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Robust Basis of Interval Multiobjective Linear and Quadratic Programming

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Abstract

In this paper we deal with multiobjective linear and quadratic programming problem with uncertain information. So far in the field of statistical analysis and data mining, e.g., mean-variance portfolio problem, support vector machine and their varieties, we have encountered various kinds of quadratic and linear programming problems with multiple criteria. Moreover coefficients in such problems have uncertainty that is expressed by interval, probabilistic distribution or possibilistic (fuzzy) distribution. In this paper, we define a robust basis for all possible perturbation of coefficients within intervals in objective functions and constraints that is regarded as secure and conservative solution under uncertainty. According to the conventional multiobjective programming literature, it is required to solve test subproblem for each basis. Therefore, in case of our interval problem excessive computational demand is estimated. In this paper investigating the properties of robust basis by means of combination of interval extreme points we obtained the result that the robust basis can be examined by working with only a finite subset of possible perturbations of the coefficients.

1 Introduction

We encounter various kinds of quadratic and linear programming problems with multiple criteria, such as mean-variance portfolio problem, support vector machine[1] and so on. In these decades with the spread and functional upgrade of information networks, practical use of information mining technology for electronically-provided (text based) information in various social activities has advanced rapidly, e.g. in higher education field[2]. Those mathematical programming problems have been closely related to the field of data mining and text mining.

Generally it is difficult to determine exactly the coefficients in such mathematical programming problems due to various kinds of uncertainties. However, it is sometimes possible to estimate the perturbations of coefficients by intervals or possibilistic distributions. In such decision making situations, pessimistic and optimistic criterion are investigated in the framework of qualitative possibility theory[3]. Interval mathematical programming or fuzzy mathematical programming with uncertain coefficients have been investigated in some literatures [4],[5],[6]. In the setting of fuzzy multiple objective programming with possibilistic coefficients, two kinds of efficient solution sets are defined as fuzzy sets.

In the interval case where all possibilistic coefficients degenerate into interval coefficients, important results for two kinds of efficiency tests were obtained [7],[8],[9],[10], i.e., efficient solutions can be examined by finite subsets of the possible perturbations of the coefficients in the interval matrix.

In this paper more general results are obtained in the framework of interval multiobjective linear and quadratic programming problem in basic form. We define a robust (or necessarily efficient) basis for all possible perturbation of coefficients within intervals in objective functions and constraints that is regarded as secure and conservative solution under uncertainty. Investigating the theoretical aspects of robust basis we discuss that the robust basis can be examined by working with only a finite subset of possible perturbations of the coefficients.

2 Multiobjective linear and quadratic programming

In this paper we define a multiobjective linear and quadratic programming as follows:

Definition 1 ($\boldsymbol{P}$**1**)

$$\begin{aligned} &\textit{maximize} && C\boldsymbol{x} \\ &\textit{minimize} && \frac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} \\ &\textit{subject to} && A\boldsymbol{x} \leqq \boldsymbol{b},\ \boldsymbol{x} \geqq \boldsymbol{0}, \end{aligned}$$

where C is a $p \times n$ matrix, Q is a positive semidefinite $n \times n$ symmetric matrix, A is an $m \times n$ matrix, and $\boldsymbol{b}$ is an m vector.

An efficient solution (or Pareto optimal solution) for this problem is defined as follows:

Definition 2 *We call $\boldsymbol{x}$ an efficient solution that is an optimal solution for the following programming problem with $\boldsymbol{\nu} \geq \boldsymbol{0}$:*

$$\begin{aligned} &\textit{maximize} && \boldsymbol{\nu}^T C\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} \\ &\textit{subject to} && A\boldsymbol{x} \leqq \boldsymbol{b},\ \boldsymbol{x} \geqq \boldsymbol{0}, \boldsymbol{\nu} \geq \boldsymbol{0}, \end{aligned}$$

where inequality $\geq$ means at least one strict inequality.

Inequality in constraint condition of this problem is rewritten as

$$A\boldsymbol{x} + \boldsymbol{\lambda} = \boldsymbol{b},\ \boldsymbol{\lambda} \geqq \boldsymbol{0}.$$

Similarly inequality in condition of the dual problem for this problem, i.e., dual feasibility, can be represented as the following equation:

$$Q\boldsymbol{x} - \boldsymbol{\mu} + A^{\mathrm{T}}\boldsymbol{y} = C^T\boldsymbol{\nu},\ \boldsymbol{\mu} \geqq \boldsymbol{0}.$$

By using the property of quadratic (linear) programming problem we can represent the optimality condition for our problem (e.g., [11]).

