taken and of the basic structures of the system such as the degree of freedom of the Langevin equations, a rich variety of bifurcations , in general, may occur [10].

Incorporating nonlinear couplings to induce phase transitions, instead of diffusive type couplings as taken in the standard ferromagnetic model, will be of interest, since such nonlinear couplings as a sigmoidal function are often dealt with in neural network models of analog neurons [11].

Furthermore, systems with time delays are ubiquitously found in the real world and have many applications in engineering problems [12, 15, 16]. Phase transitions in stochastic delay systems, however, have been less studied. In this article we deal with several types of NFPEs, including a generalized NFPE corresponding to stochastic delay differential equations, for solvable models that incorporate nonlinear couplings to study a variety of noise driven phenomena that are brought about as a result of the occurrence of various kinds of phase transitions.

II. THERMODYNAMIC TYPE PHASE TRANSITIONS

We consider a system of N-elements coupled via nonlinear global interactions subjected to white noise whose dynamics is described by a set of Langevin equations:

$$\frac{dx_i}{dt} = -x_i + \sum_{j=1}^{N} J_{ij}V(x_j) + f_i(t) \quad ,i = 1\ldots N \tag{3}$$

,where the Langevin noise $f_i(t)$ is given in the same way as in eq.(1) and $V(x)$ denotes an appropriate nonlinear function specifying the nonlinear couplings. We assume boundedness of the $V(x)$ such as in $V(x) = \tanh(\beta x)$ and $V(x) = \sin(x)$ for the sake of simplicity. The mean field coupling strengths J_{ij} of interest may be given either by the ferromagnetic type $J_{ij} = \epsilon/N$ or $J_{ij} = \epsilon/N\sum_{\mu=1}^{p}\xi_i^\mu\xi_j^\mu$, which is often used as representing synapse efficacies based on the Hebb learning rule in models of associative memory neural networks with $\xi_i^\mu(=\pm 1, \mu = 1\ldots p)$ representing memory patterns.

Choosing a sigmoidal function for the $V(x)$ leads to network model equations of analog neurons, which were extensively studied in the case without noise. In particular, by taking the coupling strengths J_{ij} with the memory patterns ξ_i^μ analyzes of the properties of associative memory models were conducted to quantitatively investigate behaviors of the retrieval state that occurs as a result of phase transitions [11, 13, 14].

In the present paper, for the sake of simplicity, we assume that $V(x)$ is odd for the purpose of investigating the phenomenon of spontaneous symmetry breaking and that the coupling strengths take the form of the ferromagnetic type : $J_{ij} = \epsilon/N$. When taking the limit $N \to \infty$, the N-body Fokker-Planck equation corresponding to eq. (3) reduces to the following single body NFPE for the empirical probability density:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[\{-x + \epsilon X(t)\}p\right] + D\frac{\partial^2 p}{\partial x^2} \tag{4}$$

$$X(t) = \int V(x)p(x,t)dx \tag{5}$$

It is noted that this equation has a characteristic feature that one can separate between the motions of the first moment $\langle x\rangle_t$ and the second moment of the probability density around the mean, although the former is affected by the latter via the term $X(t)$ arising from the couplings. Indeed, setting

$$x = \langle x\rangle_t + z \tag{6}$$

we have the time evolution equation for the mean:

$$\frac{d}{dt}\langle x\rangle_t = -\langle x\rangle_t + \epsilon X(t) \tag{7}$$

and for the probability density $\tilde{p}(z,t)$:

$$\frac{\partial \tilde{p}(z,t)}{\partial t} = -\frac{\partial}{\partial z}\left[-z\tilde{p}\right] + D\frac{\partial^2 \tilde{p}}{\partial z^2} \tag{8}$$

We see that long time behaviors of the system can be described by the dynamics of the mean $\langle x\rangle_t$, since the linear Fokker-Planck equation of the O-U process simply yields a Gaussian distribution for long times. Assuming the initial condition $\tilde{p}(z,t) = \delta(z)$ one has

$$\tilde{p}(z,t) = \frac{1}{\sqrt{2\pi S(t)}}\exp\left(-\frac{z^2}{2S(t)}\right) \tag{9}$$

with $S(t) = D\left(1 - e^{-2t}\right)$. Then $X(t)$ is given as

$$X(t) = \int V\left(z + \langle x \rangle_t\right) \frac{1}{\sqrt{2\pi S(t)}} \exp\left(-\frac{z^2}{2S(t)}\right) dz \tag{10}$$

As far as one considers the system to be started with the initial condition $p(x,t) = \delta(x - x_0)$, the time evolution of the mean $\langle x \rangle_t$ can exhaustively be described by the equations (7) and (10) with an initial condition $\langle x \rangle_t = x_0$.

