Optimal Nesting of Species for Exact Cover of Resources: Two Against One

J. Horn

Department of Mathematics and Computer Science

Northern Michigan University

Marquette, Michigan, USA

jhorn@nmu.edu

906-227-1607 (voice)

906-227-2010 (fax)

http://cs.nmu.edu/˜jeffhorn

Abstract

Experiments for resource-defined fitness sharing (RFS) show a remarkable ability to find tilings in shape nesting problems (Horn, 2002, 2005). These tilings are essentially exact covers for a set of resources, and represent a maximally sized set of co-operating (non-competing) species. This paper initiates a formal analysis of this empirical phenomenon by examining a minimal case: two species a and b "cooperate" to exactly cover the resources, while a third species c "competes" with a and b by overlapping both in terms of covered resources. The analysis reveals that in cases in which a and b *maximally compete* with c for resources, species c will become extinct, while the optimal set of species, a and b, will survive. This result is clearly proven using algebra on the niching equilibrium equations for RFS, a purely static analysis.

1 Introduction

In the 2002 introduction (Horn, 2002) of Resource-defined fitness sharing (RFS)[1], the niching method shows a remarkable ability to converge to an optimal solution on shape nesting problems if that optimal solution consists of a *tiling*. The 2002 paper provides evidence of this phenomenon for both one and two-dimension shape nesting problems. But since RFS operates with quantities defined by sets (e.g., set intersections), and does not make any direct use of the geometric properties of shape nesting, it is possible that this apparent ability to tile one arbitrary shape with another generalizes to an ability to exactly cover a set of resources with a set of subsets of those resources. That is, if an exact cover of the resources (substrate) can be found in the current population, then possibly RFS will always drive the population distribution to represent the exact cover.

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[1]RFS, applied to shape nesting problems, has patent pending status in the United States.

It therefore seems important to investigate the ability of RFS to evolve (select) exact covers by applying a theoretical analysis to minimal size cases of tiling (exact cover) problems. In this paper, we begin with the *two against one* case: two species cover all of the resources, while a third competes for coverage. Two empirical papers on RFS (Horn, 2002; 2005) precede this paper, which is the first analytical treatment of RFS.

2 Background

We provide here very brief summaries of the RFS algorithm and the problem domain of shape nesting, which is a subset of resource covering problems in general. RFS was developed as a compromise between *fitness sharing* (FS) and *resource sharing* (RS), but we do not have space here to compare and contrast these three different approaches.

2.1 Shape Nesting

The general problem at hand involves "nesting" (that is, placing) shaped pieces on a finite substrate so as to maximize the number of such pieces on the substrate. The objective is often stated, equivalently, as the minimization of "trim" (i.e., unused substrate) (Dighe & Jakiela, 1996; Kendall, 2000). No overlaps among the placed pieces are allowed, and all such pieces must be placed so as to be completely within the boundaries of the substrate. Figure 1 illustrates a typical shape nesting problem. The layout of pieces is actually the result of a run of the RFS algorithm.

In this paper the focus is on a very common subdomain of shape nesting problems. We assume a finite, two-dimensional problem, which means a flat substrate of fixed size, and flat pieces to be nested (placed). We assume identical shapes, which means there is only one
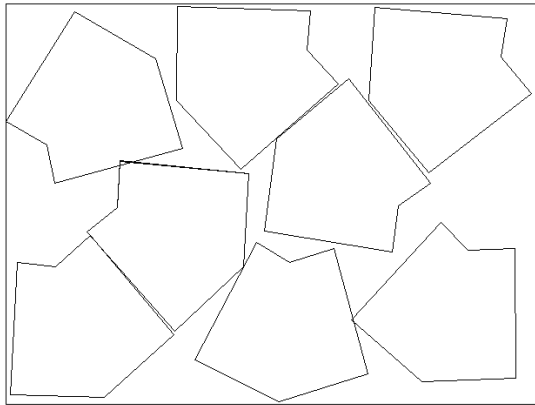
Figure 1: RFS can nest arbitrary shapes.

Figure 2: RFS can find tilings (i.e., exact covers).

shape we are nesting. We limit ourselves to polygons, to simplify area calculations, but we allow arbitrary polygons, which means that the pieces as well as the substrate can be non-convex. In addition, the pieces do not have to be axis-aligned; they can be rotated into any orientation. There are no constraints on the separation or any other relationship between shaped pieces or between the shaped pieces and the substrate boundaries, other than the normal exclusion of overlap.

