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On the Role of Numerical Preciseness for Generalization, Classification, Type-1, and Type-2 Fuzziness

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Abstract – **When performing data analysis on a computing device no mathematically idealized real number set $I\!R$ is available. A basic resolution is given, so that a fuzzy model is in fact always a discrete model and not a continuous one. Due to the limited preciseness the computing device offers only a limited number of decimals in a limited discrete number space $I\!I\!R$. This contribution considers effects on the generalization and fuzziness of data when replacing $I\!R$ by $I\!I\!R$. The effects are studied for data, that are numerically rounded or when intervals are considered. Often part of the data is missing or is of limited quality, so that it is of practical interest to consider the exact underlying space $I\!I\!R$ and not the hypothetical space $I\!R$. We calculate precisely type-1 and type-2 fuzzy membership functions under preciseness assumptions of the elements in $I\!I\!R$.**

I. Introduction

Numerical data (in one dimension) are commonly assumed to be a subset of the real number space $I\!R$. This assumption is an ideal one. A computing device might handle numbers with a fixed or varying precision (fixed-point or floating point numbers), a software might handle numbers with a different accuracy, and the measurements could have again another preciseness due to noise, although many decimals were measured. Generally spoken, the mere assumption that all these data spaces are equally the set of real numbers $I\!R$ is wrong and might be the cause for errors or unreliable results. Such problems are usually considered in numerical mathematics, but with respect to fuzzy data analysis not much work has been done so far. Classification analysis or fuzzy modeling is not being as precise as possible without any preciseness assumptions for the data in the involved data spaces. Often, we use intuitively terms like 'classification,' 'generalization,' and 'fuzziness,' although they can be defined exactly [1]. Several approaches exist that are concerned with modeling imprecise data in an exact manner. Besides interval arithmetic [2], granular computing [3] is the main research area where data is divided for analysis in so called granules, i.e., parts of the underlying data space. The granularity can be given in many ways: [4] describes three-valued shadowed sets composed of certain regions with and without a property and an uncertain region, [5] compares the partitioning of data with different strategies, [6] investigates effects of granularity in fuzzy systems, [7] partitions data with morphological operators. Many approaches for modeling fuzzy systems are available [8]-[14] for example. An extension of fuzzy sets are type-2 fuzzy sets and systems [15]-[17], that allow for modeling uncertainty in (type-1) fuzzy sets and systems.

We develop an approach that allows for the exact modeling of numerical data in an underlying exact space $I\!I\!R$ by calculating membership degrees for with elementary elements. The next section is concerned with the formalization of the space $I\!I\!R$ and the preciseness of numerical data, including generalization and classification. Section III demonstrates the relations of preciseness to the generalization of data. Fuzzy models are considered in Section IV where the models are investigated in $I\!R$ and $I\!I\!R$. Fuzzy membership functions are calculated for different examples. Due to the uncertain preciseness assumptions, we extend the approach to general type-2 fuzzy membership functions.

II. Preciseness in the Exact Space $I\!I\!R$

Mathematical abstraction led to the definition of the field of the real numbers $I\!R$. Although this continuous model $I\!R$ is useful for mathematical calculations, numerical calculations on a computing device are not based on real numbers, but on a subset of the rational numbers only. The binary digits can only represent rational numbers. For example the number π is approximated when using a computer. Thus, we have to assume a basic exactness of 2^{-m} with a fixed $m \in I\!N$. On a physically-biologically inspired computer the exactness might be finer or based on objects like atoms, but it would be limited as well. Since we are used to the basis 10 we can assume a basic exactness of $10^{-m'}$. To avoid writing commas, decimals, and signs it is no loss of generality if we multiply the numbers by $10^{m'}$ or to shift negative numbers to positive ones. Then, the basic exactness is $g = 1$.

