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Towards an Operational Interpretation of Membership Grades – On H-Valued Fuzzy Sets and Their Use for Fuzzy Quantification

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***Abstract*— Recent advances in fuzzy quantification have rendered possible a consistent interpretation of quantifying expressions involving vague quantifiers and fuzzy arguments (Glöckner 2006, Díaz-Hermida et al 2005). However, the assumption of these approaches that the modeller is able to specify precise [0,1]-valued membership functions for the involved fuzzy sets and fuzzy quantifiers can be too strong in certain cases. To alleviate this problem, we extend the existing theory of fuzzy quantification to lattice-valued fuzzy sets which no longer require a specification of precise numerical membership grades. The paper focuses on a special type of so called $\mathscr{H}$-lattices whose Hasse diagram has an hourglass shape. In this setting, we can achieve an operational interpretation of membership values, which can be calculated automatically provided that the modeller (a) decides on the basic tendency of the membership assessments and (b) specifies the salient ordering relationships between the confidence levels. The generalization of the existing theory of fuzzy quantification to $\mathscr{H}$-valued fuzzy sets is a straightforward task and few properties of the models will be lost when turning from $[0,1]$ to the generalized valuations. It is even possible to devise a generic construction which assigns a plausible model of fuzzy quantification to any given $\mathscr{H}$-lattice.**

I. Introduction

A. Preliminaries

Probability theory has a clear operational foundation in empirical probabilities, and there is even progress in operationalizing subjective probabilities in terms of gambling (see e.g. [1]). By contrast, the difficulty of establishing and defending a particular choice of numeric membership grades is still one of the major drawbacks of fuzzy set theory both from methodological perspective and from the point of view of applications, which depend crucially on the chosen membership functions. It is a popular view that focusing on the (relative) order of membership grades rather than their particular values might alleviate this problem. While the ordinal nature of the connectives min and max supports this view, we must not forget about the negation $\neg x = 1 - x$, however, which associates membership grades with their negation correlates. In this way, an additional symmetry is defined on the membership grades, and the value $\frac{1}{2}$, which has the distinguished property that $\neg\frac{1}{2} = \frac{1}{2}$, can be taken to represent a neutral position. This additional structure is missing in the purely ordinal picture. More sophisticated solutions must therefore be developed which also account for the negation of membership grades.

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Several proposals are described in the literature which generalize from precise $[0,1]$-valued membership functions, e.g. interval-valued and Type II fuzzy sets (see e.g. [2]), intuitionistic fuzzy sets [3], $\mathscr{L}$-fuzzy sets [4], and intuitionistic $\mathscr{L}$-fuzzy sets [5].

In this paper, we seek a solution which is compatible to the models of fuzzy quantification developed by Glöckner [6], [7]. As opposed to the probabilistic model described by Díaz-Hermida et al [8], these models are essentially based on the notions of three-valued cuts and supervaluation, which means that they incorporate Kleene-Dienes logic. In order to successfully generalize these models to lattice-valued fuzzy sets, we must therefore consider special kinds of lattices which are compatible with the supervaluationist and essentially three-valued view of fuzzy quantification.

B. Basic idea of decomposing membership grades into a polarity and confidence aspect

It has been pointed out that the negation $\neg x = 1 - x$ adds some additional structure to $\mathbf{I} = [0,1]$. In order to capture the symmetry around $\frac{1}{2}$, we will separate the tendency aspect of a membership assessment (i.e. YES/NO, true/false) from the degree of confidence that can be ascribed to this judgement. Both aspects together constitute the membership grade. The tendency of $\alpha \in \mathbf{I}$ is defined by $\mathrm{t} : \mathbf{I} \longrightarrow \{0,1\}$ with

$$\mathrm{t}(\alpha) = \begin{cases} 1 & : \alpha \geq \frac{1}{2} \\ 0 & : \alpha < \frac{1}{2} \end{cases} \tag{1}$$

The confidence grade of $\alpha \in \mathbf{I}$ is given by the following mapping $\mathrm{c} : \mathbf{I} \longrightarrow \mathbf{I}$,

$$\mathrm{c}(\alpha) = \begin{cases} 2\alpha - 1 & : \alpha \geq \frac{1}{2} \\ 1 - 2\alpha & : \alpha < \frac{1}{2} \end{cases} \tag{2}$$

for all $\alpha \in \mathbf{I}$. Given a tendency $\omega \in \{0,1\}$ and a confidence grade $\gamma \in \mathbf{I}$, we can reconstruct the corresponding membership grade as follows:

$$\mathrm{h}(\omega,\gamma) = \begin{cases} \frac{1}{2} + \frac{1}{2}\gamma & : \omega = 1 \\ \frac{1}{2} - \frac{1}{2}\gamma & : \omega = 0 \end{cases} \tag{3}$$

resulting in a mapping $\mathrm{h} : \{0,1\} \times \mathbf{I} \longrightarrow \mathbf{I}$. Let us notice that the stipulation $\mathrm{t}(\frac{1}{2}) = 1$ in the undecided case does not express any bias toward $\omega = 1$ because in this case, we have $\gamma = 0$, and $\mathrm{h}(0,0) = \mathrm{h}(1,0) = \frac{1}{2}$, so the choice of ω is inessential.

It should be apparent how this method can be used to assess membership grades. The modeller is first asked to decide on the basic tendency of "$e \in X$?", where 'undecided' is also an option. If there is no tendency, then we have $\omega = 1$ and $\gamma = 0$, resulting in $\alpha = \mathrm{h}(1,0) = \frac{1}{2}$. If there is a tendency towards NO, then the modeller is asked about the confidence in this judgement, thus assessing the γ level. This permits the reconstruction of the membership grade $\mathrm{h}(0,\gamma)$. Similarly we query for the confidence in the YES decision when there is a positive tendency, and the final membership grade is $\mathrm{h}(1,\gamma)$ in this case.

Let us briefly compare this analysis to existing work on modeling membership assignments with different degrees of commitment. It is customary to describe underspecified membership assignments in terms of interval-valued fuzzy sets, see e.g. [2, Sect. 4.5]. The membership intervals $[\mu_\ell, \mu_u]$ can be represented by ordered pairs $\langle \mu_\ell, \mu_u \rangle \in \mathbf{I} \times \mathbf{I}$ with $\mu_\ell \le \mu_u$ where $\langle 0,0 \rangle$ means completely false, $\langle 1,1 \rangle$ means fully true, and $\langle 0,1 \rangle$ represents indecision.

What our analysis reveals is that the interval representation is actually richer than needed. The smaller system of membership grades $\langle \omega, \gamma \rangle$ with tendency $\omega \in \{0,1\}$ (rather than $\mathbf{I} = [0,1]$) and confidence $\gamma \in \mathbf{I}$ is already sufficient to describe underspecified membership under the proposed interpretation.

Therefore the interval approach offers excess degrees of freedom which potentially make it even more difficult to ascertain the intended membership assignment.

C. Overview of the paper

The paper starts from the proposed analysis of membership assessments in terms of tendency and confidence which also underlies the models of fuzzy quantification described in [6]. However, in order to simplify the specification of the confidence aspect, we now drop the assumption that the confidence grades are expressed numerically. The new approach admits arbitrary choices of confidence values and merely assumes that the modeller is able to compare *some* of the confidence values, and to group those confidence assignments deemed to belong to the same level. These comparisons result in a quasi-order on the confidence grades which forms the basis for computing an associated confidence lattice. The description of membership grades in terms of basic tendency and confidence level is then translated into a membership grade in a so-called $\mathscr{H}$-lattice whose structure mimicks the negation symmetry of the unit interval. We show how fuzzy quantification can be carried out in this framework and present a canonical model which assigns meaningful interpretations to fuzzy quantifiers regardless of the assumed (complete) lattice of confidence levels.[1]

II. Main results

A. Definition of $\mathscr{H}$-lattices

We refer to [10] for the basic notions of a lattice, complete lattice, lattice homomorphism, and $\{0,1\}$-homomorphism which preserves the top and bottom elements.

[1]The proofs of all theorems presented in this work are published in a technical report [9].

