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Stackelberg solutions to stochastic two-level linear programming problems

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Abstract— **This paper considers a two-level linear programming problem involving random variable coefficients to cope with hierarchical decision making problems under uncertainty. Two decision making models are provided to optimize the mean of the objective function value or to minimize the variance. It is shown that the original problem is transformed into a deterministic problem. The computational methods are constructed to obtain the Stackelberg solution to the two-level programming problems. An illustrative numerical example is provided to understand the geometrical properties of the solutions.**

I. Introduction

In this article, we consider two-level programming problems, which are hierarchical decision making problems with two decision makers (DMs) who are noncooperative. The DM who has priority on the order of decision is called *leader*. On the other hand, the DM who makes a decision after the leader does is called *follower*. The Stackelberg solution is one of reasonable solutions of two-level programming problems, in which the leader makes a decision to optimize his/her objective function under the assumption that the follower makes a decision to optimize his/her objective function for a given decision of the leader.

There have been proposed the computational methods for Stackelberg solution, such as the method of enumerating the endpoints in the feasible region [5], the methods of solving a single-level problem of the leader which includes the constraints with respect to optimality conditions of the follower's problem [9], [3], [2], [4], and the method of solving the reformulate problem with penalty in the objective function of the leader's problem so as to satisfy the optimality condition of the follower's problem [17]. Since the two-level linear programming problem to obtain Stackelberg solution is NP-hard [16], computational methods using genetic algorithms [1] were developed for large-scale problems. Moreover, two-level programming problems with 0-1 or integer decision variables were investigated to deal with facility location problems or budget drafting problems [13].

In previous studies, the coefficients in the formulated problems are constant. However, in real-world decision making, they are not constant but uncertain. For example, the profit per unit product often depends on economic conditions or weather conditions. In such a case, the coefficients in the problems are represented by random variables. Therefore, we consider two-level linear programming problem with random variable coefficients.

Stochastic programming was developed as mathematical programming under stochastic environments. There are several decision making approaches and models. Dantzig discussed two-stage problems [8]. Charnes and Cooper considered chance constrained programming [6], including E-model, V-model and P-model [7]. E-model is the most basic model which is used to optimize the mean of the objective function value. V-model is useful for a decision maker who takes account of risk, the variance of the objective function value. P-model is used to maximize the probability that the objective function value is better than or equals to a given aspiration level. On the other hand, the aspiration level is optimized in another model [10], [11]. We call the model F-model after the term of *fractile* used in the literature [10] .

In this paper, we shall consider two-level programming problems with random variable coefficients, which has been yet to be investigated. We shall construct stochastic two-level programming models based on E-model and on V-model in stochastic programming, i.e., Two-Level Expectation optimization Model (TLE-model) and Two-level Variance minimization Model (TLV-model). The purpose of this article is to provide useful decision making models for the leader.

This paper is organized as follows: In the next section, we formulate stochastic two-level programming problems and propose two decision making models which take account of mean or variance of objective function values. In addition, we show that the original problem involving randomness is transformed into a deterministic equivalent problem. Section 3 develops the exact solution method for obtaining the Stackelberg solution to the problem based on TLV-model. In Section 4, we provide a numerical example to examine the properties of the solutions corresponding to the proposed models. Finally, we conclude this paper and discuss the future research.

II. Two-level stochastic programming problems

A. Problem formulation

Let $\boldsymbol{x} \in \mathbb{R}^{n_1}$ be a decision variable column vector of the leader and $\boldsymbol{y} \in \mathbb{R}^{n_2}$ that of the follower. Then we consider

