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Collective behaviour of sparsely connected oscillator network

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***Abstract*— We study a sparsely connected oscillator network by using the methods of statistical mechanical, in particular, the replica method. First, we study the conjecture of Ichinomiya on the equivalence of a sparsely connected oscillator network with ferromagnetic interactions to a fully connected network with disordered interactions. Then, we study the dynamical behaviour of the oscillator network on the Bethe lattice by using dynamical replica theory. Theoretical results are confirmed by numerical simulations.**

I. INTRODUCTION

In recent years, much attention has been paid to interacting systems where the network of interactions between the different units is of a sparse (i.e. finite average connectivity in the infinite system limit), random and often complex nature [1], [2], [3]. This current enthusiasm has been fueled by the abundance of natural systems whose interactions appear to be generally of this sparse, random type. Among these many and varied models, we have a particular interest in systems where elements have their own dynamics, i.e. the nodes of the network are dynamic objects themselves. As an example, we have been studying a immune network dynamical system model. In this model, each element or node is composed of B-cells and antibodies with the same idiotype, and the dynamics of each element is described by a pair of ordinary differential equations (odes). One particular feature of the immune system is that each element interacts with only a finite number of other elements irrespective of the number of elements (i.e. number of idiotypes in the system), N. We have derived a partial differential equation to describe the population of B-cells and antibodies by using the dynamical replica theory[4]. However, this model is rather complicated and is difficult to analyze theoretically. It is desirable to study a model in which each element has a dynamical nature is simple enough to be able to analyze theoretically.

In this paper, as an example of such a tractable system, we study N coupled phase oscillators as introduced by Kuramoto[5]. In this model, each oscillator has a definite amplitude, and the state of a given oscillator is described by its phase $\phi \in R$. The evolution equation for phase ϕ_i of i-th oscillator is given by

$$\frac{d}{dt}\phi_i = \omega_i + \sum_{j\neq i} J_{ij}\sin(\phi_j - \phi_i) + \eta_i, \tag{1}$$

where $\eta_i(t)$ is Gaussian white noise with variance $2T$,

$$\langle \eta_i(t)\eta_j(t')\rangle = 2T\delta_{ij}\delta(t-t'). \tag{2}$$

In this paper, we examine the case where $\omega_i = \omega$ for any i. Then, without loss of generality, we may assume $\omega = 0$. Further, we assume $J_{ij} = J_{ji}$. Then eq. (1) can be rewritten as

$$\frac{d}{dt}\phi_i = -\frac{\partial}{\partial\phi_i}H + \eta_i, \tag{3}$$

$$H = -\sum_{i<j} J_{ij}\cos(\phi_i - \phi_j). \tag{4}$$

Thus our assumptions allow us to to investigate a Hamiltonian system.

Now, we define three models by giving different specifications to the $\{J_{ij}\}$,

Model A $\quad P(J_{ij}) = \frac{c}{N}\delta_{J_{ij},J} + (1-\frac{c}{N})\delta_{J_{ij},0} \quad i<j \qquad (5)$

Model B $\quad P(J_{ij}) = J(\frac{c}{N} + \sqrt{\frac{c}{N}}z_{ij}) \quad i<j \qquad (6)$

Model C $\quad \sum_j J_{ij} = cJ \quad \text{for any } i, J_{ij} = J_{ji}, \quad J_{ij} \in \{0, J\}. \qquad (7)$

where $z_{ij} = z_{ji}$ are identically independently distributed Gaussian zero mean, unit variance random variables.

In particular, we focus on the following result of Ichimomiya [6]: the sparse random network with finite connectivity (model A) behaves similarly to the fully connected model with disordered bonds (model B). This can be seen intuitively, since model B describes interactions on an Erdös-Reyni random graph, which has average connectivity c and variance in connectivity c so the first two moments of the interaction strength agree with model A. In fact, in the large c limit, the Poisson distributed number of bonds can be approximated by a Gaussian distributed number of bonds and since the first two moments agree the distributions will converge (as $c \to \infty$).