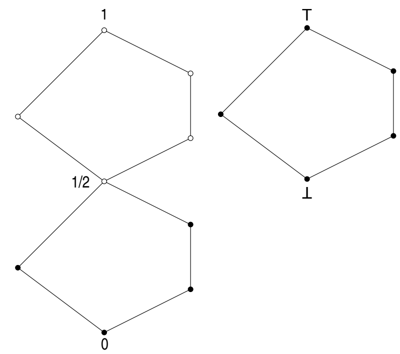

Fig. 1. Hasse diagram of an $\mathscr{H}$-lattice (shown left) and confidence lattice

Definition 1 ($\mathscr{H}$-lattice) *Let H be a non-empty set. A structure $\mathscr{H} = \langle H; \wedge, \vee, \neg, 0, 1, \frac{1}{2} \rangle$ with two binary operations $\wedge$, $\vee$, a unary operation $\neg$ and constants $0, \frac{1}{2}, 1 \in H$ is called an $\mathscr{H}$-lattice if the following hold:*

a. $\langle H; \wedge, \vee, 0, 1 \rangle$ *is a complete lattice with top element* 1 *and bottom element* 0*;*
b. $\forall x \in H$, $\neg\neg x = x$, *i.e.* $\neg$ *is an involution;*
c. $\forall x, y \in H$, $x \le y \Rightarrow \neg y \le \neg x$, *i.e.* $\neg$ *is order-reversing;*
d. $\forall x \in H$, $x \le \frac{1}{2}$ *or* $x \ge \frac{1}{2}$*;*
e. $\neg \frac{1}{2} = \frac{1}{2}$.

Notes. (a) It should be pointed out that $\neg$ is an order-reversing involution (similar to $1-x$) and *not* a complement in the sense of lattice theory. (b) The letter 'H' in $\mathscr{H}$-lattice reminds of the *hourglass shape* of the Hasse diagram for these lattices. This shape is clearly visible in the example of an $\mathscr{H}$-lattice shown in Fig. 1.

Lemma 1 *Let $\mathscr{H} = \langle H; \wedge, \vee, \neg, 0, 1, \frac{1}{2} \rangle$ be an $\mathscr{H}$-lattice. Then*

1) $\neg 0 = 1$ *and* $\neg 1 = 0$.
2) $(\forall \alpha \in H)$ *If* $\neg\alpha = \alpha$*, then* $\alpha = \frac{1}{2}$.
3) $(\forall \alpha, \alpha')$ *If* $\alpha \le \frac{1}{2}$ *and* $\alpha' \ge \frac{1}{2}$*, then* $\alpha \vee \alpha' = \alpha'$ *and* $\alpha \wedge \alpha' = \alpha$.
4) $(\forall A \subseteq H)$ $\bigvee\{\neg\alpha : A \in A\} = \neg \bigwedge A$.
5) $(\forall A \subseteq H)$ $\bigwedge\{\neg\alpha : A \in A\} = \neg \bigvee A$.
6) $(\forall A \subseteq H)$ *If* $\bigwedge A < \frac{1}{2}$*, then* $\bigwedge A = \bigwedge\{\alpha \in A : \alpha < \frac{1}{2}\}$.
7) $(\forall A \subseteq H)$ *If* $\bigvee A > \frac{1}{2}$*, then* $\bigvee A = \bigvee\{\alpha \in A : \alpha > \frac{1}{2}\}$.
8) $(\forall A \subseteq H)$ $\bigwedge\{\alpha \vee \frac{1}{2} : \alpha \in A\} = \frac{1}{2} \vee \bigwedge A$.
9) $(\forall A \subseteq H)$ $\bigvee\{\alpha \wedge \frac{1}{2} : \alpha \in A\} = \frac{1}{2} \wedge \bigvee A$.

We define the notions of $\mathscr{H}$-lattice homomorphism and $\mathscr{H}$-lattice isomorphism in the obvious ways:

Definition 2 *Let $\mathscr{H}, \mathscr{H}'$ be $\mathscr{H}$-lattices. A mapping $\phi : H \longrightarrow H'$ is called an $\mathscr{H}$-*lattice homomorphism *if*

1) ϕ *is a* $\{0,1\}$*-homomorphism;*
2) $\neg' \phi(x) = \phi(\neg x)$ *for all* $x \in H$.

*A bijective $\mathscr{H}$-lattice homomorphism is called an $\mathscr{H}$-*lattice isomorphism. *We then write $\mathscr{H} \cong \mathscr{H}'$.*

Lemma 2 *Let $\phi : \mathscr{H} \longrightarrow \mathscr{H}'$ be an $\mathscr{H}$-lattice homomorphism. Then $\phi(\frac{1}{2}) = \frac{1}{2}'$.*

B. The confidence lattice $\mathscr{C}$ for an $\mathscr{H}$-lattice

As motivated in the introduction, we assume a separation of membership assessment into a decision on the basic tendency of the membership grade (true or false) and the subsequent specification of the confidence level for this decision. We are therefore interested in decomposing a given $\mathscr{H}$-lattice into a basic truth value assignment and the underlying lattice of confidence levels.

Definition 3 (Confidence lattice) *Let $\mathscr{H}$ be an $\mathscr{H}$-lattice and suppose that the lattice $\mathscr{C} = \langle C; \sqcap, \sqcup, \top, \bot\rangle$ is isomorphic to the sublattice $\mathscr{H}^+ = \mathscr{H} \cap \{\alpha \in \mathscr{H} : \alpha \geq \frac{1}{2}\}$, with lattice isomorphism $c_+ : \mathscr{H}^+ \longrightarrow \mathscr{C}$. Then $\mathscr{C}$ is called a* confidence lattice for $\mathscr{H}$.

Notes. (a) The operations of $\mathscr{C}$ are symbolized $\sqcap, \sqcup$ and $\top$, $\bot$ rather than $\wedge$, $\vee$, 1 and 0 in order to avoid confusion with the operations of $\mathscr{H}$. For the same reason, the symbol '$\sqsubseteq$' is used for the partial order relation of $\mathscr{C}$, while the partial order relation of $\mathscr{H}$ is written as '$\leq$'. (b) The shaded points in Fig. 1 (left-hand side) correspond to the $\mathscr{H}^+$ part of the lattice. An isomorphic confidence lattice is shown in Fig. 1 on the right. (c) Obviously, $\mathscr{H}^+$ itself (with the sublattice order) qualifies as a confidence lattice for $\mathscr{H}$. In this case, $\bot = \frac{1}{2}$, $\top = 1$, $\sqcap = \wedge$ (restricted to $\mathscr{H}^+$) and $\sqcup = \vee$ (also restricted to $\mathscr{H}^+$). However, as we have seen in the introductory example where $\mathscr{C}$ was defined on $[0,1]$, it can be practical and more suggestive to use a confidence lattice different from (though isomorphic to) the canonical choice $\mathscr{H}^+$.

Let us now generalize the analysis in terms of h, c and t presented in the introduction for $[0,1]$ to arbitrary $\mathscr{H}$-lattices and the corresponding confidence lattices $\mathscr{C}$.

In order to express the operations in the $\mathscr{H}$-lattice in terms of operations on the confidence lattice, we extend c_+ to $c : H \longrightarrow C$, defined by

$$c(\alpha) = \begin{cases} c_+(\alpha) & : \quad \alpha \geq \frac{1}{2} \\ c_+(\neg\alpha) & : \quad \alpha < \frac{1}{2} \end{cases} \qquad \forall \alpha \in H.$$

We define the tendency map $t : H \longrightarrow \{0,1\}$ by

$$t(\alpha) = \begin{cases} 1 & : \quad \alpha \geq \frac{1}{2} \\ 0 & : \quad \text{else} \end{cases} \quad \forall \alpha \in H.$$

Moreover, we introduce a mapping $h : \{0,1\} \times C \longrightarrow H$ by

$$h(\omega, \gamma) = \begin{cases} c_+^{-1}(\gamma) & : \quad \omega = 1 \\ \neg c_+^{-1}(\gamma) & : \quad \omega = 0 \end{cases} \qquad \forall \omega \in \{0,1\},\ \gamma \in C.$$

The following observations can be made concerning the interactions of $\mathscr{C}$ and $\mathscr{H}$.

