**Proceedings of the 2007 IEEE Symposium on
Foundations of Computational Intelligence (FOCI 2007)**

Fuzzy Possibility Space and Type-2 Fuzzy Variable

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***Abstract*— In this paper, we present an axiomatic approach to developing the theory of type-2 (T2) fuzziness, called *fuzzy possibility theory*. We first introduce some fundamental concepts in this theory, such as fuzzy possibility measure, fuzzy possibility space, and T2 fuzzy variable. The fuzzy possibility space includes three parts: the universe, an ample field, and a fuzzy possibility measure; and the fuzzy possibility measure is defined as a set function on the ample field taking on regular fuzzy variable (RFV) values. Then, we define a T2 fuzzy vector as a measurable map from a fuzzy possibility space (FPS) to the space of real vectors, and present several concepts associated with T2 fuzzy vectors, such as secondary possibility distribution function and T2 possibility distribution function. Finally, to characterize the properties of T2 fuzzy vectors via possibility distributions, we propose the marginal secondary possibility distribution function and mutually independent T2 fuzzy variables.**

I. Introduction

Zadeh [1] introduced *T2 fuzzy sets* as an extension of ordinary fuzzy sets in 1975. But T2 fuzzy set didn't became popular immediately, it were only investigated by a few researchers; for instance, Mizumoto and Tanaka [2] discussed what kinds of algebraic structures the grades of T2 fuzzy sets form under join, meet and negation, and showed that normal convex fuzzy grades form a distributive lattace under the join and meet; Nieminen [3] studied on the algebraic structure of T2 fuzzy sets; Dubois and Prade [4] investigated the operations in a fuzzy-valued logic, and Yager [5] applied the T2 fuzzy set to decision making.

Recently, T2 fuzzy sets have been applied successfully to T2 fuzzy logic systems to handle linguistic and numerical uncertainties [6], [7], [8], [9], [10]. A T2 fuzzy logic system [9] includes a fuzzifier, rule base, fuzzy inference engine, and output processor; it is characterized by IF-THEN rules, but its antecedent or consequent sets are T2. As described in [11], a T2 fuzzy set represents the uncertainty in terms of secondary membership function and footprint of uncertainty. In pattern recognition, to enhance the hidden Markov models expressive power for uncertainty by T2 fuzzy set, Zeng and Liu [12] have recently presented the T2 fuzzy hidden Markov models, and applied the models to phoneme classification and recognition on the TIMIT speech database. To measure the similarity between two T2 fuzzy sets, Mitchell [13] introduces a similarity measure and with it to show that T2 fuzzy sets provide indeed a natural language for formulating classification problems in pattern recognition. In addition, T2 fuzzy sets have found applications in overcoming time-varying co-channel interference and equalization of a nonlinear time-varying channel [14], [15], processing questionnaire surveys [16], inferencing and knowledge representation [17], and neural-fuzzy clustering for classification of sports injuries in the lower leg [18].

In this paper we explore a theoretical framework from which a T2 theory is constructed, which is referred to as the *fuzzy possibility theory*. We introduce several fundamental concepts in this theory, such as fuzzy possibility measure, FPS, T2 fuzzy variable, T2 possibility distribution function, marginal T2 possibility distribution function, and mutually independent T2 fuzzy variables. An FPS consists of three parts: the universe, an ample field, and a fuzzy possibility measure. We define a fuzzy possibility measure as a set function from the ample field to a collection of RFVs; and a T2 fuzzy variable as a measurable map from the FPS to the set of real numbers. The fuzzy possibility theory is a generalization of the usual possibility theory [19], [20], [21], [22], [23], [24].

The paper is organized as follows. Section II reviews some concepts in the possibility theory. In Section III, we present fuzzy possibility space, in which we introduce fuzzy possibility measure, and define it as an RFV-valued set function on an ample field. In Section IV, we first define a T2 fuzzy vector as a measurable map from a fuzzy possibility space to the space of real vectors. then propose several fundamental concepts associated with T2 fuzzy vector, such as secondary fuzzy possibility distribution function and T2 possibility distribution function. The properties of T2 fuzzy vectors are discussed via possibility distributions in Sections V and VI, respectively; Section V discusses the marginal secondary possibility distribution function and marginal T2 possibility distribution function, while Section VI deals with the independence of T2 fuzzy variables. Finally, we draw conclusions in Section VII.

II. Preliminaries

Let Γ be the universe of discourse, and an ample field $\mathcal{A}$ on Γ is a class of subsets of Γ that is closed under arbitrary unions, intersections, and complement in Γ. Let $\mathrm{Pos} : \mathcal{A} \mapsto [0, 1]$ be a set function on the ample field $\mathcal{A}$. Pos is said to be a possibility measure [21] if it satisfies the following conditions:

1) $\mathrm{Pos}(\emptyset) = 0$, and $\mathrm{Pos}(\Gamma) = 1$;
2) For any subclass $\{A_i \mid i \in I\}$ of $\mathcal{A}$ (finite, countable

or uncountable),

$$\mathrm{Pos}\left(\bigcup_{i\in I} A_i\right) = \sup_{i\in I} \mathrm{Pos}(A_i).$$

The triplet $(\Gamma, \mathcal{A}, \mathrm{Pos})$ is referred to as a *possibility space*, in which a *fuzzy vector* is defined as follows.