We examine Ichinomiya's result in the more interesting regime of finite c comparing phase diagrams and the order parameters in these three models. Further, we examine the dynamical behaviour of model C using dynamical replica theory. Also, we perform numerical simulations and compare our theoretical and numerical results.

The paper is organized as follows. In §2, we analyze statics of three models and examine Ichinomiya's result. In §3, we investigate dynamical behaviour of the model C. A summary is given in §4.

II. Static behaviour of our three models

The probability density of $\boldsymbol{\phi}$ at time t, $P_t(\boldsymbol{\phi})$ obeys the following Fokker-Planck equation,

$$\frac{\partial}{\partial t}P_t(\boldsymbol{\phi}) = \sum_i \frac{\partial}{\partial \phi_i}(\frac{\partial}{\partial \phi_i}H)P_t(\boldsymbol{\phi}) + T\sum_i \frac{\partial^2}{\partial \phi_i^2}P_t(\boldsymbol{\phi}). \quad (8)$$

Thus, the stationary state of the equation is given by the canonical distribution $P_{\rm eq}(\boldsymbol{\phi})$,

$$P_{\rm eq}(\boldsymbol{\phi}) = \frac{1}{Z}e^{-\beta H}, \quad (9)$$

$$Z = \int d\boldsymbol{\phi} e^{-\beta H}, \quad (10)$$

where $\beta = 1/T$. Thus, to examine the statics of the models, we can use the methods of statistical mechanics.

We define order parameters as follows,

$$m = \sqrt{m_c^2 + m_s^2}, \quad (11)$$

$$m_s = \frac{1}{N}\sum_i \overline{\langle \sin(\phi_i)\rangle}, \quad (12)$$

$$m_c = \frac{1}{N}\sum_i \overline{\langle \cos(\phi_i)\rangle}, \quad (13)$$

and

$$Q_{cc} = \frac{1}{N}\sum_i \overline{\langle \cos^2(\phi_i)\rangle}, \quad Q_{ss} = \frac{1}{N}\sum_i \overline{\langle \sin^2(\phi_i)\rangle},$$
$$Q_{cs} = Q_{sc} = \frac{1}{N}\sum_i \overline{\langle \sin(\phi_i)\cos(\phi_i)\rangle},$$
$$q_{cc} = \frac{1}{N}\sum_i \overline{\langle \cos(\phi_i)\rangle^2}, \quad q_{ss} = \frac{1}{N}\sum_i \overline{\langle \sin(\phi_i)\rangle^2},$$
$$q_{cs} = q_{sc} = \frac{1}{N}\sum_i \overline{\langle \sin(\phi_i)\rangle\langle \cos(\phi_i)\rangle}.$$

where $\langle\cdots\rangle$ denotes the thermal average and $\overline{\cdots}$ denotes the average over the quenched randomness $\{J_{ij}\}$.

The statics of the models A and C have been studied previously[7], [8] and in terms of m, the transition temperatures between the ordered phase of $m > 0$ (F) to the disordered phase $m = 0$ (P) are given by

$$\text{Model A} \qquad c = \frac{I_0(\beta J)}{I_1(\beta J)}, \quad (14)$$

$$\text{Model C} \qquad c - 1 = \frac{I_0(\beta J)}{I_1(\beta J)}, \quad (15)$$

where I_n are the modified Bessel functions and are defined as

$$I_n(z) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\cos(n\phi)e^{z\cos(\phi)}.$$

So, in this section, we focus on the analysis of the model B. This model is the XY version of the Sherrington-Kirkpatrick (SK) model of the Ising spin-glass[9], [10] and can be solved by using the replica method [11]. The disorder averaged free energy per oscillator $\overline{f} = -\lim_{N\to\infty}(\beta N)^{-1}\overline{\ln Z}$ is calculated as

$$\begin{aligned}\overline{f} = & -\frac{c\beta}{4}[q_{cc}^2 + q_{ss}^2 + 2q_{sc}^2 - 1] - \frac{c}{2}(m_c^2 + m_s^2) \\ & -\frac{c\beta}{2}[Q_{cc}(Q_{cc}-1) + Q_{sc}^2] \\ & +\frac{1}{\beta}\int DxDy \ln \int d\phi M(\phi|x,y). \end{aligned} \quad (16)$$