Equilibrium solutions to eq.(7) is given by solving $\langle x \rangle_{t=\infty} = \epsilon X(t = \infty)$. Since $\langle x \rangle_t = 0$ satisfies the equation (7) with (10), we expand the nonlinear term $X(t)$ in eq.(7) with respect to $\langle x \rangle_t$ around $\langle x \rangle_{t=\infty} = 0$ to obtain the approximated equation up to third order of $\langle x \rangle_t$:

$$\begin{aligned} \frac{d\langle x \rangle_t}{dt} &= -\langle x \rangle_t + \epsilon \langle x \rangle_t \int V'(z)\tilde{p}(z,t)dz \\ &\quad + \frac{\epsilon}{6}\langle x \rangle_t^3 \int V^{(3)}(z)\tilde{p}(z,t)dz \end{aligned} \tag{11}$$

Stability switch is seen to occur, as the noise intensity D is varied, at the point $D = D_c$ where D_c satisfies

$$-1 + \epsilon \int V'(z)\tilde{p}(z, t = \infty)dz = 0. \tag{12}$$

In other words, while for

$$-1 + \epsilon \int V'(z)\tilde{p}(z, t = \infty)dz < 0,$$

$\langle x \rangle_{t=\infty} = 0$ is stable, for

$$-1 + \epsilon \int V'(z)\tilde{p}(z, t = \infty)dz > 0$$

the $\langle x \rangle_{t=\infty} = 0$ loses its stability and the system undergoing a pitchfork bifurcation exhibits nonzero value of $\langle x \rangle_t$. Then we have a ferromagnetic-paramagnetic transition at $D = D_c$.

III. CONTROLLING LIMIT CYCLE AND CHAOTIC ATTRACTORS WITH NOISE

The model based on the one- dimensional dynamical elements in the preceding section can easily be extended to a system of multi-dimensional dynamical elements to study the occurrence of nonequilibrium phase transitions. Such a model of interest with nonlinear couplings is given by the following set of coupled Langevin equations [10]:

$$\begin{aligned} \frac{dx^{(i)}}{dt} &= -b_1 x^{(i)} \\ &+ \sum_{j=1}^{N} J_1 V_1\left(a_{11}x^{(j)} + a_{12}y^{(j)} + a_{13}z^{(j)}\right) + f_1^{(i)}(t) \\ \frac{dy^{(i)}}{dt} &= -b_2 y^{(i)} \\ &+ \sum_{j=1}^{N} J_2 V_2\left(a_{21}x^{(j)} + a_{22}y^{(j)} + a_{23}z^{(j)}\right) + f_2^{(i)}(t) \\ \frac{dz^{(i)}}{dt} &= -b_3 z^{(i)} \\ &+ \sum_{j=1}^{N} J_3 V_3\left(a_{31}x^{(j)} + a_{32}y^{(j)} + a_{33}z^{(j)}\right) + f_3^{(i)}(t), \\ &\quad i = 1 \ldots N \end{aligned} \tag{13}$$

$$\begin{aligned} &\langle f_k^{(i)}(t) f_l^{(j)}(t') \rangle \\ &= 2D_k \delta(t - t')\delta_{ij}\delta_{kl}\ (D_k > 0),\ b_1, b_2, b_3 > 0, \end{aligned} \tag{14}$$