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the following two-level linear programming problem:

$$\left.\begin{array}{rl} \min\limits_{\boldsymbol{x}} & z_1(\boldsymbol{x},\boldsymbol{y}) = \tilde{\boldsymbol{c}}_1\boldsymbol{x} + \tilde{\boldsymbol{d}}_1\boldsymbol{y} \\ \text{where } \boldsymbol{y} & \text{solves} \\ \min\limits_{\boldsymbol{y}} & z_2(\boldsymbol{x},\boldsymbol{y}) = \tilde{\boldsymbol{c}}_2\boldsymbol{x} + \tilde{\boldsymbol{d}}_2\boldsymbol{y} \\ \text{s. t.} & A_1\boldsymbol{x} + A_2\boldsymbol{y} \le \tilde{\boldsymbol{b}} \\ & \boldsymbol{x} \ge \boldsymbol{0},\ \boldsymbol{y} \ge \boldsymbol{0}, \end{array}\right\} \quad (1)$$

where $z_1(\boldsymbol{x},\boldsymbol{y})$ and $z_2(\boldsymbol{x},\boldsymbol{y})$ are the objective function of the leader and that of the follower, respectively. $\tilde{\boldsymbol{c}}_1$ and $\tilde{\boldsymbol{c}}_2$ are n_1-dimensional row vector of random variable coefficients, and $\tilde{\boldsymbol{d}}_1$ and $\tilde{\boldsymbol{d}}_2$ are n_2-dimensional row vector of random variable coefficients. A_1 and A_2 are $m \times n_1$ and $m \times n_2$ coefficient matrices, respectively, and $\tilde{\boldsymbol{b}}$ is an m-dimensional row vector.
Since problem (1) includes random variables in the right-hand side of constraints, we take a chance constrained programming approach [6] to the problem. This means that the constraint involving random variables is satisfied at more than a given probability level. Let α_i be the probability level for the ith constraint. Then chance constraints are represented as follows:

$$\Pr\{A_1^i\boldsymbol{x} + A_2^i\boldsymbol{y} \le \tilde{b}_i\} \ge \alpha_i,\ i = 1,\dots,m, \quad (2)$$

where A_1^i, A_2^i and $\tilde{b}_i$ are the coefficients of the ith constraint. Let $F_i(\tau)$ be the distribution function of the random variable $\tilde{b}_i$. Then, since $\Pr\{A_1^i\boldsymbol{x} + A_2^i\boldsymbol{y} \le \tilde{b}_i\} = 1 - F(A_1^i\boldsymbol{x} + A_2^i\boldsymbol{y})$, inequality (2) is rewritten as $F(A_1^i\boldsymbol{x} + A_2^i\boldsymbol{y}) \le 1 - \alpha_i$. Let $K_{1-\alpha_i}$ be the maximum value of τ satisfying $\tau = F^{-1}(1-\alpha_i)$. Then from the monotonicity of the distribution, inequality (2) is rewritten as

$$A_1^i\boldsymbol{x} + A_2^i\boldsymbol{y} \le K_{1-\alpha_i},\ i = 1,\dots,m. \quad (3)$$

Equivalently, it follows that

$$A_1\boldsymbol{x} + A_2\boldsymbol{y} \le \boldsymbol{K}_{1-\alpha}, \quad (4)$$

where $\boldsymbol{K}_{1-\alpha} = (K_{1-\alpha_1},\dots,K_{1-\alpha_m})^T$ and the superscript index T denotes transposition.
Consequently, problem (1) is transformed into the following problem with deterministic constraints:

$$\left.\begin{array}{rl} \min\limits_{\boldsymbol{x}} & z_1(\boldsymbol{x},\boldsymbol{y}) = \tilde{\boldsymbol{c}}_1\boldsymbol{x} + \tilde{\boldsymbol{d}}_1\boldsymbol{y} \\ \text{where } \boldsymbol{y} & \text{solves} \\ \min\limits_{\boldsymbol{y}} & z_2(\boldsymbol{x},\boldsymbol{y}) = \tilde{\boldsymbol{c}}_{21}\boldsymbol{x} + \tilde{\boldsymbol{d}}_{21}\boldsymbol{y} \\ \text{s. t.} & A_1\boldsymbol{x} + A_2\boldsymbol{y} \le \boldsymbol{K}_{1-\alpha} \\ & \boldsymbol{x} \ge \boldsymbol{0},\ \boldsymbol{y} \ge \boldsymbol{0}. \end{array}\right\} \quad (5)$$