Proposition 1 *$\boldsymbol{x}$ $(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is an efficient solution for the problem* ($\boldsymbol{P}$**1**) *iff it satisfies the following (optimality) condition:*

$$\begin{aligned} &(A, I_{m\times m})\boldsymbol{v} = \boldsymbol{b}, \\ &(Q, O_{n\times m})\boldsymbol{v} + (-I_{n\times n}, A^T)\boldsymbol{w} = C^T\boldsymbol{\nu},\ \boldsymbol{\nu} \geqq \boldsymbol{0}, \\ &\boldsymbol{v}\cdot\boldsymbol{w} = 0, \\ &\boldsymbol{v} \geqq \boldsymbol{0}, \boldsymbol{w} \geqq \boldsymbol{0}, \end{aligned}$$

where $\boldsymbol{v} = (x_1, \cdots, x_n, \lambda_1, \cdots, \lambda_m)^T$, $\boldsymbol{w} = (\mu_1, \cdots, \mu_n, y_1, \cdots, y_m)^T$, and I is an identity matrix, O is a zero matrix.

We discuss a basis for our problem in *basic form*. Let B be an m-tuple of integers from $\{1, \ldots, m+n\}$ called basis, and $N = \{1, \ldots, m+n\} \setminus B$. Let $\boldsymbol{v} = (\boldsymbol{v}_B, \boldsymbol{v}_N)$ and $\boldsymbol{w} = (\boldsymbol{w}_N, \boldsymbol{w}_B)$, where $\boldsymbol{v}_B = \{v_i \mid i \in B\}$, $\boldsymbol{v}_N = \{v_i \mid i \in N\}$, $\boldsymbol{w}_N = \{w_i \mid i \in B\}$, $\boldsymbol{w}_B = \{w_i \mid i \in N\}$.

Then we represent an efficient basis in a basic form.

Proposition 2 *Basis B is an efficient basis iff the following conditions are satisfied:*

$$\begin{aligned} &A_{11}\boldsymbol{v}_B = \boldsymbol{b}, \\ &A_{21}\boldsymbol{v}_B + A_{22}\boldsymbol{w}_B = C^T\boldsymbol{\nu}, \boldsymbol{\nu} \geq \boldsymbol{0}, \\ &\boldsymbol{v}_B \geqq \boldsymbol{0},\ \boldsymbol{w}_B \geqq \boldsymbol{0},\ \boldsymbol{v}_N = \boldsymbol{w}_N = \boldsymbol{0} \end{aligned}$$

where A_{11} and A_{21} are matrices with the column vectors from $(A, I_{m\times m})$ and $(Q, O_{n\times m})$ corresponding to $\boldsymbol{v}_B$, and A_{22} is a matrix with the column vectors from $(-I_{n\times n}, A^{\mathrm{T}})$ corresponding to $\boldsymbol{w}_B$.

3 Interval coefficient problem

From a practical point of view due to various kinds of uncertainties it is usually difficult to specify the coefficients of the objective functions and constraints. However, there exist some cases where coefficients can be specified by possible ranges represented by intervals.

In this paper regarding the uncertainties represented by intervals, we consider interval multiobjective linear and quadratic programming problem.

Definition 3

$$\begin{aligned} &\textit{maximize} && C\boldsymbol{x} \\ &\textit{minimize} && \frac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} \\ &\textit{subject to} && A\boldsymbol{x} \leqq \boldsymbol{b},\ \boldsymbol{x} \geqq \boldsymbol{0}. \end{aligned}$$

where C is an element of a set of $p \times n$ matrices with elements $c_{ij} \in [\,\underline{c}_{ij}, \overline{c}_{ij}\,]$ $(i = 1, \ldots, p,\ \ j = 1, \ldots, n)$:

$$C \in \begin{pmatrix} [\underline{c}_{11}, \overline{c}_{11}] & \ldots & [\underline{c}_{1n}, \overline{c}_{1n}] \\ \vdots & & \vdots \\ [\underline{c}_{p1}, \overline{c}_{p1}] & \ldots & [\underline{c}_{pn}, \overline{c}_{pn}] \end{pmatrix},$$

