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Stochastic Model for Adaptation Using Basin Hopping Dynamics

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Abstract — **An analysis is presented for adaptive selection of modes using bifurcation of mode-hopping dynamics in a multi-basin system. The analysis utilizes a formalism based on coarse-grained state transitions. Stochastic state transition probabilities are used to derive expressions for mode search times and adaptation rates. This approach can be applied to the tuning of dynamical instabilities in flexible control systems.**

Index Terms— **multi-stability, multi-basin, multimode, chaotic mode hopping, chaotic itinerancy, search, adaptation**

I. Introduction

A feature of living systems is their flexibility - the ability to support a variety of responses. This ability can be viewed as a combination of two component abilities - the ability to support a variety of modes of behavior and the ability to select a particular behavior, either spontaneously or using external signals which depends on the environment and situation. Fluctuations associated with dynamical instabilities could play an important role in facilitating the search and selection of modes of behavior. Hence it is of interest to consider mechanisms for adaptive selection of complex modes of behavior using perspectives from dynamical systems.

From a dynamical systems perspective, the ability to show multiple modes of behavior can be modeled by the co-existence of multiple dynamical attractors. The process of changing behavior can be described as switching among the attractors. And the process of adaptation can be described as the selective switching of a particular mode which is in someway suited to the current external environment or situation as indicated by a driving or feedback signal.

One scenario for attractor switching in response to environment changes is the direct switching of an attractor, driven by an external input signal. This for example is the idea behind neural mode recall mechanisms, as described in reference [1].

However, in many situations, the correspondence between external signals and internal attractor landscape is not well-matched, in the sense that the external signals alone do not contain enough information to drive the system to an appropriate attractor basin. Then some form of intrinsic trial-and-error search mechanism is necessary to find the basin of an attractor which is fit in the sense of having features which match the external constraints. Search among multiple co-existing attractors can be realized by using either noise or intrinsic dynamical instabilities, or some combination of the two, to make basin transitions.

If the internal dynamical landscape corresponds to the search criterion in the sense that there is a lyapunov function which corresponds to the search metric, then it is possible to apply stochastic optimization mechanisms known as annealing [2-5]. On the other hand, if there is no direct connection between the internal basin structure and the external fitness metric, then alternative mode selection mechanisms are necessary. This issue was first raised in [6]. This work and following works demonstrated that adaptive control of the onset of chaos in a multi-mode system, that is a multi-attractor system, can be used to search among the modes using arbitrary search (fitness) criteria [6-17]. Specific applications of this mechanism have been described in models of optical systems [6-8] and neural systems [16, 17]. Moreover, the mechanism has been demonstrated in physical experiments using optical devices [12-15]. Essential features of this mechanism were discussed in a general context in [9-11], where stochastic models were used to discuss characteristic search times, and a proposition that the search time could be optimized by tuning a bifurcation parameter near the edge of chaos.

In this paper we extend the theoretical analysis in a way which better facilitates analysis and comparison with other noise-driven adaptation mechanisms. We explain how the mode search and selection mechanism can be described as a stochastic state transition process. And we present a number of prototypical cases for which we derive expressions for search times and adaptation rates.

II. Adaptive Multi-Mode System

In this section we introduce the type of multi-mode systems that we will consider in this paper, and explain their key features.

We consider a nonlinear dynamical system has multiple co-existing attractors at parameter value P_0 , but only a single attractor at parameter value P_1. We shall call the co-existing attractors the "modal attractors" and the single attractor at P_1 the "base attractor". We further assume that the base attractor spans all the domains which are basins of the modal attractors. The spanning property means that the measure of the base attractor is non-zero in each modal basin.

Examples of systems which are known to have these features include nonlinear optical resonators and lasers with delayed feedback [12, 13]. These devices can have large numbers of oscillation modes which co-exist stably in separate basins at value of control parameter, but merge into a single basin with a chaotic attractor at another parameter value. Similar behavior is known in neural systems and various dynamical models such as coupled-map lattices [16-21]. The coexistence of multiple oscillatory states in neural systems has also been called "multirythmicity" [23].

We are interested in searching among the modal attractors by switching between the two parameter regimes P_0 and P_1. The reason for introducing the spanning property for the basal attractor is that we want that

(i) when we switch from P_1 to P_0, there is a finite probability of landing in any one of the multiple modal basins, and

(ii) it is possible to make the system visit all the co-existing attractors by repeatedly switching between the two regimes.