Lemma 3 *Let $\mathscr{H}$ be an $\mathscr{H}$-lattice on a set $H \neq \varnothing$ and $\mathscr{C} = \langle C; \sqcap, \sqcup, \top, \bot\rangle$ a complete lattice of confidence grades for $\mathscr{H}$ connected by $c_+ : \mathscr{H}^+ \cong \mathscr{C}$ and $h : \{0,1\} \times C \longrightarrow H$. Then*

a. $0 = h(0, \top)$;
b. $1 = h(1, \top)$;
c. $\frac{1}{2} = h(0, \bot) = h(1, \bot)$;
d. *For all* $\alpha \in H$, $h(t(\alpha), c(\alpha)) = \alpha$;
e. *For all* $\alpha \in H$, $c(\neg\alpha) = c(\alpha)$;
f. *For all* $\alpha \neq \frac{1}{2}$, $t(\neg\alpha) = \neg t(\alpha)$;
g. $\neg\alpha = h(\neg t(\alpha), c(\alpha))$;
h. *For all* $\alpha, \alpha' \in H$,

$$\alpha \wedge \alpha' = \begin{cases} h(1, c(\alpha) \sqcap c(\alpha')) & : \quad t(\alpha) = t(\alpha') = 1 \\ h(0, c(\alpha)) & : \quad t(\alpha) = 0 \wedge t(\alpha') = 1 \\ h(0, c(\alpha')) & : \quad t(\alpha) = 1 \wedge t(\alpha') = 0 \\ h(0, c(\alpha) \sqcup c(\alpha')) & : \quad t(\alpha) = t(\alpha') = 0. \end{cases}$$

i. *For all* $\alpha, \alpha' \in H$,

$$\alpha \vee \alpha' = \begin{cases} h(1, c(\alpha) \sqcup c(\alpha')) & : \quad t(\alpha) = t(\alpha') = 1 \\ h(1, c(\alpha)) & : \quad t(\alpha) = 1 \wedge t(\alpha') = 0 \\ h(1, c(\alpha')) & : \quad t(\alpha) = 0 \wedge t(\alpha') = 1 \\ h(0, c(\alpha) \sqcap c(\alpha')) & : \quad t(\alpha) = t(\alpha') = 0. \end{cases}$$

j. *For all* $\alpha, \alpha' \in H$, $\alpha \leq \alpha' \Leftrightarrow (t(\alpha) < t(\alpha') \vee (t(\alpha) = t(\alpha') = 0 \wedge c(\alpha) \sqsupseteq c(\alpha')) \vee (t(\alpha) = t(\alpha') = 1 \wedge c(\alpha) \sqsubseteq c(\alpha')$.

These relationships reveal that the lattice $\mathscr{H}$ is uniquely determined by the lattice of cautiousness grades $\mathscr{C}$. We capture this by the following proposition.

Proposition 1 *Let $\mathscr{H}$ be an $\mathscr{H}$-lattice with a confidence lattice $\mathscr{C}$, and let $\mathscr{H}'$ be another $\mathscr{H}$-lattice. Then $\mathscr{H} \cong \mathscr{H}'$ if and only if $\mathscr{C}$ is also a confidence lattice for $\mathscr{H}$'.*

Corollary 1 *Suppose that $\mathscr{H}, \mathscr{H}'$ are $\mathscr{H}$-lattices. Then $\mathscr{H} \cong \mathscr{H}'$ if and only if $\mathscr{H}^+ \cong \mathscr{H}'^+$.*

C. Construction of $\mathscr{H}$-lattices from $\mathscr{C}$-lattices

At this point, we know that each $\mathscr{H}$-lattice comes with an associated confidence lattice. Let us now show that an $\mathscr{H}$-lattice can be constructed for any given (complete) lattice of confidence levels.

Definition 4 *Let $\mathscr{C} = \langle C; \sqcap, \sqcup, \top, \bot\rangle$ be a complete lattice. Then $\mathscr{H}(\mathscr{C}) = \langle H; \wedge, \vee, \neg, 0, 1, \frac{1}{2}\rangle$ is defined as follows: The base set is $H = (\{0,1\} \times C)/\langle 0, \bot\rangle \sim \langle 1, \bot\rangle$, i.e.*

$$H = \{\{\langle \omega, \gamma\rangle\} : \omega \in \{0,1\}, \gamma \in C \setminus \{\bot\}\} \cup \{\{\langle 0, \bot\rangle, \langle 1, \bot\rangle\}\}.$$

The constants are given by $0 = \{\langle 0, \top\rangle\}$, $1 = \{\langle 1, \top\rangle\}$ and $\frac{1}{2} = \{\langle 1, \bot\rangle, \langle 0, \bot\rangle\}$.
We introduce an auxiliary mapping $h : \{0,1\} \times C \longrightarrow H$,

$$h(\omega, \gamma) = \begin{cases} \{\langle \omega, \gamma\rangle\} & : \quad \gamma \neq \bot \\ \{\langle 0, \bot\rangle, \langle 1, \bot\rangle\} & : \quad \textit{else} \end{cases}$$

The unary operation $\neg$ is then given by

$$\neg\alpha = \begin{cases} \{\langle \neg\omega, \gamma\rangle\} & : \quad \alpha = \{\langle \omega, \gamma\rangle\} \\ \{\langle 0, \bot\rangle, \langle 1, \bot\rangle\} & : \quad \alpha = \{\langle 0, \bot\rangle, \langle 1, \bot\rangle\} \end{cases}$$

for all $\alpha \in H$. For $\wedge$, we set

$$\alpha \wedge \alpha' = \begin{cases} h(1, \gamma \sqcap \gamma') & : \quad \langle 1, \gamma\rangle \in \alpha \wedge \langle 1, \gamma'\rangle \in \alpha' \\ \alpha & : \quad \alpha = \{\langle 0, \gamma\rangle\} \wedge \langle 1, \gamma'\rangle \in \alpha' \\ \alpha' & : \quad \langle 1, \gamma\rangle \in \alpha \wedge \alpha' = \{\langle 0, \gamma'\rangle\} \\ h(0, \gamma \sqcup \gamma') & : \quad \alpha = \{\langle 0, \gamma\rangle\} \wedge \alpha' = \{\langle 0, \gamma'\rangle\} \end{cases}$$

for all $\alpha, \alpha' \in H$. *The dual construction is used for* $\vee$, *i.e.*

$$\alpha \vee \alpha' = \begin{cases} \mathrm{h}(1, \gamma \sqcup \gamma') & : \quad \langle 1, \gamma \rangle \in \alpha \wedge \langle 1, \gamma' \rangle \in \alpha' \\ \alpha & : \quad \langle 1, \gamma \rangle \in \alpha \wedge \alpha' = \{\langle 0, \gamma' \rangle\} \\ \alpha' & : \quad \alpha = \{\langle 0, \gamma \rangle\} \wedge \langle 1, \gamma' \rangle \in \alpha' \\ \mathrm{h}(0, \gamma \sqcap \gamma') & : \quad \alpha = \{\langle 0, \gamma \rangle\} \wedge \alpha' = \{\langle 0, \gamma' \rangle\}. \end{cases}$$

for all $\alpha, \alpha' \in H$.

Proposition 2 *Let* $\mathscr{C}$ *be any complete lattice. Then* $\mathscr{H}(\mathscr{C})$ *is an* $\mathscr{H}$*-lattice and* $\mathscr{C}$ *is a confidence lattice for* $\mathscr{H}(\mathscr{C})$.

Thus the construction of an $\mathscr{H}$-lattice from an arbitrary (complete) lattice of confidence grades was successful. This is important because we envision that the membership grades will only indirectly be specified in terms of the basic tendency and the confidence level.
The following corollary is obvious from Prp. 1 and Prp. 2. It states that the construction of $\mathscr{H}(\mathscr{C})$ is consistent with the existing structure on a given $\mathscr{H}$-lattice.

Corollary 2 *Let* $\mathscr{H}$ *be an* $\mathscr{H}$*-lattice and suppose that* $\mathscr{C}$ *is a confidence lattice for* $\mathscr{H}$. *Then* $\mathscr{H}(\mathscr{C}) \cong \mathscr{H}$.

D. $\mathscr{H}$*-fuzzy sets and the construction of membership grades*

Definition 5 ($\mathscr{H}$**-fuzzy subset)** *Let* $\mathscr{H}$ *be an* $\mathscr{H}$*-lattice with support H, and E a given set. An* $\mathscr{H}$*-fuzzy subset X of E is characterized by its membership function* $\mu_X : E \longrightarrow H$. *The set of all* $\mathscr{H}$*-fuzzy subsets of E is denoted* $\widetilde{\mathscr{P}_{\mathscr{H}}}(E)$.

