Proceedings of the 2007 IEEE Symposium on Computational
Intelligence in Multicriteria Decision Making (MCDM 2007)

Finding Representative Nondominated Points in Multiobjective Linear Programming

Lizhen Shao, Matthias Ehrgott
Department of Engineering Science
University of Auckland
New Zealand
{l.shao,m.ehrgott}@auckland.ac.nz

Abstract- **In this paper we address the problem of finding well distributed nondominated points for an MOLP. We propose a method which combines the global shooting and normal boundary intersection methods. It overcomes the limitation of normal boundary intersection method that parts of the nondominated set may be missed. We prove that this method produces evenly distributed nondominated points. Moreover, the coverage error and the uniformity level can be measured. Finally, we apply this method to an optimization problem in radiation therapy and show results for some clinical cases.**

1 Introduction

The goal of multiple objective optimization is to simultaneously minimize $p \geq 2$ objectives. The objectives are conflicting and a feasible solution optimizing all the objectives simultaneously does not exist. Therefore, the purpose of multiobjective optimization is to obtain the nondominated set, i.e., the collection of all nondominated points in objective space. A nondominated point corresponds to an efficient solution in decision space. An efficient solution is a feasible solution for which an improvement in one objective will always lead to a deterioration in at least one of the other objectives. The nondominated set conveys trade-off information to a decision maker (DM) who prefers less to more in each objective. In this paper, we focus on multiobjective linear programming problems.

For a decision maker, it is nearly impossible to study the infinite set of nondominated points to identify the most preferred solution. A discrete representation of the nondominated set by finitely many distinguishable points that cover the whole nondominated set simplifies this task. The decision maker can interactively navigate through the nondominated points to choose the most preferred solution. Therefore, it is of interest to find a good discrete representative subset of the nondominated set.

In Section 2 and 3, we review quality attributes of discrete representations and summarize current methods for computing discrete representations of the nondominated set. In Section 4, we propose a method which combines the global shooting method [2] and the normal boundary intersection (NBI) method [3]. We give an example that shows that the NBI method may miss parts of the nondominated set if $p \geq 3$ objectives are present. We then analyse our proposed method and show that the obtained points are evenly distributed and that the quality of the representation in terms of coverage and uniformity can be guaranteed. Neither the global shooting method, nor the normal boundary intersection method have this property. In Section 5, we apply the proposed method to a radiation therapy treatment planning problem. The results we obtain for some clinical cases illustrate the quality of our method.

2 Discrete Representations of Sets

In this section, let $Z \subset \mathbb{R}^p$ be a set and let $R \subset Z$ be a finite subset. Sayin [11] defines coverage, uniformity, and cardinality as the three attributes of quality of discrete representations. According to these three attributes, a good representation needs to contain a reasonable number of points, should not miss large portions of the nondominated set, and should not contain points that are very close to each other.

Moreover, Sayin proposes measures to quantify these attributes. The number of points contained

1-4244-0702-8/07/$20.00 ©2007 IEEE

in a representation is used to measure the cardinality. The coverage error ϵ and uniformity level δ are defined as follows.

Definition 1. Let $\epsilon \geq 0$ be a real number and d be a metric. R is called a d_ϵ-representation of Z if for any $z \in Z$, there exists $r \in R$ such that $d(z, r) \leq \epsilon$.

Definition 2. Let R be a d_ϵ-representation of Z. R is called a δ-uniform d_ϵ-representation if

$$\min_{r^1, r^2 \in R, r^1 \neq r^2} \{d(r^1, r^2)\} \geq \delta.$$

The coverage error ϵ is a parameter that signifies how precisely the set Z is being represented by the discrete representative subset R, it can be mathematical written as:

$$\epsilon = \max_{z \in Z} \min_{r \in R} d(z, r).$$

How well a fixed $z \in Z$ is covered is determined by the closest point to z in the representation R. For the entire set Z, the coverage error depends on how well an arbitrary element of Z is covered. Therefore, the coverage error ϵ is equal to the maximum of coverage error for individual points in Z.