,where a_{kl} and b_k are constants, and V_k representing nonlinear couplings is assumed to be bounded functions as before . We assume the mean field coupling strengths J_k to be given by

$$J_k = \frac{1}{N}\left(\epsilon_k + \epsilon_k^{(i)}\right)\ k = 1 \ldots 3 \tag{15}$$

where ϵ_k are constants, and $\epsilon_k^{(i)}$ represent appropriately defined colored noise in the coupling strength. For simplicity we take $\epsilon_1^{(i)} = \epsilon_2^{(i)} = 0$ and assume the Ornstein-Uhlenbeck process for $\epsilon_3^{(i)}$:

$$\begin{aligned} \frac{d}{dt}\epsilon_3^{(i)} &= -\gamma \epsilon_3^{(i)} + f_\epsilon^i(t) \\ \langle f_\epsilon^{(i)}(t) f_\epsilon^{(j)}(t') \rangle &= 2D_4 \delta(t - t')\delta_{ij}\ (\gamma > 0, D_4 \geq 0) \end{aligned} \tag{16}$$

In the absence of noise, an appropriate parameter setting for a_{kl} in eq.(13) yields limit-cycle or chaotic oscillations. Introduction of any small amount of external noise ($D_k > 0, k = 1, 2, 3$) into the system would bring about ergodicity of the stochastic system to prevent oscillatory motions of averaged physical quantities from occurring, if N is finite. Taking the thermodynamic limit $N \to \infty$ transforms the above set of equation into the NFPE describing the time evolution of the empirical probability density [10]:

$$\begin{aligned} &\frac{\partial p(t, x, y, z, \epsilon)}{\partial t} \\ &= -\frac{\partial}{\partial x}\left[\left(-b_1 x + \epsilon_1 \langle V_1 \rangle\right) p\right] - \frac{\partial}{\partial y}\left[\left(-b_2 y + \epsilon_2 \langle V_2 \rangle\right) p\right] \\ &\quad - \frac{\partial}{\partial z}\left[\left(-b_3 z + (\epsilon_3 + \epsilon)\langle V_3 \rangle\right) p\right] - \frac{\partial}{\partial \epsilon}\left[-\gamma \epsilon p\right] \\ &\quad + \left(D_1 \frac{\partial^2}{\partial x^2} + D_2 \frac{\partial^2}{\partial y^2} + D_3 \frac{\partial^2}{\partial z^2} + D_4 \frac{\partial^2}{\partial \epsilon^2}\right) p \end{aligned} \tag{17}$$

with

$$\langle V_k \rangle = \int V\left(a_{k1}x + a_{k2}y + a_{k3}z\right) p(t,x,y,z,\epsilon) dx dy dz d\epsilon \tag{18}$$

The long time behavior of the solutions to the above NFPE can be confirmed to be described by Gaussian distributions, using an H-theorem [10]. Then it suffices to evaluate the first and second moments of the variables for an exhaustive description of the macroscopic properties of the system for large times. Choosing $V_k(x) = \sin(x)$ for simplicity and numerically solving the resultant differential equation of those moments [11], we observe the occurrence of bifurcation phenomena with a change in the noise levels $D_k > 0, k = 1,2,3,4$ for two cases: (1)$D_1 = D_2 = D_3 (= D), D_4 = 0$ and (2) $D = 0, D_4 > 0$. Figure 1 depicts the variation of the largest Lyapunov exponent λ_M plotted against D in the case (1). As D is increased from $D = 0$, for which the system exhibits chaotic oscillations, the system repeatedly undergoes transitions from chaos to limit cycle as well as those of the reverse direction to eventually settle into fixed point type attractors. For sufficiently large D the system exhibits the paramagnetic phase that occurs with D passing through the ferromagnetic-paramagnetic transition point $D_{p-F} \approx 0.337$. In the case (2), where the coupling noise D_4 is present, the system also repeatedly undergoes transitions from limit cycle to chaos and then to limit cycle before finally entering a chaotic phase, as is shown in Fig.2. It is worth noting that no matter how large the value of D_4, the ultimate chaotic phase remains in existence.