where

$$\begin{aligned} M(\phi|x,y) = & \exp[c\beta m_c\cos(\phi) + c\beta\sin(\phi) \\ & +\frac{1}{2}c\beta^2(1 - Q_{cc} - q_{ss}) \\ & +\frac{1}{2}c\beta^2\cos^2(\phi)(2Q_{cc} - q_{cc} + q_{ss} - 1) \\ & +c\beta^2(Q_{sc} - q_{sc})\sin(\phi)\cos(\phi) \\ & +\beta x\sqrt{c\frac{q_{cc}q_{ss} - q_{sc}^2}{q_{ss}}}\cos(\phi) \\ & +\sqrt{q_{ss}}\sin(\phi)], \end{aligned} \quad (17)$$

and

$$\begin{aligned} & m_s = [\langle\sin(\phi)\rangle], \quad m_c = [\langle\cos(\phi)\rangle], \\ & Q_{cc} = [\langle\cos^2(\phi)\rangle] \quad Q_{ss} = [\langle\sin^2(\phi)\rangle], \\ & Q_{cs} = Q_{sc} = [\langle\sin(\phi)\cos(\phi)\rangle], \\ & q_{cc} = [\langle\cos(\phi)\rangle^2], \quad q_{ss} = [\langle\sin(\phi)\rangle^2], \\ & q_{cs} = q_{sc} = [\langle\sin(\phi)\rangle\langle\cos(\phi)\rangle], \\ & [\cdots] = \int DxDy\cdots, \\ & \langle\cdots\rangle = \frac{\int d\phi M(\phi|x,y)\cdots}{\int d\phi M(\phi|x,y)}. \end{aligned}$$

We set $J = 1$. The critical temperatures for P $\to$ F and P$\to$ SG are given by

$$\text{P} \to \text{F} \qquad T = \frac{c}{2}, \quad (18)$$

$$\text{P} \to \text{SG} \qquad T = \frac{\sqrt{c}}{2}, \quad (19)$$

where SG denotes the spin-glass phase and it is characterized by $m = 0$ and $q_{cc} > 0$. We display the critical temperatures for models A, B and C in Figs.1 and 2.

In Figs.3 and 4, we display the temperature dependence of the order parameter m for the three models.

Thus we see that by restricting our investigation to these Hamiltonian models we can measure the quality of Ichinomiya's analytically. We have control over the model in terms of both temperature and average connectivity and studies in this regime will aid inference in the more challenging case where the system is non-equilibrium in nature.

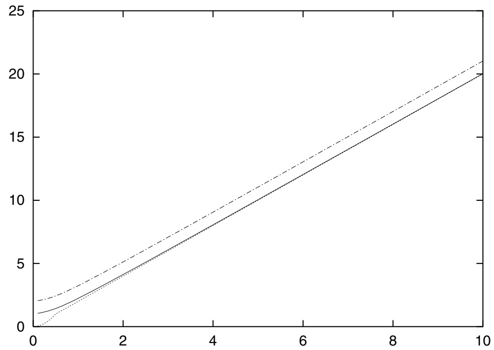

Fig. 1. Phase diagram in $T-c$ plane. Solid curve: sparse model (A model), dotted curve: fully connected model (B model), dotted-dashed curve: Bethe lattice (C model). The dotted-dashed curve is obtained by shifting the solid curve in the vertical direction by 1.