Similarly, the uniformity of a representation can be measured by the distance between a pair of closest points of R. Thus it can be expressed as

$$\delta = \min_{r^1, r^2 \in R} d(r^1, r^2).$$

For a discrete representation, a small number of points, low coverage error, and high uniformity level are desirable.

3 Existing Methods

3.1 Preliminaries

Consider a multiple objective linear programming problem (MOLP),

$$\min \{Cx : x \in X\}, \tag{1}$$

where $C \in \mathbb{R}^{p \times n}$ is the $p \times n$ matrix whose rows $c^k, k = 1, 2, \ldots, p$, are the coefficients of p linear functions $< c^k, x >, k = 1, 2, \ldots, p$ and $X \subseteq \mathbb{R}^n$ is a nonempty compact polyhedral set of feasible solutions. The feasible set Y in objective space is defined by

$$Y = \{Cx : x \in X\}. \tag{2}$$

Rockafellar [9] has shown that the image Y of a convex polyhedron X by a linear map C is also a convex polyhedron. Y is of course also compact.

Definition 3. x^0 is an efficient solution to problem (1), if $x^0 \in X$ and there exists no $x \in X$ such that $Cx \leq Cx^0$ and $Cx \neq Cx^0$. The set of all efficient solutions of problem (1) will be denoted by X_E, it is called the efficient set in decision space. Correspondingly, $y^0 = Cx^0$ is called a nondominated point and $Y_N = \{Cx : x \in X_E\}$ is the nondominated set in objective space for problem (1).

Definition 4. A feasible solution $\hat{x} \in X$ is called weakly efficient if there is no $x \in X$ such that $Cx < C\hat{x}$, i.e. $c^k x < c^k \hat{x}$, for all $k = 1, 2, \ldots, p$. The point $\hat{y} = C\hat{x}$ is then called weakly nondominated.

The following theorem is fundamental in multiple objective linear programming. The reader is referred to [4] for a proof.

Theorem 5. *A feasible solution $x^0 \in X$ is an efficient solution of the MOLP (1) if and only if there exists a $\lambda \in \mathbb{R}^p_>$ such that*

$$\lambda^T C x^0 \leq \lambda^T C x \tag{3}$$

for all $x \in X$.

Definition 6. Let $F \subset Y$ be a face of Y. F is called nondominated face, if $F \subset Y_N$. F is called maximal nondominated face if it is nondominated and there is no other nondominated face that contains F.

3.2 Survey of Existing Methods

The nondominated set of an MOLP is the union of the (maximal) nondominated faces and these nondominated faces are polyhedral due to Y being a convex polytope. Therefore, finding discrete representations of Y_N is equivalent to finding discrete representations of a union of polyhedra.

There are two groups of methods for finding representations of the nondominated set, one is based on the knowledge of X_E and the other works without the knowledge of X_E.

Based on the knowledge of X_E, Sayin [12] proposes a procedure to find discrete representations with specified coverage errors. The procedure also specifies the uniformity level of the representations. Knowledge of X_E can, however, not be assumed when solving an MOLP.

Most of the methods work without the knowledge of X_E.

Benson and Sayin [2] propose a global shooting method to find a representation of the nondominated set. This method has the coverage property, but it can not directly control the uniformity of the representations it generates.

Das and Dennis [3] propose a normal boundary intersection method for finding several nondominated points for a general multiple objective nonlinear programming problem. It uses the convex hull of the individual minima (CHIM) as reference plane. Equidistant reference points are placed on the CHIM and for each reference point a corresponding nondominated point is found by solving a scalar optimization problem. This method can produce evenly distributed nondominated points, however, some parts of the nondominated set may be missed, a problem caused by the use of the CHIM. We will illustrate this limitation in Section 3.5.

Based on the NBI method, Messac, Ismail-Yahaya and Mattson [7] propose the normalized normal constraint (NC) method. NC works in a normalized objective space and uses an inequality constraint to reduce the feasible region in objective space. However, it has the same problem as the NBI method because it uses the CHIM as reference plane. Realizing this limitation of using the CHIM, Messac and Mattson [8] improve the NC method by using an extended CHIM instead of CHIM as reference plane. They use examples to illustrate that their method provides an even representation of the entire nondominated set but they do not give any mathematical proof.