B. Two-level stochastic programming models

In this section, we propose two decision making models for two-level stochastic programming problems, i.e., Two-Level Expectation optimization Model (TLE-model) and Two-level Variance minimization Model (TLV-model), which are constructed based on E-model and V-model in stochastic programming [7], respectively. E-model is used to optimize the mean of the objective functions of the leader and follower. Then problem (5) is transformed into the following deterministic problem:

$$\left.\begin{array}{rl} \min\limits_{\boldsymbol{x}} & E[\tilde{\boldsymbol{c}}_1\boldsymbol{x} + \tilde{\boldsymbol{d}}_1\boldsymbol{y}] = \boldsymbol{m}_1^c\boldsymbol{x} + \boldsymbol{m}_1^d\boldsymbol{y} \\ \text{where } \boldsymbol{y} & \text{solves} \\ \min\limits_{\boldsymbol{y}} & E[\tilde{\boldsymbol{c}}_2\boldsymbol{x} + \tilde{\boldsymbol{d}}_2\boldsymbol{y}] = \boldsymbol{m}_2^c\boldsymbol{x} + \boldsymbol{m}_2^d\boldsymbol{y} \\ \text{s. t.} & A_1\boldsymbol{x} + A_2\boldsymbol{y} \le \boldsymbol{K}_{1-\alpha} \\ & \boldsymbol{x} \ge \boldsymbol{0},\ \boldsymbol{y} \ge \boldsymbol{0}, \end{array}\right\} \quad (6)$$

where $E[f]$ denotes the mean of f. $(\boldsymbol{m}_1^c, \boldsymbol{m}_1^d)$ and $(\boldsymbol{m}_2^c, \boldsymbol{m}_2^d)$ are mean vectors of $(\tilde{\boldsymbol{c}}_1, \tilde{\boldsymbol{d}}_1)$ and $(\tilde{\boldsymbol{c}}_2, \tilde{\boldsymbol{d}}_2)$, respectively.
Since (6) is regarded as a usual two-level linear programming problem, a Stackelberg solution of problem (6) is obtained by using existing solution methods [9], [3], [2], [5], [4], [17].
It should be noted here that problem (6) does not take account of risk or dispersion of objective function values. V-model in stochastic programming is especially useful for a decision maker who attaches importance to risk or the variance of the objective function. Therefore, we propose TLV-model which is used to minimize the variance of the objective function under the condition that the mean is better than some given satisficing level. The problem based on TLV-model is formulated as follows:

$$\left.\begin{array}{rl} \min\limits_{\boldsymbol{x}} & Var[\tilde{\boldsymbol{c}}_1\boldsymbol{x} + \tilde{\boldsymbol{d}}_1\boldsymbol{y}] = \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix}^T V_1 \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} \\ \text{where } \boldsymbol{y} & \text{solves} \\ \min\limits_{\boldsymbol{y}} & Var[\tilde{\boldsymbol{c}}_2\boldsymbol{x} + \tilde{\boldsymbol{d}}_2\boldsymbol{y}] = \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix}^T V_2 \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} \\ \text{s. t.} & A_1\boldsymbol{x} + A_2\boldsymbol{y} \le \boldsymbol{K}_{1-\alpha} \\ & \boldsymbol{m}_1^c\boldsymbol{x} + \boldsymbol{m}_1^d\boldsymbol{y} \le \beta_1 \\ & \boldsymbol{m}_2^c\boldsymbol{x} + \boldsymbol{m}_2^d\boldsymbol{y} \le \beta_2 \\ & \boldsymbol{x} \ge \boldsymbol{0},\ \boldsymbol{y} \ge \boldsymbol{0}, \end{array}\right\} \quad (7)$$

where $Var[f]$ denotes the variance of f. V_1 and V_2 are variance-covariance matrices of $(\tilde{\boldsymbol{c}}_1, \tilde{\boldsymbol{d}}_1)$ and $(\tilde{\boldsymbol{c}}_2, \tilde{\boldsymbol{d}}_2)$, respectively. Without loss of generality, let us assume that V_1 and V_2 are positive-definite. β_1 and β_2 denote the satisficing levels of objective function values for the leader and follower, respectively.