Definition 1: Let $(\Gamma, \mathcal{A}, \mathrm{Pos})$ be a possibility space. An m-ary function $X = (X_1, \cdots, X_m) : \Gamma \mapsto \Re^m$ from the universe Γ to the space of real vectors is called a fuzzy vector if for every $t = (t_1, \cdots, t_m) \in \Re^m$, the set $\{\gamma \in \Gamma \mid X(\gamma) \le t\}$ is an element of $\mathcal{A}$, i.e.,

$$\begin{array}{l}\{\gamma \in \Gamma \mid X(\gamma) \le t\} \\ \quad = \{\gamma \in \Gamma \mid X_1(\gamma) \le t_1, \cdots, X_m(\gamma) \le t_m\} \in \mathcal{A}.\end{array} \quad (1)$$

As $m = 1$, X is usually called a fuzzy variable.
The possibility distribution function of the fuzzy vector X is defined as

$$\mu_X(t) = \mathrm{Pos}(\{\gamma \in \Gamma \mid X(\gamma) = t\}), \quad t \in \Re^m. \quad (2)$$

In this paper, we often make use of a special class of fuzzy vectors, called *regular fuzzy vector*, which is formally defined as follows.

Definition 2: Let $(\Gamma, \mathcal{A}, \mathrm{Pos})$ be an FPS. An m-ary regular fuzzy vector $X = (X_1, X_2, \cdots, X_m)$ is defined as a fuzzy vector from the FPS to the set $[0, 1]^m$, i.e., for any $\gamma \in \Gamma$,

$$X(\gamma) = (X_1(\gamma), X_2(\gamma), \cdots, X_m(\gamma)) \in [0, 1]^m.$$

As $m = 1$, X is called a *regular fuzzy variable* (RFV).

In the following, we denote by $\mathcal{R}([0, 1])$ as the collection of all RFVs on $[0, 1]$.

Example 1: An RFV which only takes on value 0 with possibility 1 is denoted by

$$\tilde{0} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

while an RFV which only takes on value 1 with possibility 1 is denoted by

$$\tilde{1} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Example 2: The following function X is a discrete RFV

$$X \sim \begin{pmatrix} 0.2 & 0.4 & 0.6 & 0.8 & 1 \\ 0.3 & 0.7 & 1 & 0.9 & 0.5 \end{pmatrix}$$

which takes on values $0.2, 0.4, 0.6, 0.8$, and 1 with possibility $0.3, 0.7, 1, 0.9$, and 0.5, respectively.

Definition 3 ([25]): Let $X_i, i = 1, \cdots, n$ be m_i-ary regular fuzzy vectors defined on a possibility space $(\Gamma, \mathcal{A}, \mathrm{Pos})$, respectively. They are said to be mutually independent if

$$\begin{array}{l}\mathrm{Pos}\{\gamma \in \Gamma \mid X_1(\gamma) = u_1, \cdots, X_n(\gamma) = u_n\} \\ \quad = \min_{1\le i\le n} \mathrm{Pos}\{\gamma \in \Gamma \mid X_i(\gamma) = u_i\}\end{array} \quad (3)$$

for any $u_i = (u_1^{(i)}, \cdots, u_{m_i}^{(i)}) \in [0, 1]^{m_i}$, and $i = 1, 2, \cdots, n$.

Moreover, a family of regular fuzzy vectors $\{X_i, i \in I\}$ is said to be mutually independent if for each integer n, and $i_1 < i_2 < \cdots < i_n$, the regular fuzzy vectors $X_{i_k}, k = 1, 2, \cdots, n$ are mutually independent.

III. Fuzzy Possibility Space

It is known that in possibility theory, the possibility measure and the possibility distribution function of a fuzzy variable can be determined from each other.

More specifically, if Pos is a possibility measure, then the possibility distribution function of X can be determined by Eq.(2); Conversely, if $\mu : \Re \mapsto [0, 1]$ is a map from $\Re$ to $[0, 1]$ such that $\sup_{x\in\Re} \mu(x) = 1$, then the set function Pos defined by

$$\mathrm{Pos}(A) = \sup_{t\in A} \mu(t), \quad A \in \mathcal{P}(\Re) \quad (4)$$

is a possibility measure [19].

We now suppose that $\mu : \Re \mapsto \mathcal{R}([0, 1])$ is a map from $\Re$ to a collection of RFVs. In this case, the set function Pos defined by Eq. (4) is not a crisp number in $[0, 1]$ but an RFV. Consequently, to deal with T2 fuzziness, it is required to extend $[0, 1]$-valued set function to the case of RFV-valued one, which motivates us to present the following novel concept.