Notes. (a) It is customary to identify fuzzy subsets and their membership functions. We do not enforce this identification because we would like to treat a crisp set Y (rather than its characteristic function χ_Y) as a special kind of fuzzy set. (b) Notice that $\mathscr{H}$-fuzzy sets in our sense, i.e. fuzzy sets with membership grades in a given $\mathscr{H}$-lattice, have nothing to do with Heyting-valued fuzzy sets also described in the literature.
The $\mathscr{H}$-fuzzy lattice avails us with binary connectives $\wedge$ (which serves as the $\mathscr{H}$-fuzzy conjunction), $\vee$ (the $\mathscr{H}$-fuzzy disjunction) and the unary $\mathscr{H}$-fuzzy negation $\neg$. The $\mathscr{H}$-fuzzy set operations $\cap : \widetilde{\mathscr{P}_{\mathscr{H}}}(E)^2 \longrightarrow \widetilde{\mathscr{P}_{\mathscr{H}}}(E)$ ($\mathscr{H}$-fuzzy intersection), $\cup : \widetilde{\mathscr{P}_{\mathscr{H}}}(E)^2 \longrightarrow \widetilde{\mathscr{P}_{\mathscr{H}}}(E)$ ($\mathscr{H}$-fuzzy union), and $\neg : \widetilde{\mathscr{P}_{\mathscr{H}}}(E) \longrightarrow \widetilde{\mathscr{P}_{\mathscr{H}}}(E)$ ($\mathscr{H}$-fuzzy complement) are defined element-wise in terms of $\wedge$, $\vee$, and $\neg$, respectively. The subset relation of $\mathscr{H}$-fuzzy sets is declared by $X \subseteq X' \Leftrightarrow \mu_X(e) \leq \mu_{X'}(e)\ \forall e \in E$.

Let me now explain how the $\mathscr{H}$-lattice of confidence grades can be constructed incrementally along with the assessment of membership values of fuzzy sets of interest. Consider a set Φ_0 of propositions to be interpreted in an $\mathscr{H}$-lattice. Membership assessments asking for the result of $\mu_X(e)$ for a given $e \in E$ are a special case of such propositions. For example, Φ_0 might contain propositions like $\mathsf{happy}(\mathsf{Jan}), \mathsf{bald}(\mathsf{Jan}), \mathsf{bald}(\mathsf{Tom})$ describing the happiness and baldness of several persons. We assume that the modeller (i.e. knowledge engineer, user of the software system,...) decides on the basic tendency $\mathrm{t}(\phi) \in \{0,1\}$ of each proposition $\phi \in \Phi_0$ with respect to being true or false (abstaining from decision is also possible and results in the undecided value $\mathrm{t}(\phi) = 1$, $\mathrm{c}(\phi) = \bot$). In the normal case that the modeller commits to specific tendency, however, the confidence $\mathrm{c}(\phi)$ into this judgement must be specified separately. To this end, the modeller must compare the confidence into the decision that "$\mathrm{t}(\phi) = 0$" or "$\mathrm{t}(\phi) = 1$" to the other cases already integrated into the evolving confidence order, and place the new item at an appropriate position in the graph. It is possible that several items share the same confidence assignment. This means that at this level of analysis, we must describe the confidence structure specified by the user by a *quasi-order*, i.e. by a reflexive and transitive relation $\trianglelefteq$. In order to make sure that there is a top element representing total confidence and a bottom element representing complete indecision, we define $\trianglelefteq$ on $\Phi = \Phi_0 \cup \{\top, \bot\}$, stipulating that $\bot \trianglelefteq \phi \trianglelefteq \top$ for all $\phi \in \Phi$. The quasi-order $\trianglelefteq$ determines an equivalence relation $\bowtie$ on Φ, defined by $\phi \bowtie \phi' \Leftrightarrow \phi \trianglelefteq \phi' \wedge \phi' \trianglelefteq \phi$. We then work with the quotient $\Phi/\bowtie = \{[\phi]_{\bowtie} : \phi \in \Phi\}$ which contains all equivalence classes under $\bowtie$. The relation $\sqsubseteq$ on $\Phi/\bowtie$ defined by $[\phi]_{\bowtie} \sqsubseteq [\phi']_{\bowtie} \Leftrightarrow \phi \trianglelefteq \phi'$ qualifies as a partial order (i.e. it is reflexive and transitive, like the original $\trianglelefteq$, but in addition antisymmetric). We apply the Dedekind-McNeille completion [10, p. 166] in order to turn $\Psi = \langle \Phi/\bowtie, \sqsubseteq \rangle$ into a complete lattice, the lattice of confidence grades $\mathscr{C} = \mathrm{DM}(\Psi)$. From the confidence lattice, we finally obtain the $\mathscr{H}$-lattice of membership grades by $\mathscr{H} = \mathscr{H}(\mathscr{C})$, see Def. 4.

The described construction of membership values is visualized by the Hasse diagrams in Fig. 2. The diagram for $\Psi = \langle \Phi/\bowtie, \sqsubseteq \rangle$ on the left-hand side shows the equivalence classes in $\Phi/\bowtie$ of those propositions to which the modeller assigns the same level of confidence. In the example, the partial order Ψ shown in the Hasse diagram is already a complete lattice. This means that the confidence lattice $\mathscr{C} = \mathrm{DM}(\Psi)$ is isomorphic to Ψ in this case. The Hasse diagram on the left then shows the corresponding $\mathscr{H}$-lattice $\mathscr{H}(\mathscr{C})$ which results from the construction of Def. 4. The nodes of the diagram, which correspond to the available membership grades, are labelled with those propositions ϕ to which a corresponding membership grade $\alpha = \mathrm{h}(\mathrm{t}(\phi), \mathrm{c}(\phi))$ has been assigned. The fuzzy sets which evolve from this construction then provide the basis for subsequent knowledge processing.

It should be pointed out that the results of logical operations can also be displayed graphically in the diagram of the confidence order so that a person working with $\mathscr{H}$-fuzzy sets will never need to directly manipulate membership grades, or even see their actual representation. For example, the conjunction $\mathsf{happy}(\mathsf{Jan}) \wedge \mathsf{bald}(\mathsf{Jan})$ which has $\mathrm{t}(\mathsf{happy}(\mathsf{Jan})) = 1$, $\mathrm{t}(\mathsf{bald}(\mathsf{Jan})) = 0$, and $\mathrm{c}(\mathsf{bald}(\mathsf{Jan})) = \mathrm{c}(\mathsf{bald}(\mathsf{Tom}))$, will evaluate to $\mathrm{c}(\mathsf{happy}(\mathsf{Jan}) \wedge \mathsf{bald}(\mathsf{Jan})) = 0$ and $\mathrm{c}(\mathsf{happy}(\mathsf{Jan}) \wedge \mathsf{bald}(\mathsf{Jan})) = \mathrm{c}(\mathsf{bald}(\mathsf{Jan}))$. This means that the result of $\sim(\mathsf{happy}(\mathsf{Jan}) \wedge \mathsf{bald}(\mathsf{Jan}))$ should be displayed at the position currently labeled $\{\mathsf{bald}(\mathsf{Tom}), \sim\mathsf{bald}(\mathsf{Jan})\}$ in the graphical representation of the confidence lattice on the left of Fig. 2. This is enough to indicate to the user that the proposition has

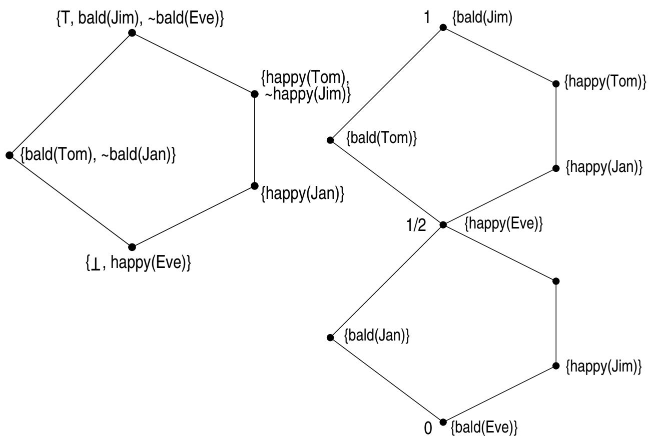

Fig. 2. Membership assessment in the $\mathscr{H}$-lattice framework: The user decides on the tendency of membership (falsity is marked by '$\sim$') and incrementally builds the confidence order by integrating membership assessments according to the relative confidence of the tendency decision (see Hasse diagram of $\Phi/\bowtie$ on the left, which is isomorphic to $\mathscr{C}$ in this case). The Hasse diagram of the resulting $\mathscr{H}$-lattice $\mathscr{H}(\mathscr{C})$ is shown on the right.

a tendency toward being false and that the confidence into this judgement is comparable to the modeller's confidence that Tom is bald, and also to the confidence that Jan is not bald.