In order to assure reachability of all the basins we require a source of stochasticity in the system to spread the distribution over the basins even if the initial states are localized on the modal attractors. This could be done by adding external noise to the multi-stable system so that basin transitions occur. Alternatively, the base attractor could be a limit-cycle or chaotic base attractor which spans the basins of the modal attractors. In the case of a spanning limit cycle the phase of the cycle should be randomized by noise. A chaotic base attractor can be expected to more quickly spread an initially localized distribution over the whole base attractor.

In order to achieve the adaptation, we want to couple the parameter switching dynamics with another module which tests the performance or fitness of the system behavior. We assume a response signal is returned to the system. For simplicity, we assume the response signal is a binary signal which can be used to directly switch the parameter between P_0 and P_1. This response function can be defined without loss of generality as a binary function of the position in phase space. Note that since the external fitness test is arbitrary, the relation between the response function and the attractors or basins of the system is arbitrary. In this paper we will restrict the discussion to cases where the response does not depend on position in the modal attractor. However, the discussion could be extended to include this possibility.

III. State Transition Model

In this section we set-up a transition model describing the dynamical structure of the system and its interaction with an external response. We identify sets of points in phase space corresponding to dynamical structures and label them with symbols. In the following sections we will use these symbols to describe the coarse grained dynamics of the system during the adaptive mode search process.

A. Coarse-grained State Definitions

We define domains in phase space, and their corresponding symbols, as follows.

b_i *(i=1,.., N):* Modal basins at P_0, which partition the phase space, excluding measure-zero separatrices.

a_i *(i=1,..,N)*: Modal attractors at P_0. Each a_i is within its corresponding modal basin.

B: Base basin at P_1 . Since we assume there is only one basin at parameter value P_1 , *B* is the whole space, except for possibly a set of measure zero.

A: Base attractor at P_1 . It is assumed that *A* is the only attractor at parameter value P_1.

Y: Positive Response set. The set of points in phase space which correspond to a positive value ("Yes") of the response signal.

N: Negative Response set. The set of points in phase space which correspond to the negative value ("No") of the response signal. N is the complement of Y. We assumed that all points in a basal attractor belong to the same set Y or N.

S_T: Trap states: The subset of *Y* which at parameter P_0 lie on trajectories which subsequently stay within *Y*. (For example if a modal attractor is within Y, then it will be in the trap set S_T.)

S_{NT}: False-Alarm states. S_{NT} is the complement of S_T in *Y*. Any state in S_{NT} is on an orbit at P_0 which eventually leaves *Y*.

Note that these definitions are not all mutually exclusive, and we will on occasion, for convenience, use more than one symbol to describe a set. For example *(a_i , Y)* is used to represent the set of points in phase space which are in a_i and have response *Y*.

B. State transition rules

We will use the above symbols to describe the coarse grained dynamics of the system during the adaptive mode search process. Specifically, we will write state transition rules using a table format as follows:

<table>
<tr><td rowspan="2">x</td><td>x</td></tr>
<tr><td>y</td></tr>
<tr><td>y</td><td>y</td></tr>
</table>

This table shows that there is a transition possible from points in set x to points in set x or points in set y, and points in set y only makes a transition to points in set y.

IV. SEARCH DYNAMICS

In this section we consider the dynamics induced by changing the value of parameter between values P_0 and P_1. The idea is to use the dynamics of the base attractor A at P_1 to move among the sets b_i which are basins at parameter value P_0. For simplicity we assume that there is only one modal attractor which corresponds to a positive response. (Extending to the case of multiple good attractors is straightforward if we assume that the search ends when *any* good attractor is reached.)

A. Search by Periodic Parameter Switching

We first consider the case where parameter P is switched between P_0 and P_1 with a square wave of period $2T$, where T is much larger than the time required for convergence of the invariant measure from an initially localized distribution located anywhere in the basin. We test the response at the end of each P_0 interval, continuing the modulation of parameter until a modal attractor with positive response is reached.

Observing the state of the system at intervals $2T$, starting immediately after the response has been detected at the end of the first P_0 interval, the state transition rules can be written as follows.

(a_k,N) $k \neq k^*$	(a_j,N) $j \neq k^*$
	(a_{k^*},Y)
(a_{k^*},Y)	(a_{k^*},Y)

Here the label k^* indicates the modal attractor which corresponds to a positive response *"Y"*.