Q is an element of a set of positive semidefinite $n \times n$ symmetric matrices with elements $q_{ij} \in [\,\underline{q}_{ij}, \overline{q}_{ij}\,]$ $(i = 1, \ldots, n,\ \ j = 1, \ldots, n)$:

$$Q \in \begin{pmatrix} [\underline{q}_{11}, \overline{q}_{11}] & \ldots & [\underline{q}_{1n}, \overline{q}_{1n}] \\ \vdots & & \vdots \\ [\underline{q}_{n1}, \overline{q}_{n1}] & \ldots & [\underline{q}_{nn}, \overline{q}_{nn}] \end{pmatrix},$$

A is an element of a set of $m \times n$ matrix with elements $a_{ij} \in [\,\underline{a}_{ij}, \overline{a}_{ij}\,]$ $(i = 1, \ldots, m,\ \ j = 1, \ldots, n)$:

$$A \in \begin{pmatrix} [\underline{a}_{11}, \overline{a}_{11}] & \ldots & [\underline{a}_{1n}, \overline{a}_{1n}] \\ \vdots & & \vdots \\ [\underline{a}_{m1}, \overline{a}_{m1}] & \ldots & [\underline{a}_{mn}, \overline{a}_{mn}] \end{pmatrix}.$$

This problem can be regarded as a set of multiobjective linear and quadratic programming problems each of which has a matrix C, Q, and A in the interval matrices respectively. We denote

$$\underline{C} = \{\underline{c}_{ij}\}, \overline{C} = \{\overline{c}_{ij}\}, \underline{Q} = \{\underline{q}_{ij}\}, \overline{Q} = \{\overline{q}_{ij}\},$$
$$\underline{A} = \{\underline{a}_{ij}\}, \overline{A} = \{\overline{a}_{ij}\}.$$

For this kind of interval coefficient problems, two kinds of solution concepts, i.e., optimistic and pessimistic solutions, have been investigated [7],[8]. In this paper we define a *robust efficient basis* as pessimistic or secure solution (necessarily efficient solution).

Definition 4 *We call B a robust efficient basis, if it is an efficient basis for all $c_{ij} \in [\,\underline{c}_{ij}, \overline{c}_{ij}\,]$, $q_{ij} \in [\,\underline{q}_{ij}, \overline{q}_{ij}\,]$, and $a_{ij} \in [\,\underline{a}_{ij}, \overline{a}_{ij}\,]$.*

Unfortunately the cardinality of this subset, combination of lower and upper bound of intervals is $2^{pn+nn+mn}$.

4 Robust efficient basis

According to Proposition 2 we can represent a robust efficient basis in a basic form.

Proposition 3 *Basis B is a robust efficient basis iff the following conditions are satisfied for all $c_{ij} \in [\,\underline{c}_{ij}, \overline{c}_{ij}\,]$, $q_{ij} \in [\,\underline{q}_{ij}, \overline{q}_{ij}\,]$ and $a_{ij} \in [\,\underline{a}_{ij}, \overline{a}_{ij}\,]$:*

$$\begin{aligned} &A_{11}\boldsymbol{v}_B = \boldsymbol{b} \\ &A_{21}\boldsymbol{v}_B + A_{22}\boldsymbol{w}_B = C^T\boldsymbol{\nu},\ \boldsymbol{\nu} \geqq \boldsymbol{0} \\ &\boldsymbol{v}_B \geqq \boldsymbol{0},\ \boldsymbol{w}_B \geqq \boldsymbol{0},\ \boldsymbol{v}_N = \boldsymbol{w}_N = \boldsymbol{0} \end{aligned}$$

where A_{11} and A_{21} are matrices with the column vectors from $(A, I_{m\times m})$ and $(Q, O_{n\times m})$ corresponding to $\boldsymbol{v}_B$, and A_{22} is a matrix with the column vectors from $(-I_{n\times n}, A^{\mathrm{T}})$ corresponding to $\boldsymbol{w}_B$.

Now we define the following two matrix sets:

Definition 5 (Matrix set M_1) *We denote a subset by M_1 for (A_{11}) having all elements of each row at the upper bound or at the lower bound. Hence, if $(A_{11}) \in M_1$, for $i = 1, \ldots, m$ either $A_{11i\cdot} = \underline{A_{11}}_{i\cdot}$ or $A_{11i\cdot} = \overline{A_{11}}_{i\cdot}$. The maximum number of elements of M_1 is 2^m.*