Definition 1. Let g be the basic exactness. We define the space $I\!I\!R(g,n)$ as the finite set $\{g,2g,\ldots,(n-1)g,ng\}$ with a fixed $n \in I\!N$. If $g = 1$, then we write shortly $I\!I\!R = \{1,2, ,\ldots,(n-1),n\}$. We

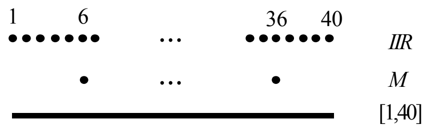

Fig. 1. The exact space $I\!I\!R = \{1,2,\ldots,39,40\}$, the 10-precise set M, and the hypothetical real valued interval [1,40].

name $I\!I\!R(g,n)$ the *exact space* (with exactness g and maximal multiplier n). To the symbol $I\!R$ another I is added in front to distinguish $I\!I\!R$ from $I\!R$.

The basic exactness needs not to be a number representing a binary computer value, but could also be an artificial property like a minimal possible distance or another elementary unit. The exact space contains unique, distinguishable elements. We will consider technical aspects of generalization and fuzzy modeling in $I\!R$ compared to $I\!I\!R$. Since we cannot distinguish elements in the exact space $I\!I\!R$, that are more precisely defined than the elementary exactness g, we have to define what we mean with 'preciseness' in the context of exact spaces.

Definition 2. A subset M of the exact space $I\!I\!R = \{1,2,\ldots,(n-1),n\}$ with exactness 1 is said to be *k-precise* ($k \in I\!N$) (with respect to $I\!I\!R$) if there exist a partition $\{P_i\}$ of M with k elements $\in I\!I\!R$ in each P_i, so that every $a \in I\!I\!R$ can be assigned to a P_i. Each element in P_i is not distinguishable from a in M. An element $a \in I\!I\!R$ is said to be *k-precise* if k elements are assigned.

Example 1. Consider the set $M = \{1,11,21,31\}$ and the exact space $I\!I\!R = \{1,2,3,\ldots,38,39,40\}$. If we assign $P_1 = \{1,2,\ldots,9,10\}$ to 1, $P_2 = \{11,12,\ldots,19,20\}$ to 11, $P_3 = \{21,22,\ldots,29,30\}$ to 21, and $P_4 = \{31,32,\ldots,39,40\}$ to 31, and set $M = P_1 \cup P_2 \cup P_3 \cup P_4$, then M is 10-precise. Remember, that number 40 in $I\!I\!R$ means the basic element number 40. The exact space $I\!I\!R$ itself is 1-precise. In Example 1 the mapping G: $P_i \subset I\!I\!R \rightarrow a \in M$ can be interpreted as a kind of rounding operation, increasing the preciseness from 1 to 10. The mapping G generalizes $I\!I\!R$ to M. The situation is depicted in Fig. 1. A noise operator, applied to M decreases the preciseness. For example, if the measurement of 1, 11, 21, 31 is possible in M only with random noise of $\pm$ 5 elements, then the measured set M' could be $M' = I\!I\!R$, when the elements are measured repeated times. Noise lead to a finer exactness than it is originally given.

When considering a data analysis task (in one dimension at the moment) like clustering, where elements of a set D are generalized to clusters, the data in the set need not to be k-precise in the sense of Def. 2 for any k.

Example 2. Consider the set $D = \{3,5,7,10,11,12,13,20,25,33,39\}$ and the exact space $I\!I\!R = \{1,2,3,\ldots,38,39,40\}$. D is not k-precise in $I\!I\!R$ for any k. If we consider only element 3 in D, then it could be up to 4-precise, representing $\{1,2,3,4\}$ in $I\!I\!R$. The element 5 could be up to 3-precise, element 12 is 1-precise, etc. If we list all possibilities we could assume that the elements 3, 5, and 7 are 2-precise, but not knowing if 3 represents $\{2,3\}$ or $\{3,4\}$, 10 to 13 seem to be 1-precise, 20, 25, 33, and 39 could be 4-precise, for example with 25 representing $\{24,25,26,27\}$.

The problem in generalizing properties of a dataset is the missing information about elements in the exact space. With missing information it cannot be determined uniquely how k-precise an element is (if it is not known a-priori).

If we assign any property to the set D in Example 2 like colors, then we can only assume which colors should be assigned to the elements of $I\!I\!R \backslash D$. If 3, 5, and 7 are assigned to 'blue,' then we could assume that the elements 4, 6 and 8 should be assigned to 'blue,' too, under the assumption that the data is 2-precise. If it is 1-precise, the approximation might be wrong, if it is 2-precise, it would be correct locally.