III. Computational method for Stackelberg solution of V-model

Suppose that the leader has made a decision $\hat{\boldsymbol{x}}$. Then, the rational response of the follower is an optimal solution $\boldsymbol{y}$ of the following problem:

$$\left.\begin{array}{rl} \min & Var[\tilde{\boldsymbol{c}}_2\hat{\boldsymbol{x}} + \tilde{\boldsymbol{d}}_2\boldsymbol{y}] = \begin{bmatrix}\hat{\boldsymbol{x}}\\ \boldsymbol{y}\end{bmatrix}^T V_2 \begin{bmatrix}\hat{\boldsymbol{x}}\\ \boldsymbol{y}\end{bmatrix} \\ \text{s. t.} & A_1\hat{\boldsymbol{x}} + A_2\boldsymbol{y} \le \boldsymbol{K}_{1-\alpha} \\ & \boldsymbol{m}_1^c\hat{\boldsymbol{x}} + \boldsymbol{m}_1^d\boldsymbol{y} \le \beta_1 \\ & \boldsymbol{m}_2^c\hat{\boldsymbol{x}} + \boldsymbol{m}_2^d\boldsymbol{y} \le \beta_2 \\ & \boldsymbol{y} \ge \boldsymbol{0}. \end{array}\right\} \quad (8)$$

Let $R(\boldsymbol{x})$ be a set of rational response of the follower, i.e., a set of optimal solutions $\boldsymbol{y}$ of problem (8). Then a Stackelberg solution is an optimal solution $(\boldsymbol{x}, \boldsymbol{y})$ of the following problem:

$$\left.\begin{array}{ll} \min & Var[\tilde{\boldsymbol{c}}_1\boldsymbol{x} + \tilde{\boldsymbol{d}}_1\boldsymbol{y}] = \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix}^T V_1 \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} \\ \text{s. t.} & \boldsymbol{y} \in R(\boldsymbol{x}) \\ & A_1'\boldsymbol{x} + A_2'\boldsymbol{y} \le \boldsymbol{b}' \\ & \boldsymbol{x} \ge \boldsymbol{0},\ \boldsymbol{y} \ge \boldsymbol{0}. \end{array}\right\} \quad (9)$$

For simplicity, we use coefficient matrices A_1', A_2' and a coefficient vector $\boldsymbol{b}'$ instead of A_1, A_2, $\boldsymbol{m}_1^c$, $\boldsymbol{m}_1^d$, $\boldsymbol{m}_2^c$, $\boldsymbol{m}_2^d$, $\boldsymbol{K}_{1-\alpha}$, β_1 and β_2 in problem (8).

Theorem 1: Without loss of generality, $R(\boldsymbol{x})$ is a singleton.

Proof: Since V_2 is positive-definite, the objective function of problem (8) is strictly convex. Moreover, the constraints of the problem are linear. This means that problem (8) is a convex programming problem where the objective function is strictly convex. Therefore, the optimal solution of the problem is unique. Consequently, $R(\boldsymbol{x})$ consists of a single element. ■

Since problem (8) is convex programming, $\boldsymbol{y} \in R(\boldsymbol{x})$, which is a condition of rational response in problem (9), is replaced by the Kuhn-Tucker condition. Let $\boldsymbol{x}^*$ be a decision of the leader. Then, the Lagrange function is

$$L(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\omega}; \boldsymbol{x}^*) = \begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{y}\end{bmatrix}^T V_2 \begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{y}\end{bmatrix} + \boldsymbol{\lambda}(A_1'\boldsymbol{x}^* + A_2'\boldsymbol{y} - \boldsymbol{b}') - \boldsymbol{\omega}\boldsymbol{y} \quad (10)$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\omega}$ are Lagrange multiplier vectors. Therefore, the Kuhn-Tucker condition is

$$2\sum_{j=1}^{n_1} v_{2(n_1+i)j}x_j^* + 2\sum_{j=n_1+1}^{n_1+n_2} v_{2(n_1+i)j}y_{j-n_1} + \boldsymbol{\lambda}A'_{2,\cdot i} - \omega_i = 0,\ i = 1, \dots, n_2 \quad (11)$$