Definition 6 (Matrix set M_2) *We denote a subset by M_2 for (A_{21}, A_{22}, C^T) having all elements of each row at the upper bound or at the lower bound. Hence, if $(A_{21}, A_{22}, C^T) \in M_2$, for $i = 1, \ldots, n$ either $A_{21i\cdot} = \underline{A_{21}}_{i\cdot}$, $A_{22i\cdot} = \underline{A_{22}}_{i\cdot}$, $C_{\cdot i} = \underline{C}_{\cdot i}$ or $A_{21i\cdot} = \overline{A_{21}}_{i\cdot}$, $A_{22i\cdot} = \overline{A_{22}}_{i\cdot}$, $C_{\cdot i} = \overline{C}_{\cdot i}$. The maximum number of elements of M_2 is 2^n.*

Then finally we obtain the following Theorems by using the theorem of alternative [12]:

Theorem 4 *(Linear case: $A_{21} = O$) Basis B is a robust efficient basis iff the following conditions are satisfied for every $(A_{11}) \in M_1$ and every $(O, A_{22}, C^T) \in M_2$:*

$$\begin{aligned} &A_{11}\boldsymbol{v}_B = \boldsymbol{b}, \\ &A_{22}\boldsymbol{w}_B = C^T\boldsymbol{\nu},\ \boldsymbol{\nu} \geq \boldsymbol{0}, \\ &\boldsymbol{v}_B \geqq \boldsymbol{0},\ \boldsymbol{w}_B \geqq \boldsymbol{0},\ \boldsymbol{v}_N = \boldsymbol{w}_N = \boldsymbol{0}. \end{aligned}$$

Theorem 5 *(Quadratic case) Basis B is a robust efficient basis if the following conditions are satisfied for every $(A_{11}) \in M_1$ and every $(A_{21}, A_{22}, C^T) \in M_2$:*

$$\begin{aligned} &A_{11}\boldsymbol{v}_B = \boldsymbol{b}, \\ &A_{21}\boldsymbol{v}_B + A_{22}\boldsymbol{w}_B = C^T\boldsymbol{\nu},\ \boldsymbol{\nu} \geq \boldsymbol{0}, \\ &\boldsymbol{v}_B \geqq \boldsymbol{0},\ \boldsymbol{w}_B \geqq \boldsymbol{0},\ \boldsymbol{v}_N = \boldsymbol{w}_N = \boldsymbol{0}. \end{aligned}$$

If $\underline{q}_{ij} = \overline{q}_{ij}$ for $i \neq j$, then the statement in this Theorem becomes a necessary and sufficient condition.

We note that the cardinality of combination of these subsets is 2^{m+n}. This theorem can be regarded as an extension of the past results for the problem with interval coefficients [7],[8].

(Numerical example)

A simple numerical example (single objective, Q and A are 2×2 matrices) is shown as follows:

$$C = (-1, 0), \qquad Q \in \begin{pmatrix} [2,3] & 1 \\ 1 & 2 \end{pmatrix},$$
$$A \in \begin{pmatrix} [-2,-1] & -2 \\ [-2,-1] & -1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} -6 \\ -6 \end{pmatrix}.$$

We examind a basic in the case that

$$\boldsymbol{v}_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \boldsymbol{v}_N = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},$$
$$\boldsymbol{w}_B = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \boldsymbol{w}_N = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},$$

then,

$$A_{11} \in \begin{pmatrix} [-2,-1] & -2 \\ [-2,-1] & -1 \end{pmatrix},$$

$$A_{21} \in \begin{pmatrix} [2,3] & 1 \\ 1 & 2 \end{pmatrix}, A_{22} \in \begin{pmatrix} [-2,-1] & [-2,-1] \\ -2 & -1 \end{pmatrix}.$$

Therefore matrix sets M_1 and M_2 are

$$M_1 = \{\begin{pmatrix} -2 & -2 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}\},$$

$$M_2 = \{(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} -2 & -2 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}), (\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix})\}.$$

In the case that

$$A_{11} = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, A_{22} = \begin{pmatrix} -2 & -2 \\ -2 & -1 \end{pmatrix},$$

optimality condition in Theorem 5 is satisfied. However, in the case that

$$A_{11} = \begin{pmatrix} -2 & -2 \\ -2 & -1 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, A_{22} = \begin{pmatrix} -2 & -2 \\ -2 & -1 \end{pmatrix},$$

optimality condition can not be satisfied. Therefore, we find that this basis is *not* a robust efficient basis.

5 Conclusion

In this paper considering the optimality condition for quadratic and linear programming problem we investigated the properties of robust basis for multiobjective linear and quadratic programming problem with interval coefficients. By means of the obtained Theorems, robust basis can be examined by working with only a finite subset of possible perturbations of the coefficients.

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