Since we consider the elements in the exact space as elementary, they are assigned uniquely to properties when considering data analysis. A set containing more than one element could have more than one property. We remark that several exact spaces with different exactness values g can be combined, for example when considering floating point numbers. In the next section we focus on the problem of data generalization.

III. Generalization in $I\!R$ and $I\!I\!R$

The preciseness of a set is related to its generalization capability. The more precise a set is, the less general it is and vice versa. If our environmental world is seen as an exact space, the information about all 1-precise elements builds the exact model. The aim of generalization is to consider only a subset of all elements that already defines the exact space or gives an acceptable approximation to the exact model.

Definition 3. Consider a subset M of the exact space $I\!I\!R$. Let N be a subset of M with $M \neq N$. The set N is defined as being *more general* than M if at least one element $a \in N$ has an increased k-preciseness compared to $a \in M$, i.e., it becomes *more general*. – It follows that a k-precise set $N \subset M$ is more general than a k'-precise set M if $k > k'$.

Example 3. If we consider the set $D = \{3,5,7,10,11,12,13,20,25,33,39\}$ in Example 2 and another set $C = D \backslash \{11\}$, both subsets of $I\!I\!R = \{1,2,3,\ldots,38,39,40\}$, then C is more general than D. The 1-precise element 12 in D is considered as being 2-precise in the set C.

The rounding operation in Example 1 leads to more general elements and sets. If the 1-precise elements of the exact space carry an exact information, e.g., a class label, a generalized k-precise element can have more than one class label. The generalization lead to vague information.

In the following we compare the underlying spaces $I\!R$ and $I\!I\!R = \{1,2,3,\ldots,38,39,40\}$ with respect to a given finite dataset

M. While *IIR**M* is finite, *IR**M* is infinite with *M* being mathematically a null set. In *IR* there is no minimal exactness as in *IIR* with $g = 1$, because to a given exactness more decimals could always be added. Let us consider the case that the elements 5, 10, and 15 are labeled with 'class 1' and the elements 20, 25, and 30 with 'class 2.' An algorithm for generalization would try to make class assignments to elements *IIR**M* or *IR**M*, respectively. A canonical generalization could be done in the real space by assigning $I_1 = [5,15]$ (interval of all real numbers between 5 and 15) to class 1, and $I_2 = [20,30]$ to class 2. In the exact space we would canonically assign $J_1 = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$, formally defined as $[5,15]_{IIR}$, to an 11-precise element 10 with class assignment 'class 1.' $J_2 = [20,30]_{IIR}$ could be assigned to an 11-precise element 25 of 'class 2.' The difference is the following: If the set of real numbers is considered, then with 3 elements an assumption for an infinite number of elements has been made. In the exact space with 3 elements an assumption of further 8 elements has been made. For both spaces the generalized models might be totally right or totally wrong as extreme cases. This is not known a priori. In the real space the model might be correct for a null set only, meaning calculated 0% correctness. In the exact space we know that the model is at least correct for 3 out of 11 elements, meaning calculated ≈27.27% correctness.

Without a correct assumption of the exactness of the underlying space generalizations might lead to models with a higher number of wrong outputs, especially for unknown data.

IV. Fuzzy Modeling in *IIR*

In data analysis it is desired to have the complete information about the exact space of interest, i.e., knowing everything about the elements of *IIR*. Then, a generalization could be done perfectly. The generalized properties could be calculated precisely, including the probability of one generalized element having one out of many different properties. Considering real-world data, usually knowledge of a subset of the exact space is known or measured. The common task is to predict the properties of the remaining elements in the exact space. Another task is compression. If the exact space is a pixel space P in image processing, the compression task would be the finding of a space $Q \subset P$, so that the information about P can exactly or approximately reconstructed from Q. In this case, the exact space is a well defined artificial space, that makes its analysis a well defined task.