Horn (2002) applied RFS to one and two-dimensional shape nesting problems but limited his tests to axis-aligned squares for the shaped pieces. For example, in the two-dimensional example, illustrated in Figure 2, the task was to nest as many of the smaller square pieces as possible within the larger piece. The width of the substrate square was exactly four times that of the piece square, so that a single optimal solution existed, consisting of sixteen pieces exactly covering the substrate, as shown in Figure 2, right.

The RFS algorithm was given no information about the solution, instead starting with a random population of 16,000 square pieces (with random positions; the orientation of all pieces are fixed so that all were parallel to the x,y axes). There were 1600 possible piece positions (on a discrete 40 by 40 grid). With a population size of 16,000, the random initial generation contained about 10 copies of each of the 1600 possible species.. The GA with RFS was able to select and promote the sixteen *species* corresponding to the solution in Figure 2, right, where each of the 16 species is represented by approximately 1000
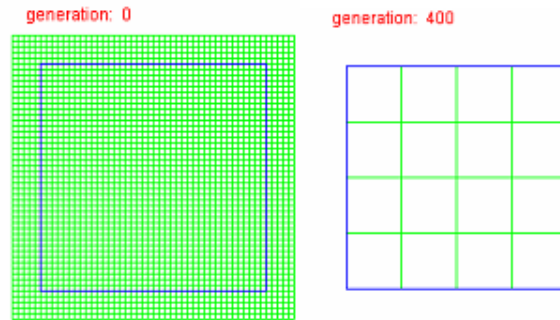
copies (individuals) in the final population.

2.2 RFS: Resource-defined Fitness Sharing

Under RFS, every individual of the current population is evaluated and assigned a fitness. In Horn (2002), each individual is a chromosome that specifies a placement of a piece. Any individual that specifies a placement that extends beyond the boundaries of the substrate is assigned a fitness of 0. All "feasible" individuals (i.e., chromosomes specifying piece placements entirely on the substrate), receive a shared fitnesses greater than 0, for use in a standard selection method (e.g., tournament selection, proportionate selection).

Under RFS, the shared fitness for each individual is a function of the resources (e.g., area) covered by the individual, and of the extent to which the individual's coverage overlaps with that of other individuals in the population. The form of the RFS shared fitness formula, $f_{sh,i}$, is that of a fraction:

$$f_{sh,i} = \frac{f_i}{niche\_count(i)} = \frac{f_i}{\sum_{j \in P} f_{ij}}, \qquad (1)$$

where i is an individual in the population P, f_i is the objective (unshared) fitness of i, and f_{ij} is the pairwise overlap in "coverage" between individuals i and j in P, and $niche\_count(i)$ measures the amount of competition for resources covered by i. Under RFS, niche count is defined as the cumulative pairwise overlap between i and other individuals in P. Figure 3 illustrates f_i and f_{ij} for two individuals *i* and *j*.

For the purposes of this paper, we normalize the objective fitness f_i to 1 $\forall i \in P$. Thus $0 \le f_{ij} \le 1, \forall i, j \in P$.

Next we define what we mean by *species* and how the term relates to individuals. We consider a species to be a
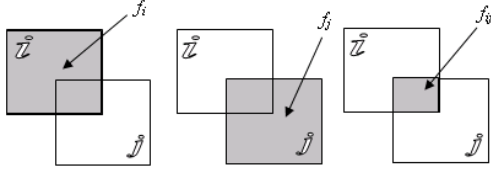
Figure 3: The basic terms used in defining RFS.

set of identical individuals (i.e., with identical coverage of resources). Thus unique chromosomes map one-to-one with unique species. There is complete overlap between any two members of the same species, while there is less than complete overlap between any two members of different species.