Analogously to [8] we revise the NBI method in a way that guarantees coverage of the whole nondominated set and that allows us to prove a uniformity guarantee.

We emphasize that representation is different from approximation of Y_N. While a representation R of Y_N is a finite set of points that must be nondominated, an approximation of Y_N may be an infinite or continuous set that has no intersection with Y_N. Quality measures for approximations are quite different from those of representations [6]. While there are many approximation methods for multiobjective programming (the reader is referred to [10] for a survey) that compute some nondominated points, they do not aim at finding evenly distributed nondominated points and may yield bad representations in terms of coverage error and uniformity. Approximation of Y_N is not considered in this paper.

3.3 The Global Shooting Method

Define $Y' = \{y \in \mathbb{R}^p : Cx \leq y \leq \hat{y} \text{ for some } x \in X\}$, where $\hat{y}$ is chosen as a point so that for all $y \in Y_N$ we have $y \leq \hat{y}$. E.g., $\hat{y}$ can be chosen as the anti ideal point y^{AI}, $y_k^{AI} = \max\{y_k, y \in Y\}, k = 1, \ldots, p$. Y' has dimension p and that Y' and Y have the same nondominated set [1].

First, a big simplex S is constructed that contains Y' and a subsimplex $\hat{S}$ of S is taken as the reference plane. Equidistant reference points are placed on $\hat{S}$ and the method "shoots" from $\hat{y}$ towards each reference point as far as possible while remaining in Y'. This is achieved by solving an LP. Thus a set of points on the boundary of Y' is calculated. Each reference point corresponds to a boundary point of Y', but not every such point is nondominated. Therefore it needs to be checked whether the intersection point is dominated or not by solving another LP.

Fig. 1 illustrates the global shooting method. Two weakly nondominated points of Y' are found. Those are shown as triangles.

The global shooting method is simple and computationally tractable for the MOLP case. It guarantees coverage because it puts equidistant reference points on $\hat{S}$ and $Y_N \subset \hat{S} + \mathbb{R}^p_{\geqq}$. However, the uniformity of the discrete representative set can not be controlled directly.

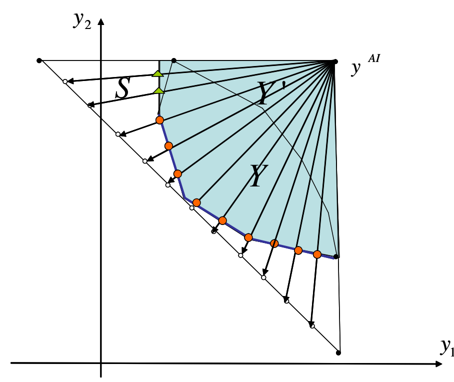

Fig. 1: Solutions obtained by the global shooting method.

3.4 The Normal Boundary Intersection Method

Consider the MOLP problem (1). Assume that individual minima of the functions $< c^k, x >$ over X are attained at x^k for $k = 1, 2, \ldots, p$. Let $y^k = Cx^k$ and let $y^I = (c^1x^1, c^2x^2, \ldots, c^px^p)^T$ be the ideal point. The points $y^1, \ldots, y^p$ define the convex hull of the individual minima (CHIM).

A set of equidistant reference points on the CHIM is generated and, for each of them, a NBI subproblem is solved to find the farthest point on the boundary of Y along the normal $\hat{n}$ of the CHIM pointing toward the ideal point. The NBI subproblem for a given reference point q is as follows:

$$\max\{t : q + t\hat{n} \in Y; t \geq 0\}. \tag{4}$$

Fig. 2 shows how the NBI method works for the same MOLP example with two objectives as in Fig. 1. For this example, all the points obtained are nondominated and no part of the nondominated set is overlooked. However, for problems with more than two objectives, even if the normal direction of the CHIM is negative, the solution method may still overlook a portion of the nondominated set. These overlooked points are likely near the periphery of the nondominated set [3]. If $\hat{n}$ is not negative, the NBI method may not find any nondominated points outside the CHIM. In Section 3.5, we will show an example to see why CHIM based algorithms do not work in some cases.