Example 4. a) Consider again the exact space *IIR* = {1,2,3,…,38,39,40} with (1,white), (2,white), (6,white), (7,white), (8,white), (14,black), (15,black), (16,black), (17,white), (18,white), (19,white), (20,white), and (s,unknown) for s = 9, 10, 11, 12, 13, 21, 22, …, 39, 40.
b) We have a look at patients $P_1,\ldots,P_{10}$ with a measured value and a certain disease: (P_1,50,ill), (P_2,55.7,ill), (P_3,75,ill), (P_4,53.337,ill), (P_5,60,ill), (P_6,58,healthy), (P_7,76.5,healthy), (P_8,81,healthy), (P_9,84.4,healthy), (P_{10},75,healthy).

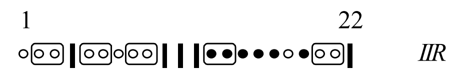

Fig. 2. The exact space is modeled with elements 2, 6, 8, 14, and 20 being 2-precise.

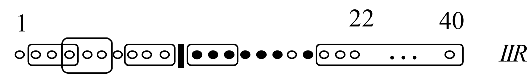

Fig. 3. The exact space is modeled with elements 2, 6, 8, and 14 being 3-precise and 20 being 21-precise.

We discuss at first Example 4a. Remember, in the exact space each element is an entity and an 1-precise element. Since we do not have class labels for 28 of the 40 elements, a model should predict these class properties. The question is which one of the models is the best one. If we assume all elements as being 1-precise, resulting in a 1-precise space, then no prediction at all is possible. Each element can either be black or white and it does not depend on the elements for which the class labels are already known. The visualization might pretend neighborhood relations, but in the exact space they need not to be considered. The 19th element is black while the 18th and 20th elements are white, so that the 19th element must be modeled 1-precisely if no uncertainty should be introduced in the model. A global 1-precise model, with 1-precise elements only, remains as it is.

The solution for finding a more sophisticated model is to allow for k-precise elements with different k's locally. The problem with considering different k's for each element is a combinatorial exploding number of models. If we assume the elements 2, 6, 8, 14, and 20 to be 2-precise, we could predict the elements 3, 5, 9, and 21 as being white, and the element 13 as being black, cf. Fig. 2. Another model could be the following: elements 2 and 6 are 3-precise, so that element 4 can be predicted uniquely as being white, too. Element 20 is modeled as being 21-precise, predicting elements 21 to 40 as being white. Element 8 is 3-precise with elements 9 and 10 predicted as being white. Element 14 is 3-precise (in the left direction), predicting elements 12 and 13 as being black, cf. Fig. 3. It is inherent to the definition of the exact space that unknown information cannot be predicted uniquely. Assumptions about the preciseness of elements can make predictions unique.

We consider again the elements in Fig. 2. We assume that the model is mainly 5-precise. Symmetrically to the left and right two elements can be predicted. We do not use this assumption for elements where 5-preciseness is not justified, i.e., it is not used for the elements 16, …, 20. Elements 3, 4, and 5 can clearly be predicted under the assumption as being white. No prediction can be made for the elements 21, …, 40. The predictions for the elements 9, …, 13 are shown in Table I. From the numbers of elements that predict each color it becomes possible to calculate a degree of preciseness. These degrees are depicted in Fig. 4. Element 11 cannot be predicted uniquely with the model assumptions. Comparing the two models in Figs. 2 and 4 element 10 is not predicted in Fig. 2

TABLE I
BASIC 5-PRECISE ELEMENTS THAT PREDICT THE ELEMENTS IN THE COLUMNS AS BEING BLACK OR WHITE

	9	10	11	12	13
black	-	14	14,15	14,15	14,15
white	6,7,8	6,7,8	7,8	8	-

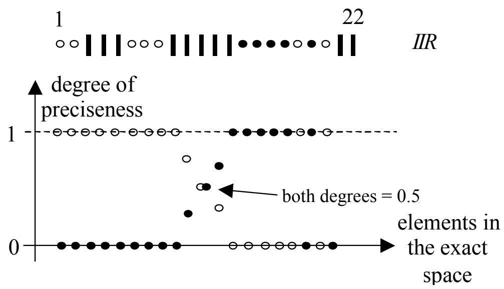

Fig. 4. Under the assumption that the elements 1, …, 15 are 5-precise the degrees of preciseness give a prediction for a priori elements without color information.