Now that we have defined "species", we can re-write Equation 1 in terms of species:

$$f_{sh,x} = \frac{f_x}{niche\_count(x)} = \frac{f_x}{\sum_{y \in S(P)} n_y f_{xy}}. \qquad (2)$$

Equations 2 and 1 are equivalent. Both have an objective fitness in the numerator, and a niche count, calculated over the entire current population, in the denominator. In Equation 1, the summation in the niche count is taken over the population of individuals (using the variable j). In Equation 2, the population is partitioned into a set $S(P)$ of species y, thus $y \in S(P)$. Each species consists of the set of all individuals with the same chromosomes (from the current population). Thus the shared fitness for any member of a species x is equal to the objective fitness of that species divided by the niche count for that species, which is computed as the sum over all species of the interaction term (f_{xy}) multiplied (weighted) by the number of members of that species (i.e., the *species count*: n_y) in the current population P.

3   Analysis: Three-Niche Scenarios

We assume exactly three species, a, b, and c. Their objective (unshared) fitnesses are f_a, f_b, and f_c, representing the amount of resource(s) they cover. There are no other species in the population P. The entire finite population P is divided up among these three species: $S(P) = \{a, b, c\}$. That is, every individual in P is a member of a, b, or c, and all individuals of a species are considered to be identical (at least for the sake of selection). If p_a, p_b,

and p_c are the proportions of the population for species a, b, and c respectively, then $0 \le p_x \le 1$, $\forall x \in S(P)$, and $p_a + p_b + p_c = 1$. Since we are dealing with proportions, we do not need to name or manipulate an explicit population size $N = |P|$.

3.1   RFS Equilibrium

A population distribution is said to be at evolutionary equilibrium if it is equal to the expected distribution of the population after application of the selection operator (Maynard-Smith, 1982):

$$E[p_x(t+1)] = p_x(t), \qquad (3)$$

where $p_x(t)$ is the proportion of species x in the population at time t, and $E[p_x(t+1)]$ is the expected proportion of x at time $t + 1$ (e.g., the subsequent generation). Under RFS and proportionate selection, the expected proportion $p_x(t)$ of species x at time $t + 1$ is a function of its proportion $p_x(t)$ at time t and of its shared fitness $f_{sh,x}(t)$ and the population's average fitness $\overline{f}(t)$ at time t:

$$
\begin{aligned}
E[p_x(t+1)] &= p_x(t)\frac{f_{sh,x}(t)}{\overline{f}(t)} & (4) \\
&= p_x(t)\frac{f_{sh,x}(t)}{\sum_{y \in S(P)} p_y(t) f_{sh,y}(t)}. & (5)
\end{aligned}
$$

Substituting the above expression into Equation 3 and cancelling $p_x(t)$, we find that at equilibrium (here we drop the notation for a specific time step t) the shared fitness $f_{sh,x}$ of all species must be equal to the average fitness,

$$\forall(x \in S(P)) : f_{sh,x} = \sum_{y \in S(P)} p_y f_{sh,y},$$

and therefore must be equal to each other:

$$\forall(x, y \in S(P)) : f_{sh,x} = f_{sh,y}.$$

3.2   The General Case

For three species, the most general situation (i.e., arbitrary pair-wise overlaps; no assumptions) has the following equilibrium equations:

$$
\begin{aligned}
f_{sh,a} &= f_{sh,c} \\
f_{sh,c} &= f_{sh,b} \\
p_a + p_b + p_c &= 1.
\end{aligned}
$$

That is, all of the shared fitnesses are the same, so that no individual (a member of a species) and hence no species,

receives any preference during selection. Under RFS (as well as under Goldberg and Richardson's (1987) original fitness sharing), the shared fitness of an individual member of species x is simply the share of its objective (static) fitness f_x when divided by the current (dynamic) *niche count* for x:

$$\frac{f_a}{niche\_count(a)} = \frac{f_c}{niche\_count(c)},$$
$$\frac{f_c}{niche\_count(c)} = \frac{f_b}{niche\_count(b)},$$
$$p_a + p_b + p_c = 1.$$

Assuming that all objective fitnesses are equal, then without loss of generality we can normalize them to one, so that $\forall x \, f_x = 1$. Cross multiplying and substituting the formula for niche count yields (basically, we are setting all niche counts, shown in Equation 2, equal to each other)

$$\sum_{x \in S(P)} (p_x f_{cx}) = \sum_{x \in S(P)} (p_x f_{ax}),$$
$$\sum_{x \in S(P)} (p_x f_{bx}) = \sum_{x \in S(P)} (p_x f_{bx}),$$
$$p_a + p_b + p_c = 1.$$