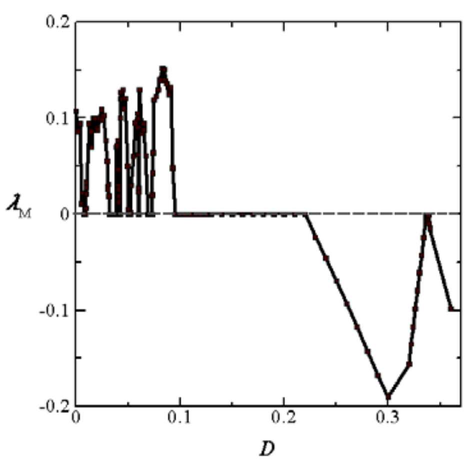

FIG. 1: The largest Liapunov exponents λ_M plotted against the noise strength D for the stochastic system (13) with $V_k(x) = \sin x$ (case(1)). A positive value of λ_M implies that the motion is chaotic. $a_{11} = 1.0, a_{12} = -1.0, a_{13} = 0.1, a_{21} = 1.0, a_{22} = 0.5, a_{23} = 0.1, a_{31} = -3.0, a_{32} = 0.6, a_{33} = 0.93, D_4 = 0, \epsilon_1 = \epsilon_2 = 1.55, \epsilon_3 = 3\epsilon_1, b_1 = b_2 = 1, b_3 = 1$.

IV. NONEQUILIBRIUM PHASE TRANSITIONS IN NONLINEARLY COUPLED STOCHASTIC SYSTEMS WITH TIME DELAYS

We turn to consider a time delayed system given by the following set of stochastic delay differential equations (SDDE)

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -b_1 x_i(t) - b_2 x_i(t-\tau) \\ &\quad + \frac{\epsilon}{N}\sum_{j=1}^{N} V(x_j(t-\tau_V)) + f_i(t),\ i = 1\dots N \\ &\langle f_i(t) f_j(t')\rangle = 2D\delta(t-t')\delta_{ij}\ (D \geq 0) \end{aligned} \tag{19}$$

where the delay times $\tau \geq 0$ and $\tau_V \geq 0$ are introduced, and $b_1 \geq 0$, $b_2 \geq 0$ and $\epsilon > 0$ are constants. We assume the initial condition to be appropriately given for -Max $\{\tau, \tau_V\} \leq t \leq 0$.

Then the generalized NFPE corresponding to eq.(19) that is obtained for the empirical probability density $p(x,t)$ in the thermodynamic limit $N \to \infty$

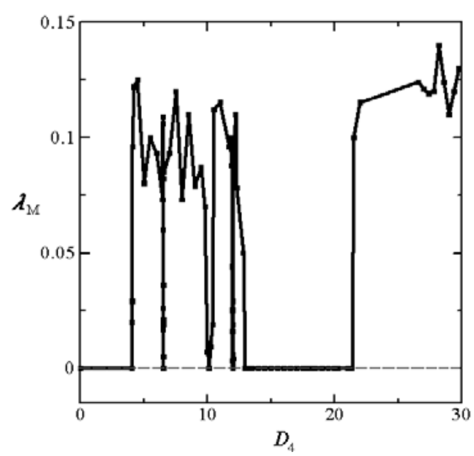

FIG. 2: The largest Liapunov exponents λ_M plotted against the noise strength D_4 for the stochastic systems (13) with $V_k(x) = \sin x$ (case(2)). $a_{ij}(i,j = 1,\cdots 3)$ are the same as in Fig.1 and $D = 0$, $\epsilon_1 = \epsilon_2 = 1.7, \epsilon_3 = 3\epsilon_1, b_1 = b_2 = 1, b_3 = 2, \gamma = 0.2$.