(degree equal to 0). In Fig. 4 it is predicted as being white with degree 0.75. In Fig. 2 element 4 could not be predicted with 2-precise elements. With 5-precise elements it was predicted as being white in Fig. 4.

The relation to fuzzy logic is obvious. One can model the 1-precise elements in the exact space as fuzzy singletons. The degrees of preciseness gives membership degrees for fuzzy membership functions. The membership function is defined on the basic elements in the exact space. In the real space the membership functions are defined in the continuum. With respect to fuzzy logic all the degrees of preciseness might be referred to as exact membership functions under certain preciseness assumptions.

Let us now consider the Example 4b. Two patients have the same measurement (P_3,75,ill) and (P_{10},75,healthy), but with different class assignments. Two reasons might be the cause for that. The measurement was not done exactly, maybe rounded, so that no decimals are available. Other measurements are available with decimals. There is no underlying common model of an exact space. Secondly, this variable is not representing the exact space needed for a unique classification. The problem space might be higher dimensional for example. If an exact space with three decimals precision is assumed (the element 50.000 could formerly be shifted to 1), then 34401 elements are in the whole exact space. With ten elements and assumed 5000-preciseness (2500 to left, element, 2499 to the right) we would obtain a model no. 1. Another possibility would be the rounding of all values, so that no decimals are remaining. Then, between 50 and 84 the number of 35 elements have to be analysed by ten not uniquely labeled elements. We assume 5-preciseness and obtain a model no. 2. We give the characteristics of the two resulting models.

TABLE II
MODEL 1: INTERVALS WITH DEGREES OF PRECISENESS AND ASSIGNED CLASS LABEL (5000-PRECISE DATA)

Interval	d_{ill}	$d_{healthy}$	Class label
$[47.500,55.499]_{IIR}$	1	0	ill
$[55.500,55.836]_{IIR}$	2/3	1/3	ill
$[55.837,57.499]_{IIR}$	1/2	1/2	-
$[57.500,58.199]_{IIR}$	2/3	1/3	ill
$[58.200,60.499]_{IIR}$	1/2	1/2	-
$[60.500,62.499]_{IIR}$	1	0	ill
$[62.500,72.499]_{IIR}$	0	0	unknown
$[72.500,73.999]_{IIR}$	1/2	1/2	-
$[74.000,77.499]_{IIR}$	1/3	2/3	healthy
$[77.500,86.899]_{IIR}$	0	0	healthy

TABLE III
MODEL 2: INTERVALS WITH DEGREES OF PRECISENESS AND ASSIGNED CLASS LABEL (5-PRECISE DATA)

Interval	d_{ill}	$d_{healthy}$	Class label
$[48,55]_{IIR}$	1	0	ill
$[56,60]_{IIR}$	1/2	1/2	-
$[61,62]_{IIR}$	1	0	ill
$[63,72]_{IIR}$	0	0	unknown
$[73,74]_{IIR}$	1/2	1/2	-
$[75,77]_{IIR}$	1/3	2/3	healthy
$[78,86]_{IIR}$	0	1	healthy

Model 1: For example 58.000 with 5000-preciseness is assigned to the interval $[55.500,60.499]_{IIR}$ with status 'healthy.' The other intervals are assigned as given in Table II. For each subinterval the degrees of preciseness for the classes 'ill' and 'healthy' are calculated. From these degrees classes can be predicted. If both degrees are 0.5, no class can be assigned. In the case that both degrees are zero, the class is unknown.
Model 2: 58 is modeled with 5-preciseness (2 left, element, 2 right) and is thus assigned in the exact space to the interval $[56,60]_{IIR}$ with elements 56, 57, 58, 59, and 60. The other intervals are given in Table III with interval degrees and class assignments.

We will discuss some values and the results using the two models as classifiers. The value 58 was measured actually on a healthy patient. In model 1 the patient would be classified incorrectly as 'ill' and in model 2 no clear assignment is possible. These effects are due to a k-preciseness with a high k > 1, introducing uncertainty to the model and possible incorrect classification decisions. If a new patient has the value 74.4, he would be classified as a healthy patient in model 1. The exact space of model 2 does not support decimals, so round(74.4) = 74 lead to no class decision in model 2.