Although Das and Dennis [3] claim that this method does compute evenly distributed nondominated points, they do not provide bounds on the spacing of the resulting points, which means the uniformity of the discrete representative is not measured.

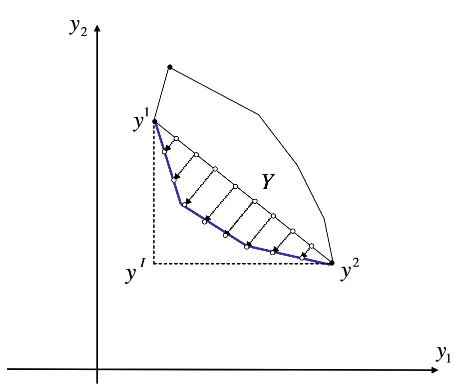

Fig. 2: Solutions obtained by the NBI method.

3.5 The Limitation of CHIM Based Algorithms

Consider a the linear relaxation of an assignment problem with three objectives. The cost matrices of the three objectives are

$$c^1 = \begin{pmatrix} 3 & 6 & 4 & 5 \\ 2 & 3 & 5 & 4 \\ 3 & 5 & 4 & 2 \\ 4 & 5 & 3 & 6 \end{pmatrix},$$

$$c^2 = \begin{pmatrix} 2 & 3 & 5 & 4 \\ 5 & 3 & 4 & 3 \\ 5 & 2 & 6 & 4 \\ 4 & 5 & 2 & 5 \end{pmatrix},$$

$$c^3 = \begin{pmatrix} 4 & 2 & 4 & 2 \\ 4 & 2 & 4 & 6 \\ 4 & 2 & 6 & 3 \\ 2 & 4 & 5 & 3 \end{pmatrix}.$$

Define $Y' = \{y \in \mathbb{R}^3 : Cx \leq y \leq \hat{y}$ for some $x \in X\}$ with $\hat{y} = (21, 21, 21)$, which is greater than the anti ideal point $(20, 20, 20)$.

In Fig. 3, the four circles which represent points $(11, 11, 14)$, $(19, 14, 10)$, $(15, 9, 17)$ and $(13, 16, 11)$ are the nondominated extreme points of Y'. The nondominated set consists of a line segment from point $(11, 11, 14)$ to point $(19, 14, 10)$ and a face which is the convex hull of $(11, 11, 14)$, $(19, 14, 10)$ and $(13, 16, 11)$.

The three dots in Fig. 3 represent the (unique) individual minima of the three objectives, $y^1 = (11, 11, 14)$, $y^2 = (15, 9, 17)$, $y^3 = (19, 14, 10)$. The normal of the CHIM is $\hat{n} = (1, -40, -28)$, which is not positive. Placing reference points on the CHIM, we can not find nondominated points on the face defined by $(11, 11, 14)$, $(19, 14, 10)$ and $(13, 16, 11)$. Therefore, for this example, the CHIM based algorithms NBI and NC do not work very well.

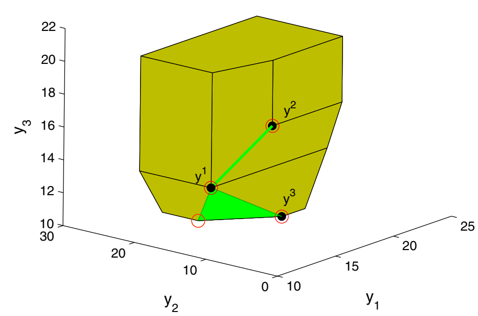

Fig. 3: Y' and the nondominated set Y_N.

4 Revised NBI Method

The global shooting method has the advantage of guaranteeing coverage, and the NBI method can produce evenly distributed nondominated points. Hence, we propose a revised NBI method which combines these two methods. This revised NBI method has the advantage of guaranteeing coverage and uniformity.