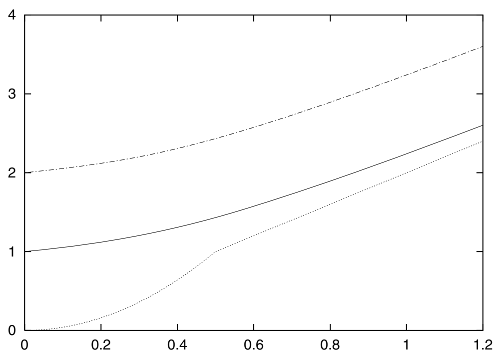

Fig. 2. Phase diagram in $T-c$ plane. Enlargement of Fig.1

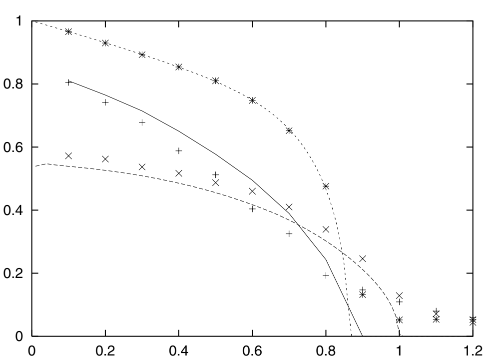

Fig. 3. T dependence of m for $c = 2$ ($c = 3$ for Bethe lattice). Curves: theoretical results. Symbols: numerical results of the Langavin eq.(1) for $N = 2000$. Solid curve and + : sparse model (A model), dashed curve and $\times$: fully connected model (B model), dotted curve and *: Bethe lattice (C model).

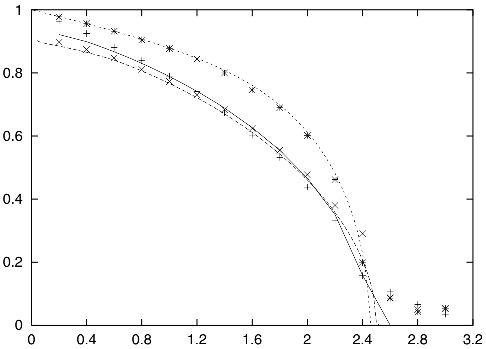

Fig. 4. T dependence of m for $c = 5$ ($c = 6$ for Bethe lattice). Curves: theoretical results. Symbols: numerical results of the Langavin eq.(1) for $N = 2000$. Solid curve and + : sparse model (A model), dashed curve and $\times$: fully connected model (B model), dotted curve and *: Bethe lattice (C model).

III. DYNAMIC BEHAVIOUR OF MODEL C

We next turn to the dynamic behaviour of model C, which we investigate using the tools of dynamical replica theory. There is no theoretical difficulty with applying dynamical replica theory to model B but the numerical solution of resulting equations is more challenging.

Dynamical replica theory provides a tool that allows us to model the dynamics approximately (in this case), but still in a finite connectivity setting, with the properties of good accuracy at short times and very long times (in fact it is exact one the system has equilibrated). Further, the level of approximation can be built up in stages allowing for further improvements, albeit at significant computational cost in terms of solving the equations.

The approach we use is to introduce a set of intensive macroscopic observables $\mathbf{\Omega}(\boldsymbol{\phi}) = (\Omega^1(\boldsymbol{\phi}), \ldots, \Omega^k(\boldsymbol{\phi}))$ and define

$$p_t(\mathbf{\Omega}) = \int \mathrm{d}\boldsymbol{\phi}\, p_t(\boldsymbol{\phi})\delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\phi})] \tag{20}$$

This distribution evolves according to a Fokker-Planck equation but it can be shown that for the choices we make for $\mathbf{\Omega}(\boldsymbol{\phi})$ the diffusion terms disappear and we are left merely with a Liouville equation for our observables describing deterministic flow in phase space. This Liouville equation is given by:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Omega^{\mu}(t) &= \\ &\langle \sum_i \left[\sum_j c_{ij}\sin(\phi_j - \phi_i) + T\frac{\partial}{\partial \phi_i} \right] \frac{\partial}{\partial \phi_i}\Omega_{\mu}(\boldsymbol{\phi}) \rangle_{\mathbf{\Omega}} \end{aligned} \tag{21}$$