Definition 4: Let $\mathcal{A}$ be an ample field on the universe Γ, and $\tilde{\mathrm{Pos}} : \mathcal{A} \mapsto \mathcal{R}([0, 1])$ a set function on $\mathcal{A}$ such that $\{\tilde{\mathrm{Pos}}(A) \mid \mathcal{A} \ni A \text{ atom}\}$ is a family of mutually independent RFVs. We call $\tilde{\mathrm{Pos}}$ a *fuzzy possibility measure* if it satisfies the following conditions:

Pos1) $\tilde{\mathrm{Pos}}(\emptyset) = \tilde{0}$;

Pos2) For any subclass $\{A_i \mid i \in I\}$ of $\mathcal{A}$ (finite, countable or uncountable),

$$\tilde{\mathrm{Pos}}\left(\bigcup_{i\in I} A_i\right) = \sup_{i\in I} \tilde{\mathrm{Pos}}(A_i).$$

Moreover, if $\mu_{\tilde{\mathrm{Pos}}(\Gamma)}(1) = 1$, then we call $\tilde{\mathrm{Pos}}$ a *regular fuzzy possibility measure*.

We call the triplet $(\Gamma, \mathcal{A}, \tilde{\mathrm{Pos}})$ as a *fuzzy possibility space* (FPS).

Remark 1: A regular fuzzy possibility measure is a generalization of the scalar possibility measure, i.e., if for any $A \in \mathcal{A}$, $\tilde{\mathrm{Pos}}(A)$ is a crisp number in $[0, 1]$ instead of an RFV, then $\tilde{\mathrm{Pos}}$ is just a possibility measure.

Remark 2: The condition $\mu_{\tilde{\mathrm{Pos}}(\Gamma)}(1) = 1$ designates that the RFV $\tilde{\mathrm{Pos}}(\Gamma)$ takes on value 1 with possibility 1. In addition, the RFV $\sup_{i\in I} \tilde{\mathrm{Pos}}(A_i)$ is the supremum of the family of RFVs $\{\tilde{\mathrm{Pos}}(A_i), i \in I\}$, which is well-defined based on the infinite-dimensional product possibility theory [26].

Remark 3: If the universe Γ is a finite set, then the axiom Pos2) in Definition 4 can be replaced by

$$\tilde{\mathrm{Pos}}\left(\bigcup_{i=1}^{n} A_i\right) = \sup_{1\le i\le n} \tilde{\mathrm{Pos}}(A_i)$$

for any finite subclass $\{A_i, i = 1, \cdots, n\}$ of $\mathcal{A}$.

Remark 4: If $\mathcal{A}$ is the power set of the universe Γ, then the atoms of $\mathcal{A}$ are all single point sets $\{\gamma\}, \gamma \in \Gamma$. Therefore, in order to define a fuzzy possibility measure on $\mathcal{A}$, it suffices to give the value of $\tilde{\mathrm{Pos}}$ at each single point set.

We now provide an example to show how to define a fuzzy possibility measure.

Example 3: Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$, and $\mathcal{A} = \mathcal{P}(\Gamma)$. Define a set function $\tilde{\mathrm{P}}\mathrm{os} : \mathcal{P}(\Gamma) \mapsto \mathcal{R}([0,1])$ as follows

$$\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_1\}) = (0.3, 0.4, 0.5),$$
$$\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_2\}) = \tilde{1},$$
$$\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_3\}) = (0.5, 0.6, 0.7),$$

and for any subset A of Γ,

$$\tilde{\mathrm{P}}\mathrm{os}(A) = \sup_{\gamma \in A} \tilde{\mathrm{P}}\mathrm{os}(\{\gamma\}),$$

where $(0.3, 0.4, 0.5)$, and $(0.5, 0.6, 0.7)$ are supposed to be mutually independent RFVs. Then, $\tilde{\mathrm{P}}\mathrm{os}$ is a fuzzy possibility measure on $\mathcal{P}(\Gamma)$, and $(\Gamma, \mathcal{P}(\Gamma), \tilde{\mathrm{P}}\mathrm{os})$ is an FPS. In Fig.1, we show the possibility distribution functions of RFVs $\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_1\}$ and $\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_3\}$, respectively.

We now show how to calculate the possibility distribution of RFV $\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_1, \gamma_2\})$. Let $X_{1,2}, X_1$, and X_2 be the RFVs of $\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_1, \gamma_2\})$, $\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_1\})$ and $\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_2\})$, respectively.

By the definition of $\tilde{\mathrm{P}}\mathrm{os}$,

$$\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_1, \gamma_2\}) = \tilde{\mathrm{P}}\mathrm{os}(\{\gamma_1\}) \vee \tilde{\mathrm{P}}\mathrm{os}(\{\gamma_2\}),$$

i.e., $X_{1,2} = X_1 \vee X_2$. Therefore, the possibility distribution function of $X_{1,2}$ is

$$\begin{aligned} &\mu_{X_{1,2}}(x) \\ &\quad = \mathrm{Pos}(\{X_{1,2} = x\}) \\ &\quad = \mathrm{Pos}(\{X_1 \vee X_2 = x\}) \\ &\quad = \mathrm{Pos}\left(\textstyle\bigcup_{x_1 \vee x_2 = x} \{X_1 = x_1, X_2 = x_2\}\right) \\ &\quad = \sup_{x_1 \vee x_2 = x} \mathrm{Pos}(\{X_1 = x_1, X_2 = x_2\}). \end{aligned}$$