Expanding the above, we find

$$p_a f_{ac} + p_b f_{bc} + p_c f_{cc} = p_a f_{aa} + p_b f_{ab} + p_c f_{ac},$$
$$p_a f_{ab} + p_b f_{bb} + p_c f_{bc} = p_a f_{ac} + p_b f_{bc} + p_c f_{cc},$$
$$p_a + p_b + p_c = 1.$$

Noting that $\forall x \, f_{xx} = 1$,

$$p_a f_{ac} + p_b f_{bc} + p_c = p_a + p_b f_{ab} + p_c f_{ac}, \quad (6)$$
$$p_a f_{ab} + p_b + p_c f_{bc} = p_a f_{ac} + p_b f_{bc} + p_c, \quad (7)$$
$$p_a + p_b + p_c = 1. \quad (8)$$

These niching equilibrium equations correspond to the most general situation with three niches/species.

3.3 Properties I and II: a and b Form an Exact Cover

Now we specialize Equations 6, 7, and 8 to deal with the "exact cover" by two out of three species. An example of this situation is shown in Figure 4.

> **Property I:** *Minimum $a \leftrightarrow b$ Competition*
> $$f_{ab} = 0$$

Now assume Property I, which is the special case in which species **a** and **b** do not overlap, so that $f_{ab} = 0$:

$$p_a f_{ac} + p_b f_{bc} + p_c = p_a + p_c f_{ac} \quad (9)$$
$$p_a f_{ac} + p_b f_{bc} + p_c = p_b + p_c f_{bc} \quad (10)$$
$$p_a + p_b + p_c = 1 \quad (11)$$

Re-arranging Equation 9 above,

$$(f_{ac} - 1)p_a + f_{bc}p_b + (1 - f_{ac})p_c = 0. \quad (12)$$

We now make the key assumption of Property II.

> **Property II:** *Maximum $(a, b) \leftrightarrow c$ Competition*
> $$f_{ac} + f_{bc} - f_{ab} = f_c$$

Under Property II species **c** is completely *covered* by **a** and **b**, which means (assuming Property I holds as well) that $f_c = f_{ac} + f_{bc} = 1$ and therefore $f_{ac} = 1 - f_{bc}$.

Substituting $1 - f_{bc}$ for f_{ac} in Equation 12 above gives

$$-f_{bc}p_a + f_{bc}p_b + f_{bc}p_c = 0 \quad (13)$$
$$\Rightarrow p_a - p_b - p_c = 0. \quad (14)$$

Similarly we re-arrange Equation 10,

$$f_{ac}p_a + (f_{bc} - 1)p_b + (1 - f_{bc})p_c = 0.$$

And substituting $1 - f_{ac}$ for f_{bc},

$$f_{ac}p_a - f_{ac}p_b + f_{ac}p_c = 0 \quad (15)$$
$$\Rightarrow p_a - p_b + p_c = 0. \quad (16)$$

From Equations 14, 16, and Equation 11, we conclude that

$$(p_a, p_b, p_c) = (\frac{1}{2}, \frac{1}{2}, 0)$$

Thus we have shown that if an exact cover of resources exists in a population, then under RFS selection only the species representing the exact cover will be expected to survive to niching equilibrium (with other species being driven to extinction), at least for the case of two-niche exact covers and a single (third) species not part of the exact cover.

We next examine how general is this result (the extinction of the third species c), by removing each of our key assumptions.
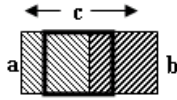
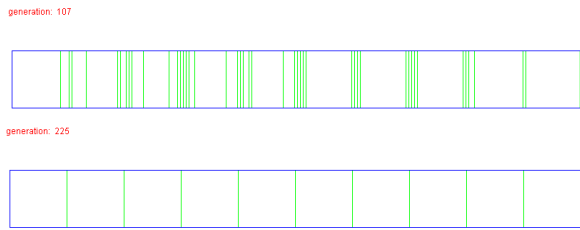Figure 4: A situation in which Properties I and II hold.

Figure 5: Properties I and II hold in this experiment from (Horn, 2002).

3.4 Property II Only

First, we remove the assumption of Property I, that species a and b do not overlap (that is, $f_{ab} = 0$). We retain the assumption of Property II, which implies that a and b completely cover the available resources, so that the union of their coverage contains all of the resources, including f_c. Figure 6 is an example of this situation.