where the $\langle \ldots \rangle$ denotes averaging over the microscopic probability measure $p_t(\boldsymbol{\phi})$ for those states where the macroscopic observables $\mathbf{\Omega}(\boldsymbol{\phi}) = \mathbf{\Omega}$. This evolution equation for our observables is exact but intractable since it still contains the full microscopic distribution $p_t(\boldsymbol{\phi})$. However, the approach of dynamical replica theory is to view these observables as parameters describing the probability distribution of the system, much as specifying the energy of the system describes the distribution in equilibrium statistical mechanics. This leads us to the approximation underlying dynamical replica theory: to use the maximum entropy distribution given the observable values as an approximation to the true non-equilibrium distribution $p_t(\boldsymbol{\phi})$. Alternatively, one could view this as projecting the non-equilibrium microscopic distribution onto an exponential family parameterised by these observables via minimising the KL divergence between the true microscopic probability measure and this exponential family. Hence, the average in eq. (21) simplifies under this approximation to:

$$\langle \ldots \rangle_{\boldsymbol{\Omega}} = \frac{\int \mathrm{d}\boldsymbol{\phi}\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\phi})](\ldots)}{\int \mathrm{d}\boldsymbol{\phi}\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\phi})]} \tag{22}$$

To begin with consider the simplest set of three observables that could be hoped to describe the system:

$$m_c(\boldsymbol{\phi}) = \frac{1}{N}\sum_i \cos(\phi_i) \tag{23}$$

$$m_s(\boldsymbol{\phi}) = \frac{1}{N}\sum_i \sin(\phi_i) \tag{24}$$

$$e(\boldsymbol{\phi}) = \frac{1}{N}\sum_{i<j} c_{ij}\cos(\phi_i - \phi_j) \tag{25}$$

the energy is an obvious choice, it means that the above set of equations will be exact at long times (i.e. in equilibrium) while m_c and m_s allow the description of overall ordering among the oscillators. Inserting these observables into the equation (21) in turn gives us coupled odes which describe the system's behaviour:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m_c(t) \quad &= \\ &-\frac{1}{N}\sum_i \langle \sum_j c_{ij}\sin(\phi_j - \phi_i)\sin(\phi_i)\rangle_{m_c(t),m_s(t),e(t)} \\ &-Tm_c(t)\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m_s(t) \quad &= \\ &\frac{1}{N}\sum_i \langle \sum_j c_{ij}\sin(\phi_j - \phi_i)\cos(\phi_i)\rangle_{m_c(t),m_s(t),e(t)} \\ &-Tm_s(t)\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e(t) \quad &= \quad \frac{1}{N}\sum_i \langle \left(\sum_j c_{ij}\sin(\phi_j - \phi_i)\right)^2 \rangle_{m_c(t),m_s(t),e(t)} \\ &\quad -2Te(t)\end{aligned} \tag{28}$$

The non-trivial aspect of these coupled odes is the measure. We have to average over all states $\boldsymbol{\phi}$ according to definition (22). To do this we move to the canonical framework, writing:

$$\begin{aligned}\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\phi})] \quad &= \\ &\exp[\hat{e}\sum_{i<j} c_{ij}\cos(\phi_i - \phi_j) + \hat{m}_c\sum_i \cos(\phi_i) \\ &+\hat{m}_s\sum_i \sin(\phi_i)]\end{aligned} \tag{29}$$

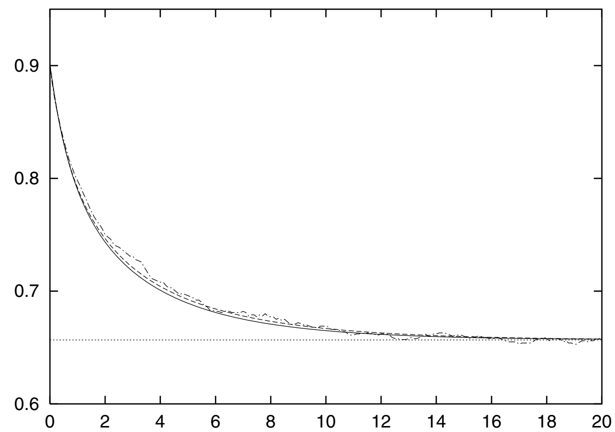

Fig. 5. Time series of the order parameter m. $T = 0.7$. Solid curve: 3DRT, dashed curve: 4DRT, dotted-dashed curve: simulation ($N = 4000$). Dotted line indicates the equilibrium value of m.