E. A framework for models of $\mathscr{H}$-fuzzy quantification

The $\mathscr{H}$-fuzzy connectives introduced so far make it possible to evaluate statements connected by propositional connectives like conjunction or disjunction. In the remainder of the paper, we will discuss how quantifying statements like *'Most bald are happy'* can be evaluated when the predicates are modeled by $\mathscr{H}$-valued sets. The framework for analysing $\mathscr{H}$-fuzzy quantification to be presented parallels the framework for $[0,1]$-valued fuzzy quantification described in [6].

Definition 6 ($\mathscr{H}$-fuzzy quantifier) *Let $\mathscr{H}$ be an $\mathscr{H}$-lattice with support H and $E \neq \varnothing$ a given set. An n-ary $\mathscr{H}$-fuzzy quantifier $\widetilde{Q}$ on E is a mapping $\widetilde{Q} : \widetilde{\mathscr{P}_{\mathscr{H}}}(E)^n \longrightarrow H$.*

For example, the $\mathscr{H}$-fuzzy universal quantifier on E can be defined by $\widetilde{\forall}_E(X) = \bigwedge\{\mu_X(e) : e \in E\}$ for all $X \in \widetilde{\mathscr{P}_{\mathscr{H}}}(E)$.

Definition 7 ($\mathscr{H}$-semi-fuzzy quantifier) *Let $\mathscr{H}$ be an $\mathscr{H}$-lattice with support H and $E \neq \varnothing$ a given set. An n-ary $\mathscr{H}$-semi-fuzzy quantifier Q on E is a mapping $Q : \mathscr{P}(E)^n \longrightarrow H$.*

For example, the $\mathscr{H}$-semi-fuzzy universal quantifier on E is defined by $\forall_E(Y) = 1$ if $Y = E$ and $\forall_E(Y) = 0$ otherwise.
A specification in terms of an $\mathscr{H}$-semi-fuzzy quantifier finds its matching $\mathscr{H}$-fuzzy quantifier by applying an $\mathscr{H}$-QFM.

Definition 8 ($\mathscr{H}$-Pre-QFM) *An $\mathscr{H}$-Pre-QFM $\mathscr{F}$ maps each $\mathscr{H}$-semi-fuzzy quantifier $Q : \mathscr{P}(E)^n \longrightarrow H$ to a corresponding $\mathscr{H}$-fuzzy quantifier $\mathscr{F}(Q) : \widetilde{\mathscr{P}_{\mathscr{H}}}(E)^n \longrightarrow H$.*

In order to express the basic compatibility of an $\mathscr{H}$-Pre-QFM with an $\mathscr{H}$-lattice, we introduce the set $\mathscr{H}$-fuzzy connectives associated with $\mathscr{F}$. These should agree with the operations $\wedge$, $\vee$ and $\neg$ of the $\mathscr{H}$-lattice.

Definition 9 (Induced $\mathscr{H}$-fuzzy truth functions) *Let $\mathscr{F}$ be an $\mathscr{H}$-Pre-QFM and $f : \{0,1\}^n \longrightarrow H$ an ($\mathscr{H}$-semi-fuzzy) truth function. The* induced $\mathscr{H}$-fuzzy truth function *$\widetilde{\mathscr{F}}(f) : H^n \longrightarrow H$ is defined by $\widetilde{\mathscr{F}}(f) = \mathscr{F}(Q_f) \circ \widetilde{\eta}$, where $Q_f = f \circ \eta^{-1}$, $\eta : \{0,1\}^n \longrightarrow \mathscr{P}(\{1,\ldots,n\})$ and $\widetilde{\eta} : H^n \longrightarrow \widetilde{\mathscr{P}_{\mathscr{H}}}(\{1,\ldots,n\})$ are defined by $\eta(y_1,\ldots,y_n) = \{i : y_i = 1\}$ and $\mu_{\widetilde{\eta}(x_1,\ldots,x_n)}(i) = x_i$, respectively.*

We impose the following minimal requirements on the compatibility of a model of quantification to an $\mathscr{H}$-lattice.

Definition 10 ($\mathscr{H}$-QFM) *An $\mathscr{H}$-Pre-QFM $\mathscr{F}$ is called an $\mathscr{H}$-QFM if it satisfies the following conditions.*

a. *If $Y_1,\ldots,Y_n \in \mathscr{P}(E)$ are crisp, then $\mathscr{F}(Q)(Y_1,\ldots,Y_n) = Q(Y_1,\ldots,Y_n)$ for every $Q : \mathscr{P}(E)^n \longrightarrow H$.*
b. *$\widetilde{\mathscr{F}}(\wedge) = \wedge$, $\widetilde{\mathscr{F}}(\vee) = \vee$ and $\widetilde{\mathscr{F}}(\neg) = \neg$.*

Thus, $\mathscr{F}$ satisfies the most elementary requirement on every fuzzification mechanism, that of proper generalization. Moreover, the induced conjunction, disjunction and negation of $\mathscr{F}$ is consistent with meet, join and negation in $\mathscr{H}$.

We will not develop an axiomatic theory of plausible $\mathscr{H}$-QFMs here, in the way that Determiner Fuzzification Schemes (DFSes) were axiomatically defined for ordinary $[0,1]$-valued fuzzy sets in [6], [7]. By contrast, we only consider a single, canonical model for the moment, which can be defined for every $\mathscr{H}$-lattice and will always be well-behaved (properties of the model will be investigated below).

F. A canonical model for fuzzy quantification in $\mathscr{H}$-lattices

A 'substitution approach' to fuzzy quantification tries to express a fuzzy quantifier by a (possibly infinitary) logical formula.[2] We propose two such circumscriptions: $\mathscr{F}_L(Q)$, the lower approximation of the target quantifier in some kind of disjunctive normal form (DNF), and the upper approximation $\mathscr{F}_U(Q)$, which (after an application of De Morgan's law, see eq. (4) in Lemma 1), corresponds to a conjunctive normal form (CNF).

Definition 11 *Let $\mathscr{H}$ be an $\mathscr{H}$-lattice, $Q : \mathscr{P}(E)^n \longrightarrow H$ an $\mathscr{H}$-semi-fuzzy quantifier and $X_1,\ldots,X_n \in \widetilde{\mathscr{P}_{\mathscr{H}}}(E)$. Then $\mathscr{F}_U(Q), \mathscr{F}_L(Q) : \widetilde{\mathscr{P}_{\mathscr{H}}}(E)^n \longrightarrow H$ are defined by:*

$$\mathscr{F}_U(Q)(X_1,\ldots,X_n) = \bigwedge\{Q^U_{V,W}(X_1,\ldots,X_n) : (V,W) \in \mathrm{D}_{E,n}\}$$
$$\mathscr{F}_L(Q)(X_1,\ldots,X_n) = \bigvee\{Q^L_{V,W}(X_1,\ldots,X_n) : (V,W) \in \mathrm{D}_{E,n}\},$$

[2]The basic idea of a substitution approach was also used by Yager [11] referring to Suppes [12].

for all $X_1,\dots,X_n \in \widetilde{\mathscr{P}}_{\mathscr{H}}(E)$, *where*

$$Q^U_{V,W}(X_1,\dots,X_n) = (\neg \Xi_{V,W}(X_1,\dots,X_n)) \vee \mathrm{U}(Q,V,W)$$
$$Q^L_{V,W}(X_1,\dots,X_n) = \Xi_{V,W}(X_1,\dots,X_n) \wedge \mathrm{L}(Q,V,W)$$
$$\Xi_{V,W}(X_1,\dots,X_n) = \bigwedge_{i=1}^{n} \Xi_{V_i,W_i}(X_i)$$
$$\Xi_{V_i,W_i}(X_i) = \bigwedge\{\mu_{X_i}(e) : e \in V_i\} \wedge \bigwedge\{\neg\mu_{X_i}(e) : e \notin W_i\}$$
$$\mathrm{U}(Q,V,W) = \bigvee\{Q(Y_1,\dots,Y_n) : (Y_1,\dots,Y_n) \in \mathrm{R}(V,W)\}$$
$$\mathrm{L}(Q,V,W) = \bigwedge\{Q(Y_1,\dots,Y_n) : (Y_1,\dots,Y_n) \in \mathrm{R}(V,W)\}$$
$$\mathrm{R}(V,W) = \{(Y_1,\dots,Y_n) \in \mathscr{P}(E)^n : V_i \subseteq Y_i \subseteq W_i,\ \text{all } i\}$$
$$\mathrm{D}_{E,n} = \{(V,W) \in \mathscr{P}(E)^n : V_i \subseteq W_i,\ \text{all } i\}.$$

Note. In the case of crisp arguments, using single points $V_i = W_i$ would be sufficient for setting up the CNF or DNF. However, we do not have a Boolean algebra here but rather an $\mathscr{H}$-lattice. Therefore *ranges* of crisp sets $V_i \subseteq W_i$ must be used in order to obtain a useful representation of the quantifier. It is not clear in advance if the results obtained from the CNF and DNF representation will agree in every $\mathscr{H}$-lattice. However, an inequality $\mathscr{F}_L \leq \mathscr{F}_U$ can be established, i.e. the lower approximation of the target quantifier is indeed smaller than the upper approximation.