Now that $f_{ab} > 0$, we must account for the fact that f_c might include f_{ab}. Returning to the set of equilibrium equations in 8, in which f_{ab} appears, we first solve the equations for the expected proportions of the three species at equilibrium, without using the assumption that a and b cover all resources. Thus the following result is general for all cases of overlap amongst a, b, and c:

$$p_a = \frac{(1 - f_{ab} - f_{ac} + f_{bc})(1 - f_{bc})}{D}$$

$$p_b = \frac{(1 - f_{ab} + f_{ac} - f_{bc})(-1 + f_{ac})}{D} \qquad (17)$$

$$p_c = \frac{(1 + f_{ab} - f_{ac} - f_{bc})(-1 + f_{ab})}{D}$$
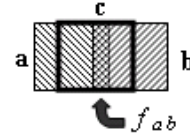

Figure 6: Here Property II, but not I, holds.

where the common denominator, D, is

$$D = f_{ab}^2 + f_{ac}^2 - 2f_{ac}(f_{bc} - 1) + 2f_{bc}$$
$$+ f_{bc}^2 - 2f_{ab}(-1 + f_{ac} + f_{bc}) - 3\,.$$

We now return to Property II, maximum competition between c and the pair (a, b). We assert here that maximum competition implies (1) c is completely covered by a and b so that there are no resources that are covered only by c, and (2) any overlap f_{ab} is completely covered[2] by c. Thus assuming maximum competition between c and (a, b),

$$f_c = f_{ac} + f_{bc} - f_{ab} = 1\,, \qquad (18)$$

since the quantity $f_{ac} + f_{bc}$ includes the intersection f_{ab} twice.

Before applying $f_{ac} + f_{bc} - f_{ab} = 1$ to the numerators on the right side of Equations 17, we check to see that that the fractions are meaningful by making sure that $D \neq 0$. Substituting $f_{ab} = f_{ac} + f_{bc} - 1$ (from Equation~\ref{eq:PropII}), we get

$$D = 4(f_{ac} - 1)(1 - f_{bc})\,.$$

Setting the above expression for D equal to zero and solving, yields $f_{ac} = 1$ and $f_{bc} = 1$. Thus the expressions in Equations 17 are useful outside of the extreme cases in which species c is identical (in coverage) to either a or b.

Substituting the Property II assumption ($f_{ab} = f_{ac} + f_{bc} - 1$) into the numerators of the expressions in Equations 17 above, and simplifying, yields

$$(p_a, p_b, p_c) = (\frac{1}{2}, \frac{1}{2}, 0),$$

the same proportions as earlier. Thus even with some overlap between the two "covering" species (a and b),

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[2]This second implication of maximum competition follows from the idea that any "private" competition between a and b would somehow detract from their mutual competition with c.

species c will be eliminated, as long as a and b *maximally compete* with c (Property II). It is clear that the dominance of the cooperative species pair a, b depends solely on Property II, the complete coverage of resources by a and b together. It is not necessary for a and b to be disjoint (i.e., minimally competitive) in their coverage; as long as both species are needed for complete coverage (i.e., $f_{ab} < 1$), and c is NOT needed (i.e., $f_c - f_{ac} - f_{bc} + f_{ab} = 0$), and c covers any competition (f_{ab}) between the two, then c will be driven to extinction while a and b will survive.

Theorem 1 *Assuming a population made up of only three distinct species (a, b, c), a* **sufficient** *condition for RFS with proportionate selection to drive species c, and only species* **c**, *to extinction is the following: $f_{ac} + f_{bc} - f_{ab} = f_c$.*

PROOF: The proof is given above.

Corollary 1 *If two species* **a** *and* **b** *form an exact cover of resources, then RFS with proportionate selection will drive any third species* **c** *to extinction.*

PROOF: Since a and b form an exact cover, then $f_{ab} = 0$ and $f_{ac} + f_{bc} = f_c$ (all resources are covered by a and b). Therefore the sufficient condition $f_{ac} + f_{bc} - f_{ab} = f_c$ from Theorem 1 holds.

Next we look at some situations in which a and b do NOT completely cover the resources; that is, c covers some resources "of its own": $f_c > f_{ac} + f_{bc}$. So we remove the assumption of Property II. Will c then survive at equilibrium?