$$A_1'\boldsymbol{x}^* + A_2'\boldsymbol{y} - \boldsymbol{b}' \le \boldsymbol{0} \quad (12)$$

$$\boldsymbol{\lambda}(A_1'\boldsymbol{x}^* + A_2'\boldsymbol{y} - \boldsymbol{b}') = 0,\ \boldsymbol{\omega}\boldsymbol{y} = 0 \quad (13)$$

$$\boldsymbol{y} \ge \boldsymbol{0},\ \boldsymbol{\lambda} \ge \boldsymbol{0},\ \boldsymbol{\omega} \ge \boldsymbol{0}, \quad (14)$$

where $A'_{2,\cdot i}$ is an ith column vector of A_2'.

The rational response $\boldsymbol{y} \in R(\boldsymbol{x})$ is replaced by the Kuhn-Tucker condition (11)–(14). Then we obtain the following quadratic programming problem with linear complementarity constraints:

$$\left.\begin{array}{ll} \min & Var[\tilde{\boldsymbol{c}}_1\boldsymbol{x} + \tilde{\boldsymbol{d}}_1\boldsymbol{y}] = \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix}^T V_1 \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} \\ \text{s. t.} & 2\sum_{j=1}^{n_1} v_{2(n_1+i)j}x_j + 2\sum_{j=n_1+1}^{n_1+n_2} v_{2(n_1+i)j}y_{j-n_1} \\ & \quad +\boldsymbol{\lambda}A'_{2,\cdot i} - \omega_i = 0,\ i = 1, \dots, n_2 \\ & A_1'\boldsymbol{x} + A_2'\boldsymbol{y} - \boldsymbol{b}' \le \boldsymbol{0} \\ & \boldsymbol{\lambda}(A_1'\boldsymbol{x} + A_2'\boldsymbol{y} - \boldsymbol{b}') = 0,\ \boldsymbol{\omega}\boldsymbol{y} = 0 \\ & \boldsymbol{x} \ge \boldsymbol{0},\ \boldsymbol{y} \ge \boldsymbol{0},\ \boldsymbol{\lambda} \ge \boldsymbol{0},\ \boldsymbol{\omega} \ge \boldsymbol{0}. \end{array}\right\} \quad (15)$$

Since problem (15) is not convex, an optimal solution of the problem is not obtained by using usual convex programming techniques. However, when eliminating the linear complementarity constraints in problem (15), it becomes a usual quadratic programming problem with linear constraints.

Bard and Moore [4] considered a solution algorithm for two-level linear programming problems, which is based on branch-and-bound method. We extend their algorithm to solve problem (15), which applies the branch-and-bound method to linear complementarity constraints and use quadratic programming techniques [18], [12] for solving the subproblems.

Now we shall construct the algorithm for obtaining a Stackelberg solution of problem (15).

Let $W = \{1, \dots, m + n_2 + 2\}$ be an index set of the linear complementarity constraints. Let W_k, S_k^+, S_k^-, S_k^0 and P_k be index sets of the kth subproblem in a tree diagram. Let us assume that $\bar{V}$ is a provisional value.

W_k denotes an index set of linear complementarity constraints which satisfies the condition that a Lagrange multiplier is 0 or that we have equality of the corresponding linear constraint in the kth subproblem. S_k^+ denotes a subset of W_k where the Lagrange multiplier equals 0. S_k^- denotes a subset of W_k where we have equality on the constraint of the original problem. S_k^0 denotes an index set not in W_k. P_k is an index set of paths in a tree diagram. X_c denotes the current solution. Then, the following is an algorithm to obtain Stackelberg solutions of the problems based on TLV-model.

An algorithm to obtain Stackelberg solutions

Step 0 Set $k = 0$, $S_k^+ = \emptyset$, $S_k^- = \emptyset$, $S_k^0 = W$ and $\bar{V} = \infty$.