Instead of the CHIM, the revised NBI method uses the subsimplex $\hat{S}$ of the simplex S that is used in global shooting method [2] as the reference plane. By doing this, we overcome the limitations of the NBI method, i.e, we have the property of coverage. By solving subproblems similar to (4), we obtain evenly spaced nondominated points.

Thus, the revised normal boundary intersection method involves choosing a reference plane, placing equidistant reference points on the plane and computing the intersection point of the normal of the plane through reference points and the boundary of Y. At last, we need to check if the intersection point is nondominated or not because not every intersection point is nondominated. In the following paragraphs, we explain the details of the revised NBI method.

Reference Plane. Here we use the subsimplex $\hat{S}$ of the simplex S used in the global shooting method [2] as the reference plane.

Let

$$\beta = \min\{< e, y >: y \in Y\}, \qquad (5)$$

where $e \in \mathbb{R}^p$ is a vector in which each entry is 1.

Define $p+1$ points $v^k \in \mathbb{R}^p$, $k = 0, 1, \ldots, p$. Let $v^0 = y^{AI}$ and, for $k = 1, 2, \ldots, p$, let

$$v_l^k = \begin{cases} y_l^{AI}, & \text{if } l \neq k, \\ \beta + y_k - < e, v^0 >, & \text{if } l = k, \end{cases} \qquad (6)$$

$l = 1, 2, \ldots, p$. Then the convex hull S of $\{v^k : k = 0, 1, \ldots, p\}$ is a p-dimensional simplex, and S contains Y, as shown by Benson and Sayin [2].

The subsimplex of S given by the convex hull $\hat{S}$ of $\{v^k : k = 1, 2, \ldots, p\}$ is the reference plane. It is a supporting hyperplane of Y_N with normal e.

Equidistant Points on the Reference Plane. We place equidistant reference points on $\hat{S}$. For $p = 2$, $\hat{S}$ is a line segment. For $p = 3$, $\hat{S}$ is an equilateral triangle in the three dimensional objective space. Therefore, we can use a triangular lattice to produce the equidistant points, see Fig. 4.

In the general case of p objectives, $\hat{S}$ is a $p-1$ dimensional simplex with equal edge length and with the normal direction e according to the construction of S. The ith reference point q^i is given by

$$q^i = \sum_{k=1}^{p} \alpha_k^i v^k$$

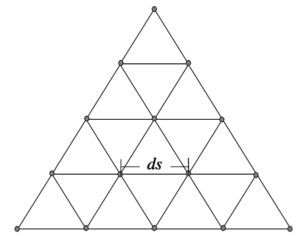

Fig. 4: Equidistant reference points on the reference plane.

where $0 \leq \alpha_k^i \leq 1$ and $\sum_{k=1}^{p} \alpha_k^i = 1$. By varying α_k from 0 to 1 with a fixed increment of η_k an evenly distributed set of points on the reference plane can be generated. For the three objective case in Fig. 4 $\eta_k = 0.25$.

Computing the Intersection Points and Checking Nondominance. Given a reference point q on $\hat{S}$, the revised NBI subproblem searches for the closest point to the reference point on the boundary of Y along the normal direction e. The revised NBI subproblem is as follows:

$$\min\{t : q + te \in Y; t \geq 0\}. \qquad (7)$$

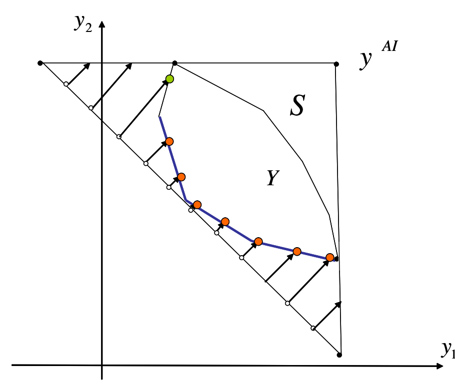

Fig. 5: Solutions obtained in the revised NBI method.

There are three scenarios for the solution of (7), as we can see in Fig. 5 (the same example as Fig. 1 and Fig. 2).

1. There is no intersection between the normal and the boundary of Y.
2. The normal and the boundary of Y intersect, but the intersection point is dominated.
3. The intersection point is nondominated.