The problem is reduced to finding the triple $\{\hat{e}, \hat{m}_c, \hat{m}_s\}$ at each time t for which the canonical measure satisfies the equalities:

$$m_c = \langle m_c(\boldsymbol{\phi})\rangle_{m_c,m_s,e} \tag{30}$$

$$m_s = \langle m_s(\boldsymbol{\phi})\rangle_{m_c,m_s,e} \tag{31}$$

$$e = \langle e(\boldsymbol{\phi})\rangle_{m_c,m_s,e} \tag{32}$$

This is a highly non-linear three dimensional inverse problem and computationally this is the most challenging part of implementing dynamical replica theory for finitely connected random systems, since this inverse problem must be solved for each time t. These equations were solved using Brent's method. By solving this inverse problem, the measure can be specified at any point in time and the odes can be solved using standard numerical techniques. In Figs. 5 and 6, we compare our numerical solutions with that given by Monte Carlo simulation of the problem:

It is natural at this stage to see if more observables can be introduced. Intuitively, the richer the parameter space spanned by the observables, the closer the maximum entropy distribution will be to the microscopic probability distribution and the better the theory. To this end we make the following choice:

$$m_{ss}(\boldsymbol{\phi}) = \frac{1}{N}\sum_{ij} c_{ij}\sin(\phi_i)\sin(\phi_j - \phi_i), \tag{33}$$

which has the appealing nature that the eq. (26) becomes more explicit in terms of the observables. This additional observable

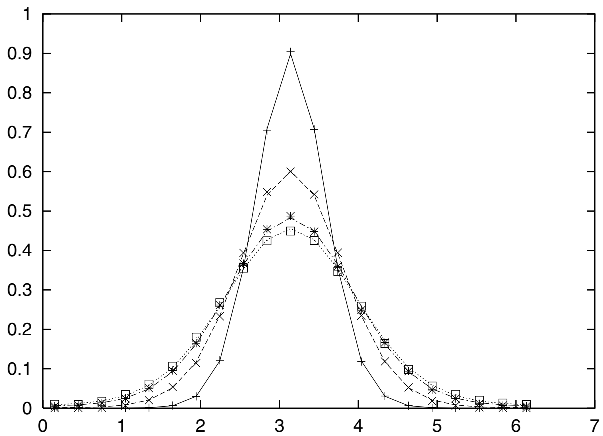

Fig. 6. Time dependence of phase distribution $P(\phi)$. $T = 0.7$. Curves are theoretical results (3DRT) and symbols are simulations for $N = 4000$. solid curve and + : $t = 0$, dashed curve and $\times$: $t = 0.1$, dotted-dashed curve and * : $t = 0.5$, dotted curve and square : $t = 2$.

changes the canonical maximum entropy distribution to:

$$\begin{aligned}\delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\phi})] \quad &= \\ &\exp[\hat{e}\sum_{i<j} c_{ij}\cos(\phi_i - \phi_j) + \hat{m}_c \sum_i \cos(\phi_i) \\ &+\hat{m}_s \sum_i \sin(\phi_i) \\ &+\hat{m}_{ss} \sum_{i<j} c_{ij}(\sin(\phi_i) - \sin(\phi_j))\sin(\phi_j - \phi_i)] \quad (34)\end{aligned}$$

In Fig.5, we see how the addition of the observable improves the theory.

The observables thus far introduced all involve single site measures or correlations between at most two neighbouring sites. It is highly likely that addition of correlations between three or more sites would significantly improve the efficacy of the method, however, this has proved to be computationally challenging to date.

IV. SUMMARY

In this paper, we studied sparsely connected oscillator networks (Model A and C) and a fully connected network (Model B) examining the conjecture by Ichinomiya on the equivalence of the Models A and B. First, we investigated statics of these models and found that the phase transition temeratures and the temperature dependence of the order parameter m for three models become more similar as the value of the connectivity c increases, and that any divergence can be measured quantitatively. We also studied the dynamical behaviour of the Model C by using dynamical replica theory. Further, we performed Monte Carlo simulations and found that our theoretical results were well confirmed by the simulations. Details of the present study will be reported elsewhere, and the dynamical behaviour for A and B models is under investigation.

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