Definition 12 *Let* $Q, Q' : \mathscr{P}(E)^n \longrightarrow H$ *be* $\mathscr{H}$*-semi-fuzzy quantifiers. Then* $Q \leq Q'$ *is defined by the condition*

$$Q(Y_1,\dots,Y_n) \leq Q'(Y_1,\dots,Y_n) \quad \text{for all } Y_1,\dots,Y_n \in \mathscr{P}(E).$$

For $\mathscr{H}$*-fuzzy quantifiers,* $\widetilde{Q} \leq \widetilde{Q}'$ *is defined analogously.*

Lemma 4 *Let* $Q : \mathscr{P}(E)^n \longrightarrow H$ *be an* $\mathscr{H}$*-semi-fuzzy quantifier and* $(V,W),(V',W') \in \mathrm{D}_{E,n}$*. If* $V_i \cup V'_i \subseteq W_i \cap W'_i$ *for all* $i \in \{1,\dots,n\}$*, then* $\mathrm{L}(Q,V,W) \leq \mathrm{U}(Q,V',W')$.

Lemma 5 *For all* $\mathscr{H}$*-semi-fuzzy quantifiers* $Q : \mathscr{P}(E)^n \longrightarrow H$ *and* $(V,W),(V',W') \in \mathrm{D}_{E,n}$*. If* $V_i \cup V'_i \subseteq W_i \cap W'_i$ *for all* $i \in \{1,\dots,n\}$*, then* $Q^L_{V,W} \leq Q^U_{V',W'}$.

Corollary 3 *For every* $\mathscr{H}$*-semi-fuzzy quantifier Q, it holds that* $\mathscr{F}_L(Q) \leq \mathscr{F}_U(Q)$.

As we shall see later in Prp. 10, $\mathscr{F}_U$ and $\mathscr{F}_L$ lack the desirable property of symmetry with respect to negation (at least if $\mathscr{F}_U$ and $\mathscr{F}_L$ differ). In order to alleviate this problem, we will combine the two models by a suitable aggregation operator. The following definition of the $\mathscr{H}$-fuzzy median generalizes the usual $[0,1]$-valued fuzzy median [13] to $\mathscr{H}$-lattices.

Definition 13 *Let* $\mathscr{H}$ *be an* $\mathscr{H}$*-lattice with support H. The* $\mathscr{H}$*-fuzzy median* $\mathrm{m}_{\frac{1}{2}} : H \times H \longrightarrow H$ *is defined by*

$$\mathrm{m}_{\frac{1}{2}}(x_1,x_2) = \begin{cases} x_1 \wedge x_2 & : & x_1 \wedge x_2 > \frac{1}{2} \\ x_1 \vee x_2 & : & x_1 \vee x_2 < \frac{1}{2} \\ \frac{1}{2} & : & \text{else} \end{cases} \qquad \forall x_1,x_2 \in H.$$

In other words, $\mathrm{m}_{\frac{1}{2}}(x_1,x_2) = \mathrm{h}(\mathrm{t}(x_1),\mathrm{c}(x_1) \sqcap \mathrm{c}(x_2))$ if $\mathrm{t}(x_1) = \mathrm{t}(x_2)$ and $\mathrm{m}_{\frac{1}{2}}(x_1,x_2) = \frac{1}{2}$ otherwise.

Proposition 3 *Consider an* $\mathscr{H}$*-lattice on a set* $H \neq \varnothing$*. Then*

(a.) $(\forall x \in H)\ \mathrm{m}_{\frac{1}{2}}(x,x) = x$.

(b.) $(\forall x_1,x_2 \in H)\ \mathrm{m}_{\frac{1}{2}}(x_1,x_2) = \mathrm{m}_{\frac{1}{2}}(x_2,x_1)$.

(c.) $(\forall x_1,x_2 \in H)\, x_1 \wedge x_2 \leq \mathrm{m}_{\frac{1}{2}}(x_1,x_2) \leq x_1 \vee x_2$.

(d.) $(\forall x_1,x_2,x'_1,x'_2 \in H)$ *If* $x_1 \leq x'_1$ *and* $x_2 \leq x'_2$*, then* $\mathrm{m}_{\frac{1}{2}}(x_1,x_2) \leq \mathrm{m}_{\frac{1}{2}}(x_1,x_2)$.

(e.) $(\forall x_1,x_2,x_3 \in H)$ $\mathrm{m}_{\frac{1}{2}}(x_1,\mathrm{m}_{\frac{1}{2}}(x_2,x_3) = \mathrm{m}_{\frac{1}{2}}(\mathrm{m}_{\frac{1}{2}}(x_1,x_2),x_3)$.

(f.) $(\forall x_1,x_2 \in H)\ \mathrm{m}_{\frac{1}{2}}(\neg x_1,\neg x_2) = \neg\,\mathrm{m}_{\frac{1}{2}}(x_1,x_2)$.

We use the $\mathscr{H}$-fuzzy median for aggregating $\mathscr{F}_U$ and $\mathscr{F}_L$ into the *canonical model* $\mathscr{F}_C$.

Definition 14 *Let* $\mathscr{H}$ *be an* $\mathscr{H}$*-lattice. The canonical* $\mathscr{H}$*-Pre-QFM* $\mathscr{F}_C$ *is defined by* $\mathscr{F}_C(Q)(X_1,\dots,X_n) = \mathrm{m}_{\frac{1}{2}}(\mathscr{F}_U(Q)(X_1,\dots,X_n),\mathscr{F}_L(Q)(X_1,\dots,X_n))$ *for all* $\mathscr{H}$*-semi-fuzzy quantifiers* $Q : \mathscr{P}(E)^n \longrightarrow H$ *and* $\mathscr{H}$*-fuzzy arguments* $X_1,\dots,X_n \in \widetilde{\mathscr{P}}_{\mathscr{H}}(E)$.

Let us now show that $\mathscr{F}_C$ is an $\mathscr{H}$-QFM. We first consider the property of *correct generalisation*.

Lemma 6 *Let* $E \neq \varnothing$ *be a set and* $n \in \mathbb{N}$*. Further let* $V,W \in \mathscr{P}(E)^n$ *such that* $V_i \subseteq W_i$*,* $i \in \{1,\dots,n\}$*. Then for all* $Y_1,\dots,Y_n \in \mathscr{P}(E)$,

$$\Xi_{V,W}(Y_1,\dots,Y_n) = \begin{cases} 1 & : & V_i \subseteq Y_i \subseteq W_i \text{ for all } i \in \{1,\dots,n\} \\ 0 & : & \text{else} \end{cases}$$

Proposition 4 *Let* $Q : \mathscr{P}(E)^n \longrightarrow H$ *be an* $\mathscr{H}$*-semi-fuzzy quantifier and* $Y_1,\dots,Y_n \in \mathscr{P}(E)$*. Then* $\mathscr{F}(Q)(Y_1,\dots,Y_n) = Q(Y_1,\dots,Y_n)$, $\forall \mathscr{F} \in \{\mathscr{F}_U,\mathscr{F}_L,\mathscr{F}_C\}$.

Next we show the compatibility with operations of $\mathscr{H}$.

Lemma 7 *Let an* $\mathscr{H}$*-lattice be given. Then* $\widetilde{\mathscr{F}}(\neg) = \neg$*,* $\widetilde{\mathscr{F}}(\wedge) = \wedge$*, and* $\widetilde{\mathscr{F}}(\vee) = \vee$*,* $\forall \mathscr{F} \in \{\mathscr{F}_U,\mathscr{F}_L,\mathscr{F}_C\}$.

Corollary 4 $\mathscr{F}_U$*,* $\mathscr{F}_L$ *and* $\mathscr{F}_C$ *are* $\mathscr{H}$*-QFMs.*

G. Important properties of the models

Let us first consider the preservation of constants.