An exact fuzzy model can be modeled based on the preciseness degrees of Tables II or III for example. Let us consider data points of one class with assumed either (symmetrical) 5- or 11-preciseness in the exact space of Figs. 5

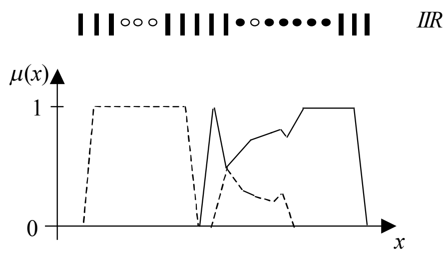

Fig. 5. Exact fuzzy modeling for the elements under the assumption that all elements are 5-precise.

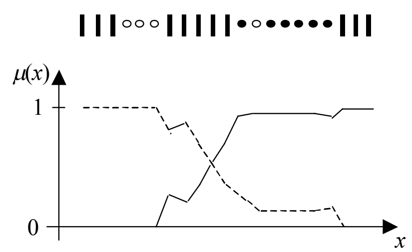

Fig. 6. Exact fuzzy modeling for the elements under the assumption that all elements are 11-precise.

and 6. In Fig. 5 the resulting fuzzy membership functions are shown with assumed 5-preciseness. The membership degrees were identified with the preciseness degrees as it was done for the models 1 and 2 before. The membership functions for class 'white' might lead to two fuzzy terms 'low' and 'high.' With 11-preciseness (Fig. 6) we have one fuzzy-term for the white and one for the black class. Due to the exact determination of the shape of the membership functions they are not ideal trapezes. Figure 7 shows an imprecise fuzzy modeling in the real space that might be based on visual exploration of the data. No assumptions about an exact space are made here. In Fig. 7 the white point at position 13 is treated as an outlier. Compared to the models with an underlying assumption of an exact space (Figs. 5 and 6) the model in the real space (Fig. 7) is more artificial and imprecise, but smoother due to missing assumptions of the *k*-preciseness for all elements.

The sum of the membership degrees for every element in the exact space is either 1 or 0. In the latter case the element was not covered by any *k*-precise element. In the literature such a fuzzy system is called *intuitionistic* [18]. For example unknown knowledge in medical applications can be modeled with intuitionistic fuzzy sets [19]. The main advantages between our approach and the approach in [7] are the definition of an exact space and of *k*-preciseness for all data elements while in [7] gaps of a certain length between the data are generated (dilatation) or deleted (erosion). This process does not support numerical preciseness since 'outliers' are deleted and not included in the calculation. With the elements 17, 18, 20, and 25, a gap between 17 and 20 is considered in the same way as a gap between 18 and 25, so that the exact space {17,18,19,20,21,22,23,24,25} and the resulting membership functions are distorted as a consequence.

V. Extension to Type-2 Fuzzy Sets

The ideas of modeling type-1 fuzzy sets in the exact space IIR can be extended to exact type-2 fuzzy sets. Originally, type-2 fuzzy sets were introduced in [15]. With this model uncertainty about uncertainty is expressed. For example different persons might model the fuzzy number 4 with different membership functions. To obtain a common model, a second membership degree is assigned to each value of the first membership function, representing the uncertainty of the first membership

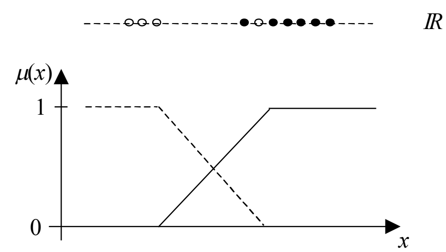

Fig. 7. Visual fuzzy modeling in IR for the elements in Fig. 5.

degree. The type-2 fuzzy sets in the real space IR are defined in [16], where examples for applying operators on such sets are given.

Definition 4. A *type-2 fuzzy set* $\tilde{M}$ is characterized by a *type-2 membership function* $\mu_{\tilde{A}}(x,u)$, where $x \in X$ and $u \in J_x \subset$ [0,1], i.e.,

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) \mid \forall x \in X, \forall u \in J_x \subset [0,1]\} \qquad (1)$$

with $0 \le \mu_{\tilde{A}}(x,u) \le 1$. Given a fixed x', if all secondary membership degrees of a secondary membership function $\mu_{\tilde{A}}(x = x', u)$ are set to 1, then the type-2 fuzzy set is called an *interval type-2 fuzzy set*.