Step 1 Let us have equality on the linear complementarity constraint in S_k^+ or S_k^-, or set the corresponding Lagrange multiplier to 0. Solve the problems which are obtained by eliminating constraints of problem (15) for each subproblem. If the subproblem is not feasible, then go to Step 5. Otherwise, set $k = k + 1$ and $X^c := (\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{\lambda}^k, \boldsymbol{\omega}^k)$.

Step 2 If $Var[\tilde{\boldsymbol{c}}_1\boldsymbol{x}^k + \tilde{\boldsymbol{d}}_1\boldsymbol{y}^k] \ge \bar{V}$, then go to Step 5.

Step 3 If all the linear complementarity constraints are satisfied, then go to Step 4. Otherwise, set $S_k^+ := S_k^+ \cup i$, $S_k^0 := S_k^0 \backslash i$ such that the amount of violation of linear complementarity constraint is largest, $S_k^- := S_k^-$, and return to Step 1.

Step 4 Set $\bar{V} := Var[\tilde{\boldsymbol{c}}_1\boldsymbol{x}^k + \tilde{\boldsymbol{d}}_1\boldsymbol{y}^k]$.

Step 5 If there is no nonexplored node, then go to Step 6. Otherwise, go to the nearest nonexplored node and update W_k, S_k^+, S_k^-, S_k^0 and P_k, and return to Step 1.

Step 6 If $\bar{V} = \infty$, then terminate the algorithm, and output "nonfeasible". Otherwise, output the solution corresponding to the current $\bar{V}$ as a Stackelberg solution.

IV. Numerical Example

Let us consider a simple stochastic two-level linear programming problem with 5 constraints, in which both the leader and the follower have only the one decision variable, repectively.

The problem is formulated as follows:

$$\left.\begin{array}{ll} \min_{x} & z_1(x,y) = \tilde{c}_1 x + \tilde{d}_1 y \\ \text{where } y \text{ solves} & \\ \min_{y} & z_2(x,y) = \tilde{c}_2 x + \tilde{d}_2 y \\ \text{s. t.} & -x+3y \le \tilde{b}_1,\ 10x - y \le \tilde{b}_2 \\ & 3x + y \ge \tilde{b}_3,\ x + 2y \ge \tilde{b}_4 \\ & 3x + 2y \ge \tilde{b}_5,\ x \ge 0, y \ge 0, \end{array}\right\} \quad (16)$$

where mean of random variables are shown as in Table I. Variance-covariance matrices of $(\tilde{c}_1, \tilde{d}_1)$ and $(\tilde{c}_2, \tilde{d}_2)$ are given as

$$V_1 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix},\ V_2 = \begin{bmatrix} 1 & -1 \\ -1 & 6 \end{bmatrix}.$$

Let us assume that all the right-hand side constant of the constraints are normal random variables. Table II shows mean and variance of the random variables, and the satisficing probability levels for the two DMs.

TABLE I
RANDOM VARIABLE COEFFICIENTS OF THE OBJECTIVE FUNCTIONS

coefficient	$\tilde{c}_1$	$\tilde{d}_1$	$\tilde{c}_2$	$\tilde{d}_2$
mean	−2.0	−3.0	2.0	1.0

TABLE II
RANDOM VARIABLE COEFFICIENTS OF THE CONSTRAINTS

coefficient	$\tilde{b}_1$	$\tilde{b}_2$	$\tilde{b}_3$	$\tilde{b}_4$	$\tilde{b}_5$
mean	50.11	113.15	15.16	13.16	25.63
variance	9.0	36.0	9.0	4.0	16.0
probability	0.85	0.70	0.90	0.70	0.80

Following the decision making procedure of the proposed model in the previous section, problem (16) is reformulated as follows:

$$\left.\begin{array}{ll} \min_{x} & \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{where } y \text{ solves} & \\ \min_{y} & \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{s. t.} & -x + 3y \le 47,\ 10x - y \le 110 \\ & -3x - y \le -19,\ -x - 2y \le -15 \\ & -3x - 2y \le -29,\ -2x - 3y \le -31 \\ & 2x + y \le 33,\ x \ge 0,\ y \ge 0, \end{array}\right\} \quad (17)$$

where $-2x-3y \le -31$ and $2x+y \le 33$ are the constraints of the leader and follower with respect to the mean of objective function values.