Proposition 5 *Suppose there exists a constant* $\alpha \in H$ *such that* $Q(Y_1,\dots,Y_n) = \alpha$ *for all* $Y_1,\dots,Y_n \in \mathscr{P}(E)$*. Then* $\mathscr{F}(Q)(X_1,\dots,X_n) = \alpha$ *for all* $X_1,\dots,X_n \in \widetilde{\mathscr{P}}_{\mathscr{H}}(E)$*,* $\mathscr{F} \in \{\mathscr{F}_U,\mathscr{F}_L,\mathscr{F}_C\}$.

Definition 15 *Let* $Q : \mathscr{P}(E)^m \longrightarrow H$ *be an m-ary* $\mathscr{H}$*-semi-fuzzy quantifier and* $\xi : \{1,\dots,m\} \longrightarrow \{1,\dots,n\}$ *a mapping,* *where* $n, m \in \mathbb{N}$. *By* $Q\xi : \mathscr{P}(E)^n \longrightarrow H$ *we denote the n-ary* $\mathscr{H}$*-semi-fuzzy quantifier defined by*

$$Q\xi(Y_1, \ldots, Y_n) = Q(Y_{\xi(1)}, \ldots, Y_{\xi(m)}) \quad \forall Y_1, \ldots, Y_n \in \mathscr{P}(E).$$

We use an analogous definition for $\mathscr{H}$*-fuzzy quantifiers.*

Lemma 8 *Let* $X_1, \ldots, X_n \in \widetilde{\mathscr{P}}_{\mathscr{H}}(E)$ *be given* $\mathscr{H}$*-fuzzy sets* $\xi : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$ *an injective mapping. Further let* $(V, W) \in \mathrm{D}_{E,n}$ *be given, and suppose that* $V_j = \varnothing$ *and* $W_j = E$ *for all* $j \in \{1, \ldots, n\}$ *with* $j \notin \operatorname{Im}\xi$*, where* $\operatorname{Im}\xi = \xi(\{1, \ldots, m\})$. *If* $(V', W') \in \mathrm{D}_{E,m}$ *are defined by* $V'_i = V_{\xi(i)}$ *and* $W'_i = W_{\xi(i)}$ *for all* $i \in \{1, \ldots, m\}$*, then*

$$\Xi_{V',W'}(X_{\xi(1)}, \ldots, X_{\xi(m)}) = \Xi_{V,W}(X_1, \ldots, X_n).$$

Proposition 6 *Let* $Q : \mathscr{P}(E)^m \longrightarrow H$ *be an m-ary* $\mathscr{H}$*-semi-fuzzy quantifier and* $\xi : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$ *an injective mapping. Then* $\mathscr{F}(Q\xi) = \mathscr{F}(Q)\xi$, $\forall \mathscr{F} \in \{\mathscr{F}_U, \mathscr{F}_L, \mathscr{F}_C\}$.

In particular, the models are compatible with permutations of the order of the arguments (see [6, Chap. 4.5]), and they are also compatible with cylindrical extensions (see [6, Chap. 4.6]), i.e. the fuzzy quantification results will not change if vacuous argument positions are added which do not affect the behaviour of the original $\mathscr{H}$-semi-fuzzy quantifier but only extend its nominal number of arguments.

Definition 16 *Let* $E \neq \varnothing$ *be some set and* $e \in E$. *The* crisp membership assessment quantifier $\pi_e : \mathscr{P}(E) \longrightarrow \{0, 1\}$ *is defined by* $\pi_e(Y) = 1$ *if* $e \in Y$ *and* $\pi_e(Y) = 0$ *otherwise. The* $\mathscr{H}$-fuzzy membership assessment quantifier $\widetilde{\pi}_e : \widetilde{\mathscr{P}}_{\mathscr{H}}(E) \longrightarrow H$ *is defined by* $\widetilde{\pi}_e(X) = \mu_X(e)$ $\forall X \in \widetilde{\mathscr{P}}_{\mathscr{H}}(E)$.

Proposition 7 *For all* $E \neq \varnothing$ *and* $e \in E$, $\mathscr{F}(\pi_e) = \widetilde{\pi}_e$, $\forall \mathscr{F} \in \{\mathscr{F}_U, \mathscr{F}_L, \mathscr{F}_C\}$.

Thus, the models are compatible with membership assessments. Next we will discuss monotonicity properties.

Lemma 9 *Let* $Q, Q' : \mathscr{P}(E)^n \longrightarrow H$ *be* $\mathscr{H}$*-fuzzy quantifiers. If* $Q \leq Q'$*, then* $\mathrm{U}(Q, V, W) \leq \mathrm{U}(Q', V, W)$ *and* $\mathrm{L}(Q, V, W) \leq \mathrm{L}(Q', V, W)$ *for all* $(V, W) \in \mathrm{D}_{E,n}$.

Proposition 8 *Let* $Q, Q' : \mathscr{P}(E)^n \longrightarrow H$ *be* $\mathscr{H}$*-semi-fuzzy quantifiers. If* $Q \leq Q'$*, then* $\mathscr{F}(Q) \leq \mathscr{F}(Q')$ $\forall \mathscr{F} \in \{\mathscr{F}_U, \mathscr{F}_L, \mathscr{F}_C\}$.

Lemma 10 *Let* $\mathscr{H}$ *be an* $\mathscr{H}$*-lattice,* $n \in \mathbb{N} \setminus \{0\}$ *and* $V, V', W, W' \in \mathscr{P}(E)^n$ *such that* $V'_i \subseteq V_i \subseteq W_i \subseteq W'_i$ *for all* $i \in \{1, \ldots, n\}$. *Then* $\Xi_{V,W}(X_1, \ldots, X_n) \leq \Xi_{V',W'}(X_1, \ldots, X_n)$ *for all* $X_1, \ldots, X_n \in \widetilde{\mathscr{P}}_{\mathscr{H}}(E)$.

Proposition 9 *Suppose that* $Q : \mathscr{P}(E)^n \longrightarrow H$ *is nondecreasing in the n-th argument, i.e.* $\forall Y_1, \ldots, Y_n, Y'_n \in \mathscr{P}(E)$ *with* $Y_n \subseteq Y'n$, $Q(Y_1, \ldots, Y_n) \leq Q(Y_1, \ldots, Y_{n-1}, Y'_n)$. *Then* $\mathscr{F}(Q)$ *is also nondecreasing in the n-argument* $\forall \mathscr{F} \in \{\mathscr{F}_U, \mathscr{F}_L, \mathscr{F}_C\}$ *(using an analogous definition of the property for fuzzy arguments).*

Next we discuss properties related to negation.

Definition 17 *The (external) negation of* $Q : \mathscr{P}(E)^n \longrightarrow H$ *is the quantifier* $\neg Q : \mathscr{P}(E) \longrightarrow H$*, defined by*

$$(\neg Q)(Y_1, \ldots, Y_n) = \neg(Q(Y_1, \ldots, Y_n)) \quad \forall Y_1, \ldots, Y_n \in \mathscr{P}(E).$$

For $\mathscr{H}$*-fuzzy quantifiers,* $\neg\widetilde{Q}$ *is defined analogously.*

Lemma 11 *Let* $Q : \mathscr{P}(E)^n \longrightarrow H$ *be an* $\mathscr{H}$*-semi-fuzzy quantifier. Then for all* $(V, W) \in \mathrm{D}_{E,n}$, $\mathrm{U}(\neg Q, V, W) = \neg \mathrm{L}(Q, V, W)$ *and* $\mathrm{L}(\neg Q, V, W) = \neg \mathrm{U}(Q, V, W)$.

Proposition 10 *Let* $Q : \mathscr{P}(E)^n \longrightarrow H$ *be an* $\mathscr{H}$*-semi-fuzzy quantifier. Then* $\mathscr{F}_U(\neg Q) = \neg\mathscr{F}_L(Q)$, $\mathscr{F}_L(\neg Q) = \neg\mathscr{F}_U(Q)$, *and* $\mathscr{F}_C(\neg Q) = \neg\mathscr{F}_C(Q)$.

Thus $\mathscr{F}_U$ and $\mathscr{F}_L$ are not compatible with negation (at least when $\mathscr{F}_U$ and $\mathscr{F}_L$ differ), while the canonical model $\mathscr{F}_C$ shows the desired symmetry.