By the independence of X_1 and X_2,

$$\begin{aligned} &\mathrm{Pos}(\{X_1 = x_1, X_2 = x_2\}) \\ &\quad = \mathrm{Pos}(\{X_1 = x_1\}) \wedge \mathrm{Pos}(\{X_2 = x_2\}) \\ &\quad = \mu_{X_1}(x_1) \wedge \mu_{X_2}(x_2), \end{aligned}$$

where $\mu_{X_1}(x_1)$ and $\mu_{X_2}(x_2)$ are the possibility distribution functions of X_1 and X_2, respectively, and given by

$$\mu_{X_1}(x_1) = \begin{cases} 10x_1 - 3, & \text{if } 0.3 \le x_1 \le 0.4 \\ 5 - 10x_1, & \text{if } 0.4 < x_1 \le 0.5 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mu_{X_2}(x_2) = \begin{cases} 1, & \text{if } x_2 = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Combining the above gives the possibility distribution function of $X_{1,2}$ as follows

$$\begin{aligned} &\mu_{X_{1,2}}(x) \\ &\quad = \sup_{x_1 \vee x_2 = x} (\mu_{X_1}(x_1) \wedge \mu_{X_2}(x_2)) \\ &\quad = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Remark 5: Although RFVs $X_{1,2}$ and X_2 have the identical possibility distribution function, they are not the same RFV. This is similar to what we encounter in probability theory, we have independent and identically distributed random variables, i.e., two different random variables may have an identical probability distribution function.

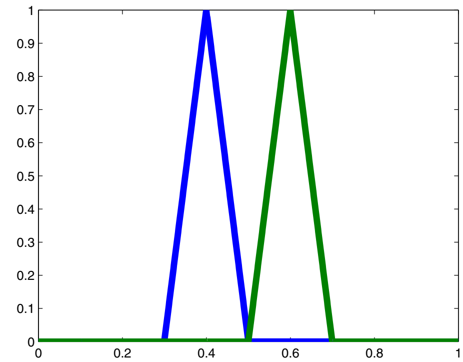

Fig. 1. The possibility distribution functions of $\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_1\})$ and $\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_3\})$ defined in Example 3.

IV. T2 Fuzzy Variable

So far, we have established the FPS. In this section, we will justify our approach by showing that certain definitions existing in the literature can be obtained from the FPS, and in addition, the new results which are obtained in the following will lead to credible interpretations for our variable-based arguments.

One of the interesting consequences of FPS is that it leads to the definition of a T2 fuzzy set on $\Re^m$ which we will call a *T2 fuzzy vector*. The T2 fuzzy vector plays the same role in fuzzy possibility theory as a random vector does in probability theory, it can be formally defined as follows.

Definition 5: Let $(\Gamma, \mathcal{A}, \tilde{\mathrm{P}}\mathrm{os})$ be an FPS. A map $\xi = (\xi_1, \xi_2, \cdots, \xi_m) : \Gamma \mapsto \Re^m$ is called an m-ary *T2 fuzzy vector* if for any $x = (x_1, x_2, \cdots, x_m) \in \Re^m$, the set $\{\gamma \in \Gamma \mid \xi(\gamma) \le x\}$ is an element of $\mathcal{A}$, i.e.,

$$\begin{aligned} &\{\gamma \in \Gamma \mid \xi(\gamma) \le x\} \\ &\quad = \{\gamma \in \Gamma \mid \xi_1(\gamma) \le x_1, \cdots, \xi_m(\gamma) \le x_m\} \in \mathcal{A}. \end{aligned} \quad (5)$$

As $m = 1$, the map $\xi : \Gamma \mapsto \Re$ is called a *T2 fuzzy variable*.

Remark 6: If the ample field $\mathcal{A}$ is replaced by the power set of Γ, i.e., $\mathcal{A} = \mathcal{P}(\Gamma)$, then the requirement in Eq. (5) can be removed.

The concept of T2 fuzzy vector has been employed in the current development since it plays the same role as a random vector does in probability theory. We are suggesting it is a more appropriate definition for a T2 fuzzy set on $\Re^m$. In the literature, a T2 fuzzy set is usually defined via its T2 membership function; whereas in this paper, we obtain the T2 membership function as the transformation of $\tilde{\mathrm{P}}\mathrm{os}$ from the universe Γ to the space $\Re^m$ via T2 fuzzy vector, which is formally defined as follows.