The probability of the transition $a_k \rightarrow a_j$ is the projection of the measure of A, at the time of the switch, on the basin b_j. Note that due to the assumption that time T is long enough for relaxation of the distribution on the base attractor, the probability of the transition $a_k \rightarrow a_j$ is independent of k and can be written $p_{A:j}$ .This transition probability $p_{A:j}$ is just the projection of the asymptotic distribution over the basal attractor onto the basin of the modal attractor with label j. The probability of transition to the target modal attractor with label k^* is just $p_{A:k^*}$. Hence the average search time t_s can be obtained as

$$t_s = 2T / p_{A:k^*} \quad , \tag{1}$$

where $2T$ is the switching period and $p_{A:k^*}$ is just the projection of the asymptotic distribution over the basal attractor onto the basin of the modal attractor with label k^*.

B. Search by Adaptive Parameter Control

In the case of periodic parameter switching described in the previous sub-section, the state of the system is tested only after the relaxation to a modal attractor. However, if there is a structural relation between the modal attractors and the basal attractor, as in the case of chaotic itinerancy [12, 19], then when the response in the basal attractor is not positive, the probability of falling into the basin of a modal attractor with a positive response may also be small. With this situation in mind, we modify the periodic parameter switching procedure to test the response at the end of each interval of T, and if the response detected in the basal attractor is not positive, the parameter is *not* switched from P_1 to P_0; the parameter is kept at P_1 so the dynamics persist on the basal attractor. In this case, the parameter switching protocol can be defined by the following control sequence [11]:

(step 1) evolve for time T
(step 2) test for fitness
(step 3) switch parameter according to the following rules, and then return to step 1:

(P_1,N)	P_1
(P_1,Y)	P_0
(P_0,N)	P_1
(P_0,Y)	P_0

Note that with this protocol, the parameter value automatically will stay at P_0 after a modal attractor with a positive response is reached.

Observing the state of the system immediately after the parameter switch we obtain the following transition rules:

(A,N)	(A,N)
	(A,Y)
(A,Y)	(a_k,N) $k \neq k^*$
	(a_{k^*},Y)
(a_k,N) $k \neq k^*$	(A,N)
	(A,Y)
(a_{k^*},Y)	(a_{k^*},Y)

Now, if there are no false alarm states, that is all states in (A, Y) lie in the basin of a_{k^*}, the process further simplifies to

(A,N)	(A,N)
	(A,Y)
(A,Y)	(a_{k^*},Y)
(a_{k^*},Y)	(a_{k^*},Y)

Let us consider the transition probability for transition $(A, N) \rightarrow (A, Y)$. Since we assumed interval T is long enough for relaxation of the measure on the attractor, this probability is independent of initial state. We can obtain the average search time t_s as

$$t_s = T / p_{A:Y} \quad , \qquad (2)$$

where T is the switching period and $p_{A:Y}$ is just the relative measure of points on the basal attractor A with positive response. We emphasize that equation (2) gives the average time required to find a modal attractor when the system does *not* have false-alarms.

C. Modified Adaptive Parameter Control

Finally, we mention an alternative parameter switching protocol in which we do not require the system to wait time T to converge to an attractor before testing. The idea behind this is that if the positive response is strongly localized on a modal attractor, when a good response state is detected in the basal attractor, then it is likely that the system is near a state corresponding to the target modal attractor and so one should switch immediately. We mention this type of control here for two reasons. Firstly, this particular type of control has been used effectively in numerical experiments and physical experiments. And secondly, it is useful to point out that coarse-grained transition rules can be defined for more complicated search protocols.

The simplest form of this control is to switch from P_1 to P_0 immediately after the response Y is received and switch from P_0 to P_1 immediately after the response N is received.

In this case, instead of the periodic timing of state transitions, we use the change in parameter as the timing to define state transitions. Since we can no longer assume that the dynamics have relaxed to attractors, the attractor symbols are not appropriate to describe the process, and we use the basin symbols instead. We obtain the following set of transition rules.