If LP (7) is infeasible, then there is no intersection between the normal and the boundary of Y, else there is an intersection point. Not every intersection point is a nondominated point. Therefore, we need to check whether it is dominated or not.

A simple nondomination filter can be used to exclude some of the dominated points [7]. This method has the advantage of being fast, but it may accept some of the dominated points which are near the boundary of Y_N as nondominated.

An exact way to check nondominance is according to the following theorem.

Theorem 7. *Assume that $\lambda \in \mathbb{R}^p_>$ and $\bar{y} \in Y$. Then $\bar{y}$ belongs to Y_N if and only if $\bar{y}$ is an optimal solution to the the following problem*

$$\min\{< \lambda, y >: y \leq \bar{y}; y \in Y\}. \tag{8}$$

The reader is referred to [4] for a proof. By solving (8), we can get rid of all the dominated points that remain after filtering. In our implementation we have used $\lambda = e$.

4.1 Analysis of the Nondominated Points

Given a nondominated face, the angle between the reference plane and the plane of the nondominated face can be calculated as follows:

$$\cos\theta = \frac{< m, n >}{||m|| \, ||n||}. \tag{9}$$

Here, $m \in \mathbb{R}^p$, $n \in \mathbb{R}^p$ are the normal vector of the reference plane and the plane of the nondominated face, respectively.

Because the normal m of the reference plane is equal to $e \in \mathbb{R}^p$ (9) can be written as:

$$\cos\theta = \frac{n_1 + \cdots + n_p}{\sqrt{(n_1)^2 + \cdots + (n_p)^2}\sqrt{p}}. \tag{10}$$

According to Theorem 5 and Definition 6, a set $F \in \mathbb{R}^p$ is a face of Y_N of the MOLP (1) if and only if F equals the optimal solution set $Y^*(\lambda)$ of the problem

$$\min\{< \lambda, y >: y \in Y\} \tag{11}$$

for some $\lambda \in \mathbb{R}^p_>$. Therefore, we know $n \in \mathbb{R}^p_>$ and we have

$$\begin{aligned} & \frac{n_1 + \cdots + n_p}{\sqrt{(n_1)^2 + \cdots + (n_p)^2}\sqrt{p}} \\ > \; & \frac{n_1 + \cdots + n_p}{\sqrt{(n_1 + \cdots + n_p)^2}\sqrt{p}} = \frac{1}{\sqrt{p}}. \end{aligned} \tag{12}$$

When $m = kn, k \neq 0$, we have $\cos\theta = 1$. So the range of $\cos\theta$ is

$$\frac{1}{\sqrt{p}} < \cos\theta \leq 1 \tag{13}$$

and θ is in the range of $0 \leq \theta < \arccos\frac{1}{\sqrt{p}}$.

If $p = 2$, $0 \leq \theta < \frac{\pi}{4}$. If $p = 3$, $0 \leq \theta < \arccos\frac{\sqrt{3}}{3}$. We can see that as p increases, the range of angles between the reference plane and the plane of a nondominated face can increase.

Suppose we have equidistant reference points with distance ds on the reference plane. This implies that the distance between the nondominated points can be calculated as $ds/\cos\theta$.

Fig. 6 shows an example with two objectives ($p = 2$). The nondominated faces are line segments. F_1 is a nondominated face, while F_2 is a weakly nondominated face. The biggest angle between the nondominated face and the reference plane is approaching $\frac{\pi}{4}$. The angle between the reference plane and the weakly nondominated face is $\theta = \frac{\pi}{4}$. The distance between the nondominated points obtained by the revised NBI method is between ds and $\sqrt{2}ds$.

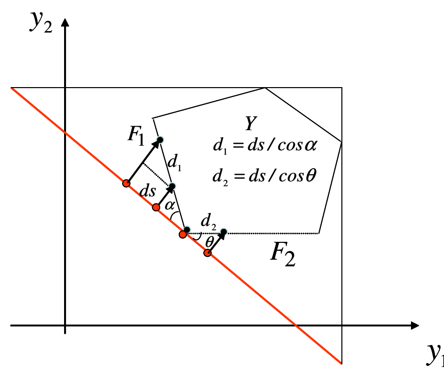

Fig. 6: Distance of nondominated points.