Definition 6: Let $\xi = (\xi_1, \xi_2, \cdots, \xi_m)$ be a T2 fuzzy vector defined on an FPS $(\Gamma, \mathcal{A}, \tilde{\mathrm{P}}\mathrm{os})$. The *secondary possibility distribution function* of ξ, denoted by $\tilde{\mu}_\xi(x)$, is a map $\Re^m \mapsto \mathcal{R}[0,1]$ such that

$$\tilde{\mu}_\xi(x) = \tilde{\mathrm{P}}\mathrm{os}\{\gamma \in \Gamma \mid \xi(\gamma) = x\}, \quad x \in \Re^m \quad (6)$$

while the *T2 possibility distribution function* of ξ, denoted by $\mu_\xi(x,u)$, is a map $\Re^m \times J_x \mapsto [0,1]$ such that

$$\mu_\xi(x,u) = \mathrm{Pos}\left\{\tilde{\mu}_\xi(x) = u\right\}, \quad (x,u) \in \Re^m \times J_x, \qquad (7)$$

where Pos is the possibility measure induced by the distribution of $\tilde{\mu}_\xi(x)$, and $J_x \subset [0,1]$ is the support of $\tilde{\mu}_\xi(x)$, i.e., $J_x = \{u \in [0,1] \mid \mu_\xi(x,u) > 0\}$.

The secondary possibility distribution function and the T2 possibility distribution function of $\xi = (\xi_1, \xi_2, \cdots, \xi_m)$ are also refereed to as the *secondary joint possibility distribution function* and the *T2 joint possibility distribution function* of $\xi_i, i = 1, 2, \cdots, m$, respectively.

Remark 7: We present the concepts of T2 fuzzy variable and T2 possibility distribution function with the intension of adopting variable-based approach to dealing with T2 fuzziness, which facilitate us to investigate fuzzy possibility theory via modern mathematical tools. For instance, by T2 fuzzy vector ξ, we can turn the study of the abstract FPS $(\Gamma, \mathcal{A}, \tilde{\mathrm{P}}\mathrm{os})$ to that of the concrete FPS $(\Re^m, \mathcal{P}(\Re^m), \tilde{\Pi})$, where $\tilde{\Pi}$ is the fuzzy possibility measure on $\mathcal{P}(\Re^m)$ induced by ξ via the formula

$$\tilde{\Pi}(A) = \tilde{\mathrm{P}}\mathrm{os}(\{\gamma \in \Gamma \mid \xi(\gamma) \in A\}), A \in \mathcal{P}(\Re^m).$$

Obviously, the FPS $(\Re^m, \mathcal{P}(\Re^m), \tilde{\Pi})$ is easier to understand and use than the abstract one, and it also allows us to apply real analysis approach to dealing with T2 fuzziness.

Definition 7: The *support* of a T2 fuzzy vector ξ is defined as

$$\mathrm{supp}\,\xi = \{(x,u) \in \Re^n \times [0,1] \mid \mu_\xi(x,u) > 0\}, \qquad (8)$$

where $\mu_\xi(x,u)$ is the T2 possibility distribution function of ξ.

Remark 8: The concept of *support* of a T2 fuzzy vector is similar to the *footprint* of a T2 fuzzy set defined in [11].

Example 4: Let $\Gamma = (0,1)$, and $\mathcal{A} = \mathcal{P}(\Gamma)$. Define a set function $\tilde{\mathrm{P}}\mathrm{os} : \mathcal{P}(\Gamma) \mapsto \mathcal{R}([0,1])$ as follows

$$\tilde{\mathrm{P}}\mathrm{os}(\{\gamma\}) = (\gamma^3, \gamma^2, \gamma), \quad \gamma \in \Gamma$$

and for any $A \in \mathcal{P}(\Gamma)$,

$$\tilde{\mathrm{P}}\mathrm{os}(A) = \sup_{\gamma \in A} \tilde{\mathrm{P}}\mathrm{os}(\{\gamma\}),$$

where $\{(\gamma^3, \gamma^2, \gamma), \gamma \in \Gamma\}$ is supposed to be a family of mutually independent regularly triangular fuzzy variables, and for fixed $\gamma \in \Gamma$, the possibility distribution function of $(\gamma^3, \gamma^2, \gamma)$ is given by

$$\mu(x) = \begin{cases} \frac{x-\gamma^3}{\gamma^2-\gamma^3}, & \text{if } \gamma \le x \le \gamma^2 \\ \frac{\gamma^2-x}{\gamma-\gamma^2}, & \text{if } \gamma \le x \le \gamma^2 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{P}(\Gamma)$ is a fuzzy possibility measure, and $(\Gamma, \mathcal{P}(\Gamma), \tilde{\mathrm{P}}\mathrm{os})$ is an FPS.

Let $\xi : \Gamma \mapsto \Re$ be a function form Γ to $\Re$ such that

$$\xi(\gamma) = \gamma, \quad \gamma \in \Gamma.$$

Then ξ is a T2 fuzzy variable on $(\Gamma, \mathcal{P}(\Gamma), \tilde{\mathrm{P}}\mathrm{os})$. In Fig. 1, we show the support of ξ.

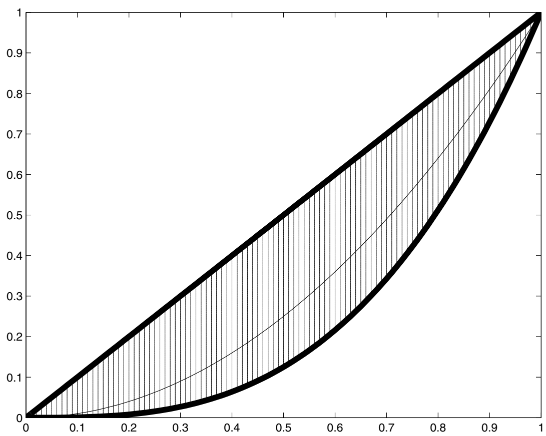

Fig. 2. The support of T2 fuzzy variable ξ defined in Example 4.