Until now mainly interval type-2 fuzzy sets were used [16], [17] due to the complexity of the general approach. In Fig. 7 a real valued type-1 membership function has been depicted. An interval type-2 modeling could be given by assigning an uncertainty region around the first membership function $\mu(x)$. Such a model is only an imprecise or approximate modeling of general type-2 uncertainty.

We consider the exact space IIR in Fig. 5, where we have modeled the type-1 membership functions under the assumption of 5-preciseness. In Fig. 6 we assumed 11-preciseness. In the following we model a type-2 fuzzy system under the assumption that 5- to 11-preciseness is possible. The minimal and maximal values define the borders of the uncertainty regions, that are visualized in Fig. 8. We notice that the uncertainty of class 'white' allows for either one or two linguistic terms when a type-1 system should be

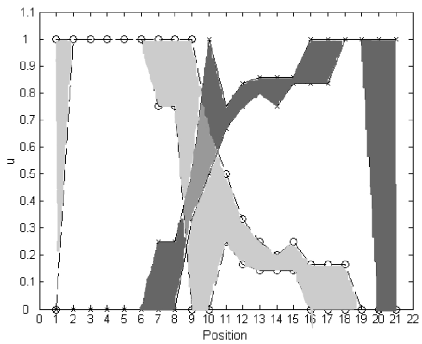

Fig. 8. Fuzzy type-2 modeling of the data in Fig. 5. The light grey area is the uncertainty region of class 'white,' the dark grey area the uncertainty region of class 'black.'

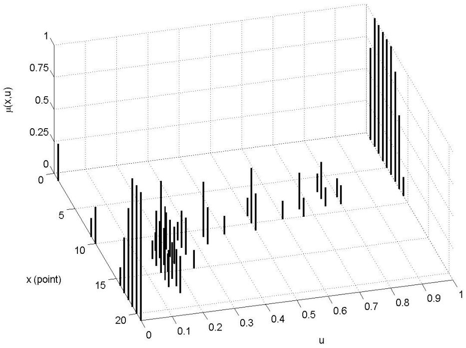

Fig. 9. The secondary membership degrees $\mu(x,u)$ for class 'white' are plotted as bars. The light grey area in Fig. 8 can be embedded in the x-u plane.

considered. The (interval) type-2 system is capable of modeling the uncertainty about the number of linguistic terms. If a general type-2 fuzzy system should be modeled, an exact model in IIR can be generated by the membership degrees, counting the number of values for each point and calculating relative proportions between 0 and 1. For example the degree 0 appears 5 times for element 18, the degree 1/6 appears 2 times (class 'white'). We obtain $\mu_{\text{white}}(x,0) = 5/7$ and $\mu_{\text{white}}(x,1/6) = 2/7$. The (general) secondary membership function $\mu_{\text{white}}(x,u)$ is depicted in Fig. 9.

VI. CONCLUSION

In numerical data analysis and algorithmic design it is mostly assumed that the underlying space is IR. But the computing device, the software, and the data is restricted to assumptions about the preciseness used for calculation or measurement. To deal with the situation, the assumption of an underlying exact space IIR was made, that can be interpreted as a lattice. In the exact space assumptions about impreciseness of elements can be modeled exactly. The generalization of data and the fuzzy modeling in the exact space lead to models that are better motivated by the given preciseness assumptions. With the k-preciseness unknown data, that cannot be classified upon the training data, can be labeled as 'unknown.' The approach helps to find more exact models for modeling impreciseness. Furthermore, it helps in modeling interval or general type-2 fuzzy systems based on k-preciseness of the data. For the case that no assumption about k-preciseness is available, a task for the future could be the finding of a data driven heuristic for it. A next step is the extension of the approach to a higher dimensional exact space IIR^n where the generalization is possible in many directions in the exact space. This might be a step towards a more universal data theory, in which a precise model of uncertainty can be given.

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