For p objectives, the distance between the nondominated points is $ds \leq d < \sqrt{p}ds$. As p increases, the range of the distance d between the representative nondominated points on Y_N increases.

We have the following result.

Theorem 8. *Let all maximal nondominated faces of Y have dimension $p-1$. Then the representative subset R obtained by the revised NBI method is a ds-uniform $d_{\sqrt{p}ds/2}$-representation of Y_N.*

This result quantifies the quality of representation in terms of coverage and uniformity. The parameter of the method is the distance ds between reference points. As ds decreases, the cardinality of R increases, the coverage error decreases, and the uniformity decreases.

If the number of objectives is not very big, then we think the revised NBI method finds good quality representations. Moreover, to the best of our knowledge, this is the first method that allows the computation of a discrete representative set with guaranteed coverage and uniformity measures.

5 Application

We apply the revised normal boundary intersection method to the beam intensity optimization problem of radiation therapy planning.

The objectives of radiation therapy are to deliver a high uniform dose to the planning target, while at the same time sparing as much as possible to the surrounding normal tissues and organs at risk. Given the number of beams and beam directions, beam intensity optimization needs to determine beam intensity profiles that yield the best dose distribution under consideration of clinical and physical constraints. Due to the conflicting objectives and properties of radiation, the beam intensity optimization problem can be formulated as an MOLP [5]. In this MOLP the objectives are to minimize the maximum deviation α, β, γ of delivered dose from tumor lower bounds, from critical organ upper bounds and from the normal tissue upper bounds, respectively.

Three clinical cases are used: an acoustic neuroma (AC), a prostate (PR) and a pancreatic lesion (PL). They are ordered from simple to complex according to the number of constraints and the number of variables.

In Table 1, we list the number of reference points (RP), the number of intersection points between the normal and the boundary of Y (IP), the number of nondominated points (NP), the distance ds between reference points, and the computation time (CPU) in seconds for calculating the nondominated points for each case. For all three cases, more than half of the reference points do not produce intersection points. No intersection means that LP (7) is infeasible. Detecting infeasibility is very simple, so the reference points that do not yield intersection points do not contribute much to the computation time. Moreover, we can see from the prostate and pancreatic lesion cases in Table 1, that not every intersection point corresponds to a nondominated point. Therefore, it is necessary to check nondominance even though it takes time.

The computation time is related to the number of reference points which corresponds to the number of LPs to be solved. Therefore, for the same case, more reference points need more computation time as we can see in Table 1.

Table 1: Data for the Application Problems.

	RP	IP	NP	ds	CPU
AC	378	72	72	1.04	56.3
PR	378	144	112	4.79	101.3
PL	378	145	129	3.31	523.9
AC	153	29	29	1.59	27.5
PR	153	62	48	7.30	44.5
PL	153	59	54	5.06	213.9

We show the nondominated points of the three clinical cases in Fig. 7 and 8. We can see from these pictures that the nondominated points are evenly distributed. The revised NBI method overcomes the deficiency of the NBI method, i.e., the calculated nondominated points cover the whole nondominated set. As long as we have enough equidistant points on the reference plane, the nondominated points produced will be a good representation of the nondominated set according to coverage, uniformity and cardinality, the three attributes of discrete representation.

6 Conclusion

In this paper, we address the problem of producing well distributed nondominated points for an MOLP. A revised normal boundary intersection method is proposed. By combining features of the normal boundary intersection method and the global shooting methods it overcomes the limitation of CHIM based algorithms. This is the first method for which quality guarantees for coverage and uniformity have been proved. Moreover, numerical results on intensity optimization problems from radiotherapy treatment planning show that the nondominated points are indeed evenly distributed in practice. The issue of choosing a final solution from amongst the discrete representation is an issue that deserves further study. It is amenable to the large variety of methods of multicriteria decision analysis.