(B,N)	(b_k,Y) $k \neq k^*$
	(b_{k^*},Y)
(b_k,Y) $k \neq k^*$	(B, N)
(b_{k^*},Y)	(B, N)

Note that with this definition of transition, i.e. we only recognize a transition when the parameter changes, there are no identical transitions in the table. Once the dynamics converge to the unique target modal attractor (a_{k^*}, Y), which is within (b_{k^*}, Y), there are no further transitions.

If we now assume that the positive responses are only in the basin of the target modal attractor, then the transition rules are modified as follows.

(B,N)	(b_{k^*},Y)
(b_{k^*},Y)	(B, N)

Further, if there are no false alarm states, then there is only one type of transition.

(B,N)	(b_{k^*},Y)

Alternatively, if we use the change in state as the timing to define state transitions then we obtain the following more complicated set of transition rules.

(A,N)	(b_k,Y) $k \neq k^*$
	(b_{k^*},Y)
	(a_{k^*},Y)
(B,N)	(A, N)
	(b_k,Y) $k \neq k^*$
	(b_{k^*},Y)
	(a_{k^*},Y)
(b_k,Y) $k \neq k^*$	(B, N)
	(A, N)
(b_{k^*},Y)	(B, N)
	(A, N)
	(a_{k^*},Y)

For simplicity, we exclude the cases where two state memberships change at the same time, considering such events to be rare. With the simplifying assumption that the positive response states only exist in the basin of the target modal attractor, then the transition rules are modified as follows.

(A, N)	(b_{k^*},Y)
	(a_{k^*},Y)
(B,N)	(A, N)
	(b_{k^*},Y)
	(a_{k^*},Y)
(b_{k^*},Y)	(B, N)
	(A, N)
	(a_{k^*},Y)

Moreover, if there are no false-alarm states, so that all points in *(A/B, Y)* states are in the trap set, converging to (a_{k^*}, Y) along trajectories which stay in Y.

(A, N)	(b_{k^*},Y)
	(a_{k^*},Y)
(B,N)	(A, N)
	(b_{k^*},Y)
	(a_{k^*},Y)
(b_{k^*},Y)	(a_{k^*},Y)

D. Rates for Persistent Adaptation

If the environment is changing in time so the fitness function and target mode k* change in time, then the adaptation process is repeated over and over again. If we assume that the target mode changes randomly, the persistent adaptation rate, expressed as modes per unit time, can be obtained as the inverse of the search time.

$$R_A = 1 / t_s \ . \qquad (3)$$

The search times are as obtained in the previous sub-sections.

Further, we can express the information generation rate in terms of the number of multistable attractors and the mode search time. Let the number of coexisting attractors be N, and the search time be t_s . The information generation rate R_I can be characterized as

$$R_I = \log N / t_s \ . \qquad (4)$$

In particular, if the search time t_s scales linearly with the number of attractors, the scaling of information generation rate R_I with number of attractors N is

$$R_I \ \alpha \ (\log N) \ / N \ . \qquad (5)$$

V. EXAMPLE OF BASIN HOPPING DYNAMICS

In this final section, we consider a particular model of basin hopping dynamics on the basal attractor, and show how properties of the dynamics determine the search times.

Specifically, we consider that the basal attractor is formed as a result of destabilization of the modal attractors. This is actually the situation in the nonlinear optical resonators and chaotic neural networks which have been previously studied in physical and numerical experiments of adaptive mode selection [12-17]. This type of chaos has also been referred to as chaotic itinerancy [12, 19].

We consider the modal attractors are limit cycles or fixed point attractors in the multistable regime, and for simplicity consider a Poincare map representation of the dynamics where the modal attractors correspond to fixed points (“modal points”) of an m-dimensional map. At P_0 the points are stable with only stable exponents λ_{0s} *(j) (j=1,2,…,m)* and at P_1 the fixed points are saddle points with both stable exponents λ_s *(j) (j=1,2,…, m-*D_u*)* and unstable exponents λ_u *(j) (j= m+1-*D_u*,…,m),* where D_u is the number of unstable dimensions. For simplicity we assume that all the stable exponents are the same value λ_s, and all the unstable exponents are the same value λ_u.

Next we assume that one of the modal points a_{k*} corresponds to a positive response Y, and that there is a trap set S_T centered on this modal point. Let L be the typical size of the area of the set of states near a_{k*} which get a positive response, relative to the size of the trap set, and consider the case that L is larger than 1, *i.e.* the positive response domain near a_{k*} is bigger than the trap set. We also assume the injection into the positive response domain is uniform random, due to strong mixing of the chaotic dynamics.