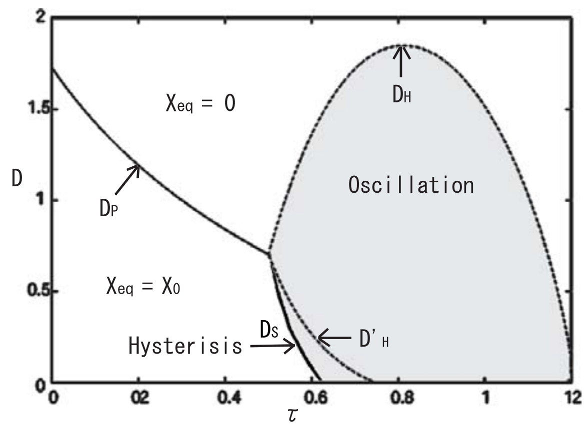

FIG. 3: Phase diagram on the time delay(τ) - noise strength(D) plane for the delayed stochastic systems (19) with $V_k(x) = \sin x,\ b_1 = 1,\ b_2 = 2,\ \epsilon = 4$. The phase boudary D_p denotes the onset of the ferromagnetic phase via a pitchfork bifurcation and D_H corresponds to a Hopf bifurcation. D'_H denotes the occurrence of a subcritical Hopf bifurcation that occurs on the branch of the nonzero fixed point solution generated via an unstable pitch-fork bifurcation. At $D = D_S$ the limit cycle oscillations undergoes a saddle node type bifurcation to vanish with a finite amplitude of oscillation. The oscillatory and ferromagnetic phases coexist in the region $D_S < D < D'_H$, leading to the occurrence of a hysterisis phenomenon.

reads

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} &= b_1\frac{\partial}{\partial x}(xp) - \frac{\partial}{\partial x}\left[\epsilon p\int V(y)p(y,t-\tau_V)dy\right] \\ &\quad -\frac{\partial}{\partial x}\left[p\int -b_2 y p(y,t-\tau|x,t)dy\right] \\ &\quad +D\frac{\partial^2}{\partial x^2}p \qquad (20)\end{aligned}$$

where $p(y,t-\tau|x,t)$ represents the conditional probability density of $X(t-\tau) = y$ at time $t-\tau$ given that $X(t) = x$ at time t [15]. The equation for the first moment $\langle x(t)\rangle$ is obtained as

$$\frac{d}{dt}\langle x(t)\rangle = -b_1\langle x(t)\rangle - b_2\langle x(t-\tau)\rangle + \epsilon\langle V\left(x(t-\tau_V)\right)\rangle \qquad (21)$$

where $\langle V\left(x(t-\tau_V)\right)\rangle$ represents the average of V with respect to $p(x, t-\tau_V)$. For large times $p(x,t)$ can be expected to be Gaussian with mean $\langle x(t)\rangle$ and variance σ^2 , which is given by [16]

$$\sigma^2 = D\frac{b_2\sin\left(\tau\sqrt{b_2^2-b_1^2}\right)+\sqrt{b_2^2-b_1^2}}{\sqrt{b_2^2-b_1^2}\left[b_1+b_2\cos\left(\tau\sqrt{b_2^2-b_1^2}\right)\right]} \qquad (22)$$

in the case of $b_1 < b_2$,with which we are concerned in the present paper. In this case the delay time τ has to satisfy

$$\tau < \tau_c = \frac{\cos^{-1}(\frac{-b_1}{b_2})}{\sqrt{b_2^2-b_1^2}} \quad ,$$

because the second moment of the fluctuation under the stationarity condition diverges to infinity as τ approaches τ_c. Setting $V(x) = \sin x$ leads to

$$\langle V(x(t-\tau_V))\rangle = \exp\left(-\frac{\sigma^2}{2}\right)\sin[x(t-\tau_V)] \quad .$$

Assuming $\tau_V = 0$ for simplicity, we solve eq.(21) for various values of τ and D with b_1, b_2 and ϵ kept fixed. We find not only fixed point type solutions but also limit cycle type ones, which are summarized into the phase diagram on the $\tau - D$ plane depicted in Fig.3.

When τ is fixed small, the system undergoes a pitchfork bifurcation as D is changed. When, on the other hand, τ is larger than a certain value a Hopf bifurcation occurs for the system to settle into a limit cycle type attractor below a critical value of D_H. The critical values of D_p and D_H where $\langle x\rangle = 0$ lose its stability can be determined from the linear stability analysis of eq.(21). We note that a hysterisis phenomenon occurs between the oscillatory phase and the ferromagnetic one.

V. CONCLUSION

We have shown that the NFPE approach is very useful for studying the occurrence of various types of phase transitions including nonequilibrium phase transitions associated with limit cycle and chaotic attractors in noisy coupled systems of large size. The results imply that noise can be employed to control the dynamical behavior of nonlinear systems in a qualitative sense by generating transitions between several types of attractors. Time delay has been found to play an important role in the sense that types of attractors expected to appear are determined by a combination of delay τ and noise level D .

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