V. Marginal T2 Possibility Distribution

When we talk about a T2 fuzzy vector ξ, we usually mean 1) ξ is defined on some FPS; and 2) ξ has known secondary possibility distribution and T2 possibility distribution functions. In this section, we will deal with the properties of T2 fuzzy vectors via possibility distributions.

Let $\xi = (\xi_1, \xi_2, \cdots, \xi_m)$ be a T2 fuzzy vector defined on an FPS $(\Gamma, \mathcal{A}, \tilde{\mathrm{P}}\mathrm{os})$. According to Definition 6, the secondary possibility distribution function of ξ is

$$\begin{aligned} &\tilde{\mu}_\xi(x_1, x_2, \cdots, x_m) \\ &\quad = \tilde{\mathrm{P}}\mathrm{os}\left\{\gamma \mid \xi_1(\gamma) = x_1, \xi_2(\gamma) = x_2, \cdots, \xi_m(\gamma) = x_m\right\}, \end{aligned}$$

where $(x_1, x_2, \cdots, x_m) \in \Re^m$. Then our question is what are the secondary possibility functions of $\xi_i, i = 1, 2, \cdots, m$; or more generally, what is the secondary possibility distribution function of a subvector of ξ. The secondary possibility distribution of a subvector of ξ is called *marginal*. In the following, we express the marginal secondary possibility distributions of $\xi_i, i = 1, 2, \cdots, m$ in terms of their joint distribution $\tilde{\mu}_\xi(x_1, x_2, \cdots, x_m)$.

Let $1 \le r \le m$, and $1 \le i_1 < i_2 < \cdots < i_r \le m$. Since

$$\begin{aligned} &\{\gamma \in \Gamma \mid \xi_{i_1}(\gamma) = x_{i_1}, \cdots, \xi_{i_r}(\gamma) = x_{i_r}\} \\ &\quad = \textstyle\bigcup_{x_{j_1}, x_{j_2}, \cdots, x_{j_{m-r}}} \{\gamma \mid \xi_1(\gamma) = x_1, \cdots, \xi_m(\gamma) = x_m\}, \end{aligned}$$

we deduce that $(\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_r})$ is a T2 fuzzy vector on the FPS $(\Gamma, \mathcal{A}, \tilde{\mathrm{P}}\mathrm{os})$. In addition, noting that $\{\gamma \in \Gamma \mid \xi_1(\gamma) = x_1, \xi_2(\gamma) = x_2, \cdots, \xi_m(\gamma) = x_m\}$ is an atom of $\mathcal{A}$ for any $(x_1, x_2, \cdots, x_m) \in \Re^m$, it follows from the definition of $\tilde{\mathrm{P}}\mathrm{os}$ that $\tilde{\mathrm{P}}\mathrm{os}\{\gamma \in \Gamma \mid \xi_1(\gamma) = x_1, \xi_2(\gamma) = x_2, \cdots, \xi_m(\gamma) = x_m\}, (x_1, x_2, \cdots, x_m) \in \Re^m$ is a family of mutually independent RFVs.

As a consequence, the *marginal secondary possibility distribution function* is formally as follows.

Definition 8: Let $\xi = (\xi_1, \xi_2, \cdots, \xi_m)$ be a T2 fuzzy vector on an FPS $(\Gamma, \mathcal{A}, \tilde{\mathrm{P}}\mathrm{os})$. For any $1 \le r \le m$, and $1 \le i_1 < i_2 < \cdots < i_r \le m$, the *marginal secondary possibility distribution function* of $(\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_r})$ with respect to ξ

is defined as

$$\begin{aligned}&\tilde{\mu}_{(\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_r})}(x_{i_1},\cdots,x_{i_r})\\ &=\sup_{x_{j_1},\cdots,x_{j_{m-r}}}\tilde{\mathrm{P}}\mathrm{os}\{\gamma\mid\xi_1(\gamma)=x_1,\cdots,\xi_m(\gamma)=x_m\}\\ &=\sup_{x_{j_1},\cdots,x_{j_{m-r}}}\tilde{\mu}_{\xi}(x_1,x_2,\cdots,x_m)\end{aligned}\tag{9}$$

for any $(x_{i_1},x_{i_2},\cdots,x_{i_r})\in\Re^r$, where $\{j_1,j_2,\cdots,j_{m-r}\}=\{1,2,\cdots,m\}\backslash\{i_1,i_2,\cdots,i_r\}$, and $\sup_{t_{j_1},t_{j_2},\cdots,t_{j_{m-r}}}$ is the supremum of $\tilde{\mu}_{\xi}(x_1,x_2,\cdots,x_m)$ over $\Re^{m-r}$.