3.5 Property I Only

To simplify our analysis, we constrain our niching situation in two ways. First, we assume disjoint coverage by returning to the assumption of Property I. That is, we now assume that a and b do NOT overlap ($f_{ab} = 0$). (While removing the assumption of Property II, we are adding the assumption of Property I). Thus our equilibrium equations are the same as in Equations 9, 10, and 11 above. Second, we assume a third property,

> **Property III**, *Symmetry of Competition*: $f_{ac} = f_{bc}$

These two assumptions, of Properties I and III, simplify our foray into situations in which Property II, maximum competition, does not hold.

Property III allows us to substitute f_{ac} for f_{bc} in Equations 9, 10, and 11. After substituting and re-arranging, we have the following set of equations:

$$(f_{ac} - 1)p_a + f_{ac}p_b + (1 - f_{ac}p_c) = 0, \quad (19)$$

$$(-f_{ac})p_a - (1 - f_{ac})p_b + (f_{ac} - 1)p_c = 0, \quad (20)$$

$$p_a + p_b + p_c = 1. \quad (21)$$

Adding the first two equations above yields $-p_a + p_b = 0$ which means that $p_a = p_b$. Substituting this last result into the Equation 21 above, and solving for p_c results in

$$p_c = 1 - 2p_a, \quad (22)$$

which can then be substituted for p_c in Equation 20 above, giving

$$(-f_{ac})p_a - (1 - f_{ac})p_b + (f_{ac} - 1)(1 - 2p_a) = 0.$$

Solving this for p_a results in

$$p_a = \frac{1 - f_{ac}}{3 - 4f_{ac}} = p_b.$$

Substituting these values of p_a and p_b into Equation 21 and solving for p_c:

$$p_c = \frac{1 - 2f_{ac}}{3 - 4f_{ac}}. \quad (23)$$

Equation 23 is plotted in Figure 7, which shows how the population proportion of c decreases with increasing overlap f_{ac} up to $f_{ac} = \frac{1}{2}$. (Recall that since $f_{ac} = f_{bc}$ the maximum value of f_{ac} is $\frac{1}{2}$ so that the denominator in Equation 23 can never be zero.) As Figure 7 reveals, when $f_{ac} = 0$, there is no overlap between any of the three species, and the equilibrium distribution divides the population equally among a, b, and c.

3.6 Visualizing the Combined Results

We can now visualize all of the above results together in a three dimensional plot of species c (that is, p_c) as a function of pairwise overlap with species a and b (that is, as a function of f_{ac} and f_{bc}).

First we re-compute the general function $p_c(f_{ac}, f_{bc})$ under only one assumption, that of Property I ($f_{ab} = 0$). Going back to the most general expression for p_c, in Equation 17, we substitute 0 for f_{ab} and get

$$p_c = \frac{1 - (f_{ac} + f_{bc})}{f_{ac}^2 + f_{bc}^2 + 2f_{ac}f_{bc} + 2f_{ac} + 2f_{bc} - 3}.$$

We note that the expression above applies only if $f_{ac} + f_{bc} \leq 1$ (otherwise, p_c becomes negative). Property I implies this condition,

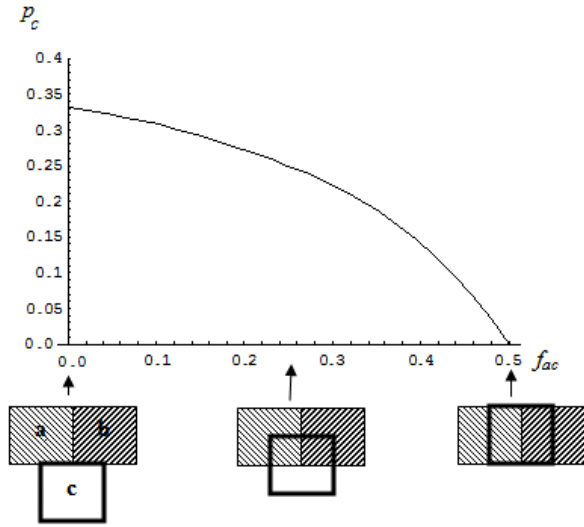$$f_{ab} = 0 \Rightarrow f_{ac} + f_{bc} \leq 1,$$