Definition 18 *Let* $Q : \mathscr{P}(E) \longrightarrow H$ *be an* $\mathscr{H}$*-semi-fuzzy quantifier of arity* $n > 0$. *The* antonym *of* Q *is the* $\mathscr{H}$*-semi fuzzy-quantifier* $Q\neg : \mathscr{P}(E)^n \longrightarrow H$ *defined by*

$$Q\neg(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_{n-1}, \neg Y_n) \quad Y_1, \ldots, Y_n \in \mathscr{P}(E),$$

where $\neg Y_n = \{e \in E : e \notin Y_n\}$ *is the complement of* Y_n. *Antonyms of* $\mathscr{H}$*-fuzzy quantifiers are defined analogously.*

Lemma 12 *Let* $X \in \widetilde{\mathscr{P}}_{\mathscr{H}}(E)$ *be an* $\mathscr{H}$*-fuzzy subset of some set* E *and suppose that* $V, W \in \mathscr{P}(E)$ *are crisp subsets of* E *with* $V \subseteq E$. *Then* $\Xi_{V,W}(\neg X) = \Xi_{\neg W, \neg V}(X)$*, where* $\neg W = \{e \in E : e \notin W\}$ *and* $\neg V = \{e \in E : e \notin V\}$.

Proposition 11 *Let* $Q : \mathscr{P}(E)^n \longrightarrow H$ *be an* $\mathscr{H}$*-semi-fuzzy quantifier of arity* $n > 0$. *Then* $\mathscr{F}(Q\neg) = \mathscr{F}(Q)\neg$, $\forall \mathscr{F} \in \{\mathscr{F}_U, \mathscr{F}_L, \mathscr{F}_C\}$.

Definition 19 *The* dual *of an* $\mathscr{H}$*-semi-fuzzy quantifier* $Q : \mathscr{P}(E)^n \longrightarrow H$, $n > 0$*, is the* $\mathscr{H}$*-semi fuzzy-quantifier* $Q\square : \mathscr{P}(E)^n \longrightarrow H$ *defined by*

$$Q\square(Y_1, \ldots, Y_n) = \neg Q(Y_1, \ldots, Y_{n-1}, \neg Y_n) \quad \forall Y_1, \ldots, Y_n \in \mathscr{P}(E),$$

i.e. $Q\square = (\neg Q)\neg = \neg(Q\neg)$. *The dual* $\widetilde{Q}\square$ *of an* $\mathscr{H}$*-fuzzy quantifier is defined analogously.*

Proposition 12 *Let* $Q : \mathscr{P}(E)^n \longrightarrow H$ *be an* $\mathscr{H}$*-semi-fuzzy quantifier,* $n > 0$. *Then* $\mathscr{F}_U(Q\square) = \mathscr{F}_L(Q)\square$, $\mathscr{F}_L(Q\square) = \mathscr{F}_U(Q)\square$, *and* $\mathscr{F}_C(Q\square) = \mathscr{F}_C(Q)\square$.

Let us now generalize Mukaidono's ambiguity relation [14].

Definition 20 *Let* $\mathscr{H}$ *be an* $\mathscr{H}$*-lattice on a set* $H \neq \varnothing$. *We define* $\preceq_c \subseteq H \times H$ *as follows,*

$$\forall \alpha, \alpha' \in H \quad \alpha \preceq_c \alpha' \Leftrightarrow (\alpha' \leq \alpha \leq \tfrac{1}{2}) \vee (\tfrac{1}{2} \leq \alpha \leq \alpha').$$

In other words, $\alpha \preceq_c \alpha'$ iff $\mathrm{c}(\alpha) = \bot \vee (\mathrm{t}(\alpha) = \mathrm{t}(\alpha') \wedge \mathrm{c}(\alpha) \sqsubseteq \mathrm{c}(\alpha'))$. Thus $\alpha \preceq_c \alpha'$ holds if α and α' have a compatible

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truth tendency (towards true or false), but there is less confidence in α than in α'.

Proposition 13 *Let* $\mathscr{F} \in \{\mathscr{F}_U, \mathscr{F}_L, \mathscr{F}_C\}$, $Q, Q' : \mathscr{P}(E)^n \longrightarrow H$ *and* $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathscr{P}}_{\mathscr{H}}(E)$ *be given.*

a. *If* $Q \preceq_c Q'$ *for all crisp arguments, then also* $\mathscr{F}(Q)(X_1, \ldots, X_n) \preceq_c \mathscr{F}(Q')(X_1, \ldots, X_n)$.

b. *If* $\mu_{X_i}(e) \preceq_c \mu_{X'_i}(e)$ *for all* $e \in E$, $i \in \{1, \ldots, n\}$, *then* $\mathscr{F}(Q)(X_1, \ldots, X_n) \preceq_c \mathscr{F}(Q)(X'_1, \ldots, X'_n)$.

In other words, the models propagate undecidedness both in quantifiers and in the arguments.

III. Conclusions

This paper has introduced so-called $\mathscr{H}$-lattices as a useful abstraction of the structure of $\mathbf{I} = [0,1]$ with the standard operations min, max and $\neg x = 1 - x$, which no longer requires the modeller to commit to numerical membership grades. We have explained how membership functions can be constructed in $\mathscr{H}$-lattices by deciding on the overall tendency $\mathrm{t}(\phi)$ of a membership assessment and by relating the confidence level of this decision to the confidence of other membership assessments. Since the confidence levels need not form a total order, the modeller can abstain from a comparison of confidence levels in unclear cases and focus on the most important discernments. This makes the proposed construction of $\mathscr{H}$-lattices a practical method of operationalizing membership assessments in fuzzy set theory. The problem of assigning membership grades is solved by eliminating the need to assign membership grades in the first place: Instead of a fixed membership assignment, we now have a system of comparisons of confidence levels, and membership grades arise only indirectly as a by-product of these comparisons.

Having introduced $\mathscr{H}$-fuzzy sets and suitable propositional connectives, the problem of fuzzy quantification for $\mathscr{H}$-valued arguments was investigated. We have presented a framework for analyzing $\mathscr{H}$-fuzzy quantification in terms of a specification of the quantifier of interest by a so-called $\mathscr{H}$-semi-fuzzy quantifier, to which a model of fuzzy quantification ($\mathscr{H}$-QFM) is then applied. Embarking on a substitution approach to fuzzy quantification, two auxiliary models $\mathscr{F}_U$ and $\mathscr{F}_L$ were introduced which circumscribe the target quantifier in terms of a conjunctive or disjunctive normal form. A *canonical model* of fuzzy quantification $\mathscr{F}_C$ was then constructed from $\mathscr{F}_U$ and $\mathscr{F}_L$ which shows improved properties like symmetry with respect to negation. Though we have started an investigation of formal properties of the model, many other criteria developed in [6] still wait for generalization to the $\mathscr{H}$-valued case, and verification for the proposed model. A research agenda for continuing this work should include: (a) generalization of all known concepts for describing $\mathbf{I}$-valued semi-fuzzy and fuzzy quantifiers to the case of $\mathscr{H}$-fuzzy quantifiers; (b) identification of a class of plausible models in terms of a system of independent axioms, similar to the axiom set proposed in [6]; (c) development of efficient algorithms for evaluating quantifying expressions involving $\mathscr{H}$-fuzzy quantifiers; (d) development of software tools which support the acquisition of membership functions in the proposed framework, in particular development of a graphical editor for displaying and manipulating the Hasse diagrams of confidence orders and for visualising the resulting $\mathscr{H}$-lattices of membership grades.

A challenging issue for further research is that of making the proposed analysis useful for large-scale problems with hundreds or thousands of membership assessments. To achieve this, one would need techniques for composing the global confidence lattice from smaller parts (e.g. a local confidence lattice for each linguistic variable) and for hiding unnecessary detail when displaying results. In any case, the proposed analysis will even be valuable if one returns to predefined lattices of confidence grades. Consider using a finite chain $\bot = \gamma_0 \sqsubseteq \gamma_1 \cdots \sqsubseteq \gamma_m = \top$ of linguistic labels for the confidence lattice (e.g. *'no confidence at all'* – *'weak confidence'* – *'medium confidence'* – *'strong confidence'* – *'total confidence'*). The resulting membership lattice $\mathscr{H}(\mathscr{C})$ then becomes a finite chain with an odd number of elements and the apparent (symmetric) negation. As in the interval-valued approach, one could also use ordered pairs of such confidence labels, like $\langle$*medium confidence*, *strong confidence*$\rangle$, so that a commitment to a particular confidence grade is no longer necessary. If applied in this way, the benefit of the proposed analysis of membership in terms of tendency and confidence is not so much that it avoids direct membership assignments but rather that the resulting system of membership grades in $\mathscr{H}(\mathscr{C})$ then forms an $\mathscr{H}$-lattice, and thus allows an interpretation of fuzzy quantification in a canonical model.

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