Moreover, the *marginal T2 possibility distribution function* of $(\xi_{i_1},\cdots,\xi_{i_r})$ with respect to ξ is defined as

$$\begin{aligned}&\mu_{(\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_r})}(x_{i_1},x_{i_2},\cdots,x_{i_r},u)\\ &=\mathrm{Pos}\left\{\tilde{\mu}_{(\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_r})}(x_{i_1},x_{i_2},\cdots,x_{i_r})=u\right\}\end{aligned}\tag{10}$$

for any $(x_{i_1},x_{i_2},\cdots,x_{i_r},u)\in\Re^{m-r}\times J_{(x_{i_1},x_{i_2},\cdots,x_{i_r})}$, where Pos is the possibility induced by the possibility distribution of $\tilde{\mu}_{(\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_r})}(x_{i_1},x_{i_2},\cdots,x_{i_r})$, and $J_{(x_{i_1},x_{i_2},\cdots,x_{i_r})}\subset[0,1]$ is the support of $\tilde{\mu}_{(\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_r})}(x_{i_1},x_{i_2},\cdots,x_{i_r})$.

We now provide an example to show the concept of marginal secondary possibility distribution function.

Example 5: Let $(\Gamma,\mathcal{P}(\Gamma),\tilde{\mathrm{P}}\mathrm{os})$ be the FPS defined in Example 3. Define a map $\xi=(\xi_1,\xi_2):\Gamma\mapsto\Re^2$ as follows

$$\xi(\gamma)=\begin{cases}(-6,1), & \text{if }\gamma=\gamma_1\\ (-5,2), & \text{if }\gamma=\gamma_2\\ (-4,1), & \text{if }\gamma=\gamma_3.\end{cases}$$

Then ξ is a T2 fuzzy vector. Find the marginal secondary possibility distribution functions of ξ_1 and ξ_2, respectively. First, according to Definition 5, the secondary possibility distribution function of ξ is

$$\begin{aligned}&\tilde{\mu}_{\xi}(x_1,x_2)\\ &=\tilde{\mathrm{P}}\mathrm{os}\{\gamma\in\Gamma\mid\xi_1(\gamma)=x_1,\xi_2(\gamma)=x_2\},\end{aligned}$$

where $(x_1,x_2)\in\Re^2$. As $(x_1,x_2)=(-6,1)$, we have

$$\tilde{\mu}_{\xi}(-6,1)=\tilde{\mathrm{P}}\mathrm{os}\{\gamma\mid\xi_1(\gamma)=-6,\xi_2(\gamma)=1\}.$$

By the definitions of ξ_1 and ξ_2, we have $\xi_1(\gamma_1)=-6$ and $\xi_2(\gamma_1)=1$. It follows from the definition of $\tilde{\mathrm{P}}\mathrm{os}$ that

$$\tilde{\mu}_{\xi}(-6,1)=\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_1\})=(0.3,0.4,0.5).$$

Using the similar method, we can obtain the following calculation result

$$\tilde{\mu}_{\xi}(x_1,x_2)=\begin{cases}(0.3,0.4,0.5), & \text{if }(x_1,x_2)=(-6,1)\\ \tilde{1}, & \text{if }(x_1,x_2)=(-5,2)\\ (0.5,0.6,0.7), & \text{if }(x_1,x_2)=(-4,1)\\ \tilde{0}, & \text{otherwise.}\end{cases}$$

Since (ξ_1,ξ_2) takes on values $(-6,1),(-5,2)$, and $(-4,1)$, respectively, we deduce that ξ_1 takes on values $-6,-5$, and -4, respectively; while ξ_2 takes its values in $\{1,2\}$.

We first calculate the marginal secondary possibility distribution function $\tilde{\mu}_{\xi_1}(t_1)$ of ξ_1. By Definition 8, we have

$$\tilde{\mu}_{\xi_1}(t_1)=\sup_{t_2\in\Re}\tilde{\mathrm{P}}\mathrm{os}\left\{\gamma\mid\xi_1(\gamma)=t_1,\xi_2(\gamma)=t_2\right\}.$$

If $t_1=-6$, then we have

$$\{\gamma\mid\xi_1(\gamma)=-6,\xi_2(\gamma)=t_2\}=\{\gamma_1\}$$

for $t_2=1$, and

$$\{\gamma\mid\xi_1(\gamma)=-6,\xi_2(\gamma)=t_2\}=\emptyset$$

for $t_2\neq1$. As a consequence, we have

$$\begin{aligned}&\tilde{\mu}_{\xi_1}(-6)\\ &=\sup\nolimits_{t_2\in\Re}\tilde{\mathrm{P}}\mathrm{os}\left\{\gamma\mid\xi_1(\gamma)=-6,\xi_2(\gamma)=t_2\right\}\\ &=\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_1\})=(0.3,0.4,0.5).\end{aligned}$$

It is similar to deduce

$$\tilde{\mu}_{\xi_1}(-5)=\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_2\})=\tilde{1},$$

$$\tilde{\mu}_{\xi_1}(-4)=\tilde{\mathrm{P}}\mathrm{os}(\{\gamma_3\})=(0.5,0.6,0.7).$$