After injection into the positive response domain, and waiting time t, the probability p_t of actually being in the trap set is

$$p_t = (1/L\lambda_u^{\,t})^{\,Du} \ , \qquad (6)$$

as determined by the ratio of orbits which do not escape from the trap set along the unstable directions. Then the number of basin visits needed to reach a trap set is expected to be

$$n_v = (L \ \lambda_u^{\,t})^{Du} \ . \qquad (7)$$

On the other hand, we can estimate the recurrence time t_r between visits to the same basin as

$$t_r = N / \log \lambda_u \ , \qquad (8)$$

where N is the number of basins, and $1/ \log \lambda_u$ is the average residence time at each basin visit. Average residence time is estimated by the time required to escape the vicinity of the saddle point along the maximally unstable direction.

Hence the search time t_s is estimated as

$$\begin{aligned} t_s &= n_v \, t_r \\ &= N(L^{Du}) \ \lambda_u^{\,t\,Du} \ / \log \lambda_u \ . \end{aligned} \qquad (9)$$

Now due to the tradeoff between reducing λ_u to increase the trap probability, and increasing λ_u to reduce the recurrence time, there is an optimal value of λ_u. which minimizes the search time. The optimal value of λ_u is given by

$$D_u \log \lambda_u = 1/t \ . \qquad (10)$$

It is reasonable to assume that the value of λ_u can be varied by changing the value of the parameter P_1. When the optimal value of λ_u applies, the average search time is

$$t_s = e \ N(L^{Du}) \ D_u \, t \ , \qquad (11)$$

or in terms of λ_u ,

$$t_s = e \ N(L^{Du}) \, / \log \lambda_u \ . \qquad (12)$$

Next we consider the case where the convergence is stronger than the divergence, given by the following condition

$$\lambda_u \, \lambda_s < 1. \qquad (13)$$

In this case, after injection into the positive response domain, the probability of being in the trap set increases up to a certain time t_w which is determined by the condition $L\lambda_s^{tw} = 1$.

$$t_w = log\ L\ /\ |log\ \lambda_s| \qquad (14)$$

If we assume that this value of optimal wait time is used, then the condition for minimal search time becomes

$$D_u\ log\ \lambda_u = |log\ \lambda_s|\ /\ log\ L\quad . \qquad (15)$$

With this condition, the minimal search time is given by,

$$t_s\ =\ e\ N(L^{Du})\ D_u\ log\ L\ /\ |log\ \lambda_s|\quad . \qquad (16)$$

As mentioned above, in the case that the chaotic basal attractor is formed by the de-stabilization of the modal attractors it is reasonable to assume that the optimal condition specified in Eqn. (15) can be achieved by suitable choice of the value of parameter P_1. The results of this section illustrate the advantage of tuning chaos in basal attractors (in this case using parameter P_1, for example) to optimize search times by balancing convergence and divergence in search dynamics. (Note that the condition Eqn. (15) was mentioned previously in references [9-10], with a different symbol definition in which the symbol λ corresponds to log λ of this paper.) This balance of convergence and divergence in search dynamics provides another perspective of the optimality of dynamics near "the edge of chaos" [22, 23].

VI. CONCLUSIONS

In this paper we have described an approach to modeling systems which achieve mode adaptation using basin-hopping dynamics. Our modeling approach is based on the specification of coarse-grained transition rules. This approach can be used to analyze and compare various different basin-hopping systems. This approach can provide insights into the search dynamics and the factors determining search times.

Systems which use this type of adaptive stochastic basin-hopping can support highly flexible behavior due to the flexible nature of the coupling between internal mode dynamics and external fitness responses. However, due to the trial-and-error nature of the search process the time to find a suitable mode is an important practical issue.

We developed the analysis for a number of different cases exploiting specific assumptions about the dynamics and the adaptive control. In particular, a common assumption is the existence of multiple attractors or one global spanning attractor, depending on parameter values. Introduction of the condition that the system should be allowed to approach the asymptotic distribution on an attractor before switching the parameter greatly simplifies the analysis. However, it can result in longer search times. Relaxing this condition reduces the search times, but makes the analysis more complicated. In the final section we introduced a number of extra assumptions about the structure of the basins and responses to allow an estimate of search times in terms of characteristic lyapunov exponents.

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