Problem (17) is transformed into the following the usual single-level quadratic programming problem with linear complementarity constraints:

$$\left.\begin{array}{ll} \min & \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{s. t.} & -2x + 12y + 3\lambda_1 - \lambda_2 - \lambda_3 - 2\lambda_4 - 2\lambda_5 \\ & \quad -3\lambda_6 + \lambda_7 - \omega = 0 \\ & -x + 3y \le 47,\ 10x - y \le 110 \\ & -3x - y \le -19,\ -x - 2y \le -15 \\ & -3x - 2y \le -29,\ 2x + 3y \le 33 \\ & -2x - 3y \le -31 \\ & \lambda_1(-x + 3y - 47) = 0 \\ & \lambda_2(10x - y - 110) = 0 \\ & \lambda_3(-3x - y + 19) = 0 \\ & \lambda_4(-x - 2y + 15) = 0 \\ & \lambda_5(-3x - 2y + 29) = 0 \\ & \lambda_6(-2x - 3y - 31) = 0 \\ & \lambda_7(2x + 3y - 33) = 0 \\ & \omega y = 0 \\ & x \ge 0, y \ge 0, \lambda_i \ge 0,\ i = 1, \dots, 7 \\ & \omega \ge 0. \end{array}\right\} \quad (18)$$

Fig. 1 shows both the feasible region and the Stackelberg solutions of the proposed models. The white circle at $(5.1667, 6.8889)$ illustrates the Stackelberg solution to problem (17), which is obtained based on TLV-model. The black circle at $(7, 4)$ indicates the Stackelberg solution when eliminating the constraints with respect to the mean of the objective function value in problem (17). The black square at $(1, 16)$ shows the Stackelberg solution to the problem based on TLE-model.

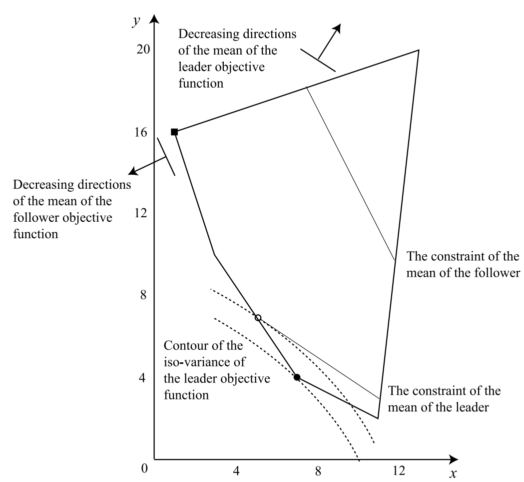

Fig. 1. The feasible region and the Stackelberg solutions

It is observed from Table III that the mean of the objective function for TLE-model is not much better than that for TLV-model. On the other hand, as for the variance of the objective function value, TLE-model is by far worse than TLV-model. Therefore, we conclude that there are cases where TLV-model is especially useful for decision makers who take account of risk.

TABLE III
THE STACKELBERG SOLUTIONS OF THE PROPOSED MODEL

model	solution	mean of the leader	mean of the follower	variance of the leader	variance of the follower
TLV-model	$(5.17, 6.89)$	-31	17	266.94	240.25
TLV-model without mean	$(7, 4)$	-26	18	202	89
TLE-model	$(1, 16)$	-50	18	802	1505

V. CONCLUSION

In this paper, we have considered a stochastic two-level programming problem to deal with hierarchical decision making problems under uncertainty. We have proposed TLE-model and TLV-model to take account of mean and risk, respectively. Our main contribution is to construct the exact solution method for obtaining an optimal solution of the problem based on TLV-model, which consists of combine use of a quadratic programming technique and the branch-and-bound method. This owes to the fact that the rational response set in TLV-model is necessarily unique without loss of generality, which has been proven in Theorem 1.

In the future, we will consider other decision making models based on P-model and on F-model.

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