Figure 7: When Properties I and III hold, but II does not, then p_c decreases with increasing overlap $f_{ac} = f_{bc}$.

therefore,

$$f_{ac} + f_{bc} > 1 \Rightarrow f_{ab} > 0.$$

If there is non-zero overlap between a and b, then there are many ways for c to interact with a, b, and the overlap of a and b. In particular, when $f_{ab} > 0$, and there are no other contraints on the species' interactions, then p_c becomes a function of f_{ac}, f_{bc}, f_{ab}, and f_{abc}. So to keep our visualization of p_c a surface plot function of just f_{ac} and f_{bc}, we assume Property II ($f_{ac} + f_{bc} - f_{ab} = f_c = 1$) when $f_{ac} + f_{bc} > 1$. We have already shown that $p_c = 0$ under the assumption of Property II. Thus we have

$$p_c = \begin{cases} \frac{1 - f_{ac} - f_{bc}}{(f_{ac}+f_{bc})^2 + 2(f_{ac}+f_{bc}) - 3} & \text{if } f_{ac} + f_{bc} \leq 1, \\ 0 & \text{if } f_{ac} + f_{bc} \geq 1. \end{cases} \quad (24)$$

In Figure 8 we plot this $p_c(f_{ac}, f_{bc})$. We can see that species c should survive at equilibrium if it covers any resources not covered by a or b (i.e., $1 - f_{ac} - f_{bc} > 1$). We can see how all three species receive an equal share (one third) of the population when there is no inter-species overlap at all (i.e., when $f_{ac} = f_{bc} = f_{ab} = 0$). And we can see that the representation of c, that is, p_c, decreases with any increase in the combined overlap with other species (that is, $f_{ac} + f_{bc}$).

To help relate the surface in Figure 8 to the various earlier analyses under different assumptions, we label a contour plot of the surface with the assumed properties (I,
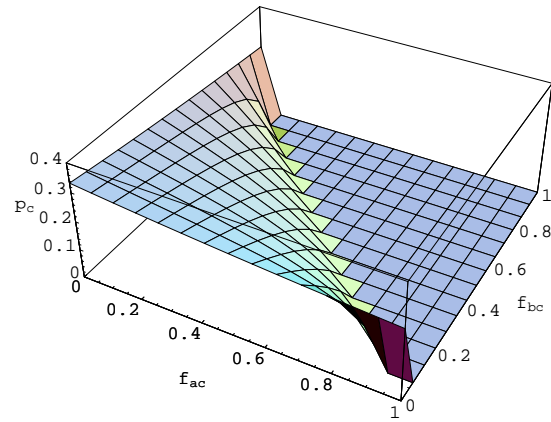

Figure 8: The more general situation in which c's overlaps with a and with b are allowed to vary independently.

II, or both) in Figure 9. In the figure, the lower left triangle of the f_{ac}-f_{bc} plane is the region in which $f_{ac} + f_{bc} \leq 1$, and so Property I, $f_{ab} = 0$, can hold.

4 Conclusions

The RFS algorithm, unlike many other effective and important co-evolutionary systems, lends itself to a static analysis. Interaction among individuals, and thus between species, is limited to pair-wise competition for resources. Furthermore, the formulation for the niche-count calculation leads to linear equations to describe niching equilibrium. These linear equations can be manipulated for analysis via simple algebra.

The analytical results for three interacting species show that if two species together exactly cover the resources of a third species, then the first two species will take over the population at niching equilibrium, driving the third, "redundant" species to extinction. Furthermore, for the three niche case, if a two-niche *complete cover* exists, even if not exact, then that cover will be "selected" by RFS niching, and will be the sole surviving *ensemble* at niching equilibrium, as long as the third species covers any overlap between the first two. If however, all three species cover some resource(s) uniquely (that is, for each species there is some resource covered only by that species), then all three species will be present when the population reaches niching equilibrium.