References

[1] H. P. Benson. An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objec-

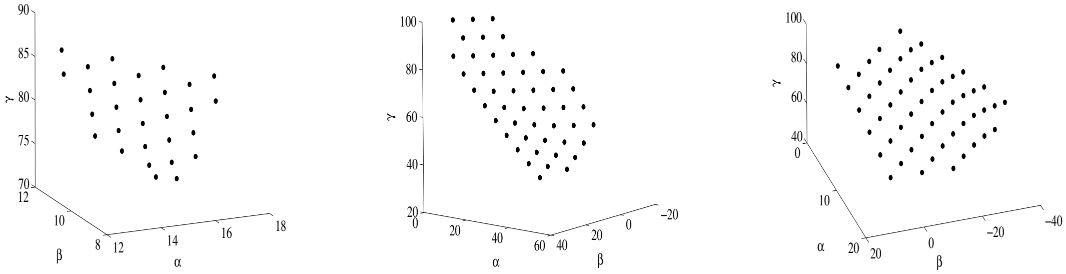

Fig. 7: Pictures from left to right are the nondominated points of acoustic, prostate and pancreatic lesion with 153 reference points.

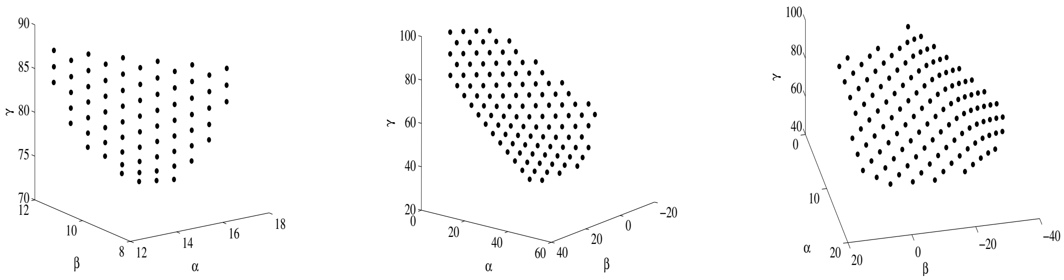

Fig. 8: Pictures from left to right are the nondominated points of acoustic, prostate and pancreatic lesion with 378 reference points.

tive linear programming problem. *Journal of Global Optimization*, 13:1–24, 1998.

[2] H. P. Benson and S. Sayin. Towards finding global representations of the efficient set in multiple objective mathematical programming. *Naval Research Logistics*, 44:47–67, 1997.

[3] I. Das and J. E. Dennis. Normal-boundary intersection: A new method for generating the pareto surface in nonlinear multicriteria optimization problems. *SIAM Journal on Optimization*, 8(3):631–657, 1998.

[4] M. Ehrgott. *Multicriteria Optimization. 2nd edition*. Springer, 2005.

[5] H.W. Hamacher and K.-H. Küfer. Inverse radiation therapy planing – A multiple objective optimization approach. *Discrete Applied Mathematics*, 118(1-2):145–161, 2002.

[6] A. V. Lotov, A. V. Bushenkov, and G. K. Kamenev. *Interactive Decision Maps. Approximation and Visualization of Pareto Frontier*. Kluwer, 2004.

[7] A. Messac, A. Ismail-Yahaya, and C. A. Mattson. The normalized constraint method for generating the pareto frontier. *Structural Multidisciplinary Optimization*, 25:86–98, 2003.

[8] A. Messac and C. A. Mattson. Normal constraint method with guarantee of even representation of complete Pareto frontier. *AIAA Journal*, 42(10):2101–2111, 2004.

[9] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, New Jersey, 1970.

[10] S. Ruzika and M. M. Wiecek. Approximation methods in multiobjective programming. *Journal of Optimization Theory and Applications*, 126(3):473–501, 2005.

[11] S. Sayin. Measuring the quality of discrete representations of efficient sets in multiple objective mathematical programming. *Mathematical Programming*, 87:543–560, 2000.

[12] S. Sayin. A procedure to find discrete representations of the efficient set with specified coverage errors. *Operations Research*, 51(3):427–436, 2003.