Combining the above gives

$$\tilde{\mu}_{\xi_1}(x_1)=\begin{cases}(0.3,0.4,0.5), & \text{if }x_1=-6\\ \tilde{1}, & \text{if }x_1=-5\\ (0.5,0.6,0.7), & \text{if }x_1=-4\\ \tilde{0}, & \text{otherwise.}\end{cases}$$

Moreover, we deduce the marginal secondary possibility distribution function of ξ_2 as follows

$$\tilde{\mu}_{\xi_2}(x_2)=\begin{cases}(0.5,0.6,0.7), & \text{if }x_2=1\\ \tilde{1}, & \text{if }x_2=2\\ \tilde{0}, & \text{otherwise.}\end{cases}$$

VI. INDEPENDENCE

In this section, we continue to study the properties of T2 fuzzy vectors. Let $\xi_i,i=1,2,\cdots,m$ be T2 fuzzy variables on an FPS $(\Gamma,\mathcal{A},\tilde{\mathrm{P}}\mathrm{os})$. We now focus our attention on the converse problem discussed in Section V, and assume that the secondary possibility distribution functions of $\xi_i,i=1,2,\cdots,m$ are known. Our problem is how to determine the secondary joint possibility distribution function of $(\xi_1,\xi_2,\cdots,\xi_m)$ according to the marginal distributions of its components.

Generally speaking, the joint secondary possibility distribution function of a T2 fuzzy vector cannot be determined by its marginal secondary possibility distribution functions before the relations among marginal secondary possibility distribution functions have been specified. Example 5 is such an example. But in the case that T2 fuzzy variables are mutually independent, their secondary possibility distribution functions can determine their joint secondary possibility distribution function. Hence, to characterize the relationship among T2 fuzzy variables via their possibility distributions, we introduce the *mutually independent T2 fuzzy variable* as follows.

Definition 9: Let $\xi_i,i=1,2,\cdots,m$ be T2 fuzzy variables on an FPS $(\Gamma,\mathcal{A},\tilde{\mathrm{P}}\mathrm{os})$. They are said to be mutually independent if

$$\begin{aligned}&\tilde{\mathrm{P}}\mathrm{os}\left(\{\gamma\mid\xi_1(\gamma)\in B_1,\cdots,\xi_m(\gamma)\in B_m\}\right)\\ &=\min_{1\leq i\leq m}\tilde{\mathrm{P}}\mathrm{os}\left(\{\gamma\mid\xi_i(\gamma)\in B_i\}\right)\end{aligned}\tag{11}$$

for any $B_i \subset \Re, i = 1, \cdots, m$, where $\tilde{\text{P}}\text{os}(\{\gamma \mid \xi_i(\gamma) \in B_i\})$, $i = 1, 2, \cdots, m$ are supposed to be mutually independent RFVs.

Moreover, a family $\{\xi_i \mid i \in I\}$ of T2 fuzzy variables is said to be mutually independent if for each integer $m \geq 2$, and $i_1 < i_2 < \cdots < i_m$, the T2 fuzzy variables $\xi_{i_k}, k = 1, 2, \cdots, m$ are mutually independent.

Remark 9: The implications of mutually independent T2 fuzzy variables include the following two aspects: For any subsets $B_i \subset \Re, i = 1, 2, \cdots, m$,

1) The values that RFV $\tilde{\text{P}}\text{os}(\{\gamma \mid \xi_1(\gamma) \in B_1, \xi_2(\gamma) \in B_2, \cdots, \xi_m(\gamma) \in B_m\})$ takes are defined by the values that RFVs $\tilde{\text{P}}\text{os}(\{\gamma \mid \xi_i(\gamma) \in B_i\})$, $i = 1, 2, \cdots, m$ take via the operations of the minimum operator.
2) The possibility distribution function of $\tilde{\text{P}}\text{os}(\{\gamma \in \Gamma \mid \xi_1(\gamma) \in B_1, \xi_2(\gamma) \in B_2, \cdots, \xi_m(\gamma) \in B_m\})$ is determined by the possibility distribution functions of $\tilde{\text{P}}\text{os}\{\gamma \in \Gamma \mid \xi_i(\gamma) \in B_i\}$, $i = 1, 2, \cdots, m$ as well as their independence.

VII. Conclusion

In this study, to deal with T2 fuzziness, we constructed a framework of fuzzy possibility theory, in which we introduced some fundamental concepts.

1) We defined fuzzy possibility measure as an RFV-valued set function on an ample field, so it generalizes the scalar possibility measure in the literature.
2) We defined a T2 fuzzy vector as a measurable map from an FPS to the space of real vectors. Several fundamental concepts associated with T2 fuzzy vectors, such as secondary possibility distribution function and T2 possibility distribution function, have been presented.
3) To characterize the properties of T2 fuzzy vectors via possibility distributions, we proposed the marginal secondary possibility distribution function and mutually independent T2 fuzzy variables.

Based on the work of this paper, we will continue to investigate the proposed fuzzy possibility theory as well as its applications, the detailed results will be presented in our future papers.

Acknowledgment

This work was supported partially by the National Natural Science Foundation of China under Grant No.70571021, and the CityUHK SRG 7001794 & 7001679.

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