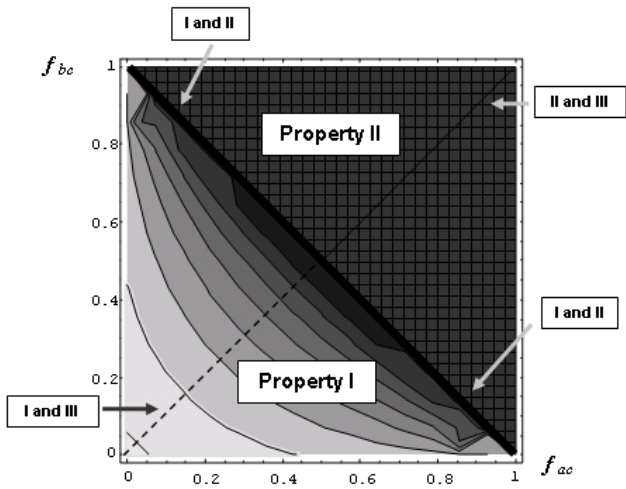One might interpret the relationship between two covering species to be "cooperative" in that they both

Figure 9: To plot a function of two independent variables (overlaps f_{ac} and f_{bc}), certain properties are assumed in certain regions.

Figure 10: Examples reveal the range of three-way niching situations in the surface plot of p_c.

"compete" against (overlap) a common competitor: the third, and losing, species. This is a remarkable result. The three species all have the same *objective* fitness (that is, unshared fitness); they all cover the same amount of resources. Yet RFS selection strongly favors two of the three. This preference must be due solely to the greater resource coverage of one particular ensemble of species over any other.

We note that the RFS algorithm analysed here, and the analysis itself, are general to all types of RFS applications. The results are not limited to axis-aligned squares. They apply to any shapes, with or without rotation, in any number of dimensions. Furthermore, the results of this paper are not limited to spatial "nesting" of geometric shapes. The RFS algorithm, and hence the current analyis of the algorithm, apply to the nesting of any kind of sets. Shape overlap is really just a special case of set intersection. Thus the most general problem domain to which this analysis is applicable is *exact k-cover* (from the general domain of *set covering*).

5   Future Work

This paper presents only a first foray into the theoretical analysis of RFS. In particular, it uses only a static analysis, looking only at niching equilibrium, and considers only three niches, in which two of the three form a complete cover of the resources.
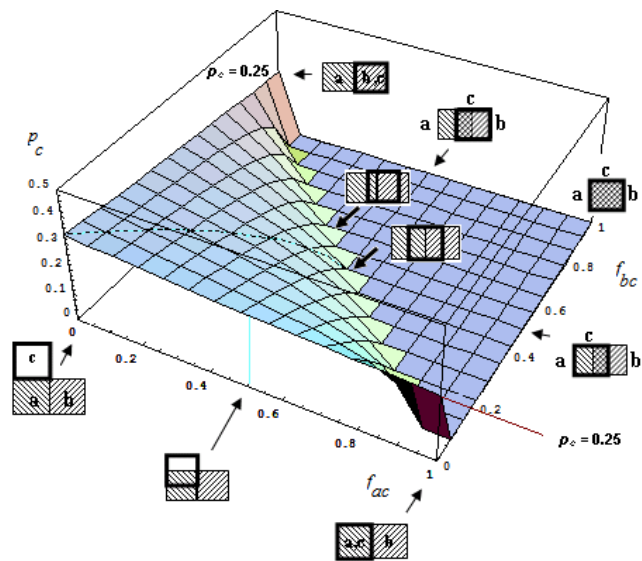
A logical next step in the analysis would be to consider the case of two-against-k. Would the two covering species still emerge as the sole surviving species at niching equilibrium, when competing against two, three, or an arbitrarily large number k of other, covered, species? If the answer is "yes", then does that result extend to h ($h > 2$) covering species against k species? If so, then we will have proven that RFS converges to the optimal tiling in shape nesting problems (e.g., Horn, 2002).

Other directions for future work include a comparison with *resource sharing* (Horn, Goldberg, & Deb, 1994). Does resource sharing behave similarly to RFS when two of three species exactly/completely cover all the resources? (We note that there seems to be little need to conduct a similar analsis of, and comparison with, *fitness sharing* (Goldberg & Richardson, 1987) because FS and RFS use the same functional structure in their sharing functions, with the caveat that FS is limited to rotationally symmetric niche shapes.) Finally, beyond static analysis lie many tools of dynamic analysis of niching and sharing methods (Deb & Goldberg, 1989; Horn, 1997), such as convergence to equilibrium, stability of equilibrium, using expected proportions over time via recurrence relations, and Markov chains, which have mostly been applied to two-niche scenarios.

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