2015 IEEE Symposium Series on Computational Intelligence

Ω-algebras

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***Abstract*—In the framework of Ω-sets, where Ω is a complete lattice, we generalize the notion of a (universal) algebra, and we investigate its basic properties. Our techniques belong to the theory of lattice-valued (fuzzy) structures and we use cut-sets. An Ω-algebra is equipped with an Ω-valued equality instead of the classical one. We investigate identities and their satisfiability by these new structures. We prove that a set of identities holds on an Ω-algebra if and only if the cut-subalgebras over the corresponding cut-congruences of the Ω-valued equality satisfy the same identities in the classical setting.**

I. Introduction

The topic of this research are Ω-valued algebraic structures, where Ω is a complete lattice.

Our research originates in both, in fuzzy structures and in Ω-sets. Formally fuzzy set theory was found in 1965., by Zadeh's known paper [21], and has become a highly developed theory since then. Ω-sets appeared 1979., in the paper [10] by Fourman and Scott. Introducing Ω-sets, they intended to model intuitionistic logic. An Ω-set is a nonempty set A equipped with an Ω-valued equality E, with truth values in a complete Heyting algebra Ω. E is a symmetric and transitive function from E^2 to Ω. Ω-sets have been further applied to non-classical predicate logics, and also to theoretical foundations of Fuzzy Set Theory ([12], [14]).

We use Ω-sets and in our approach Ω is a complete lattice (not necessarily a Heyting algebra, or some other residuated lattice). The main reason for choosing this membership values structure is our usage of lattice operations meet and join (and not some additional operations existing in residuated lattices). Namely, these operations allow us to use cut-sets as a tool. In this setting, main algebraic and set-theoretic notions and their properties can be generalized from their classical origin (appearing on cut structures) to the lattice-valued framework ('cutworthy' properties, see [16]). This is not the case if other operations in residuated lattices are applied: if the operations are not meet and join, then many properties of cuts could not be transferred, generalized to fuzzy structures. So we deal also with lattice-valued sets. These were developed within the Fuzzy Set Theory in which the unit interval has been replaced by a complete lattice (firstly by Goguen [11]). This approach is widely used for dealing with algebraic topics (see e.g., [9], then also [18], [19]), and with lattice-valued topology (starting with [15] and many others). In the recent decades, along with the development of fuzzy logic, a complete lattice as a membership (truth values) structure is often replaced by a complete residuated lattice (see e.g., [1]).

A lattice-valued equality generalizing the classical one has been introduced in fuzzy mathematics by Höhle in [13], (see also [14]), and then it was used in investigations of fuzzy functions and fuzzy algebraic structures by many authors, in particular by Demirci ([8]), Bělohlávek and V. Vychodil ([2]) and others. Compatible fuzzy relations were also investigated from the early period (see, e.g., Murali ([17]).

Identities for lattice-valued structures with fuzzy equality were introduced in [20], and then developed in [3], [4], [5], [6]. In this framework, an identity holds if the corresponding lattice-theoretic formula is fulfilled. What is new in this approach is that an identity may hold on a lattice-valued algebra, while the underlying classical algebra does not satisfy the analogue classical identity.

II. Preliminaries

A. Lattices, algebras

A partially ordered set $(\Omega, \leqslant)$, where every subset M has both a meet $\bigwedge M$ and a join $\bigvee M$ is a **complete lattice**. A complete lattice possesses the least and the greatest elements 0 and 1, respectively. A meet and a join of a two-element subset $\{a, b\}$ of Ω are binary operations, denoted by $a \wedge b$ and $a \vee b$, respectively.

A **language** (or a **type**) $\mathcal{L}$ is a set $\mathcal{F}$ of functional symbols, together with a set of natural numbers (arities) associated to these symbols. An **algebra** of type $\mathcal{L}$ is a pair (A, F), denoted by $\mathcal{A}$, where A is a nonempty set and F is a set of (fundamental) operations on A. An n-ary operation in F corresponds to an n-ary symbol in the language. A **subalgebra** of $\mathcal{A}$ is an algebra of the same type, defined on a subset of A, closed under the operations in F. **Terms** in a language are regular expressions constructed by the variables and operational symbols (see [7]). If $t(x_1, \dots, x_n)$ is a term in the language of an algebra $\mathcal{A}$, then by t^A we denote the corresponding **term-operation** $A^n \to A$ on $\mathcal{A}$ (as usual, t^A is obtained by replacing all functional symbols in t by the corresponding fundamental operations on $\mathcal{A}$, and variables by elements from A). An **identity** in a language is a formula $t_1 \approx t_2$, where t_1, t_2 are terms in the same language. An identity $t_1(x_1, \dots, x_n) \approx t_2(x_1, \dots, x_n)$ is said to be **valid** on an algebra $\mathcal{A} = (A, F)$, or that $\mathcal{A}$ **satisfies** this identity, if for all $a_1, \dots, a_n \in A$, the equality $t_1^A(a_1, \dots, a_n) = t_2^A(a_1, \dots, a_n)$ holds. An equivalence relation ρ on A which is compatible with respect to all fundamental operations, meaning that $x_i \rho y_i, i = 1, \dots, n$ imply $f(x_1, \dots, x_n) \rho f(y_1, \dots, y_n)$, is a **congruence** relation on $\mathcal{A}$.

978-1-4799-7560-0/15 $31.00 © 2015 IEEE
DOI 10.1109/SSCI.2015.141

B. Ω-valued functions and relations

Throughout the paper, $(\Omega, \wedge, \vee, \leqslant)$ is a complete lattice with the top and the bottom elements 1 and 0 respectively.

An **Ω-valued function** μ on a nonempty set A is a mapping $\mu : A \to \Omega$.

For $p \in \Omega$, a **cut set** or a **p-cut** of an Ω-valued function $\mu : A \to \Omega$ is a subset μ_p of A which is the inverse image of the principal filter in Ω, generated by p:

$$\mu_p = \mu^{-1}(\uparrow(p)) = \{x \in A \mid \mu(x) \geqslant p\}.$$

An **Ω-valued** (binary) **relation** R **on** A is an Ω-valued function on A^2, i.e., it is a mapping $R : A^2 \to \Omega$.

R is **symmetric** if

$$R(x, y) = R(y, x) \text{ for all } x, y \in A; \tag{1}$$

R is **transitive** if

$$R(x, y) \geqslant R(x, z) \wedge R(z, y) \text{ for all } x, y, z \in A. \tag{2}$$

Let $\mu : A \to \Omega$ and $R : A^2 \to \Omega$ be an Ω-valued function an Ω-valued relation on A, respectively. Then we say that R is an **Ω-valued relation on** μ if for all $x, y \in A$

$$R(x, y) \leqslant \mu(x) \wedge \mu(y). \tag{3}$$

An Ω-valued relation R on $\mu : A \to \Omega$ is said to be **reflexive on** μ if

$$R(x, x) = \mu(x) \text{ for every } x \in A. \tag{4}$$

A symmetric and transitive Ω-valued relation R on A, which is reflexive on $\mu : A \to \Omega$ is an **Ω-valued equivalence** on μ.

Observe that an Ω-valued equivalence R on A fulfills the **strictness** property (see [14]):

$$R(x, y) \leqslant R(x, x) \wedge R(y, y). \tag{5}$$

A Ω-valued equivalence R on A is an **Ω-valued equality**, if it satisfies the **strong separation property**:

$$R(x, y) = R(x, x) \text{ implies } x = y. \tag{6}$$

Remark 1: In [10] and then also in [14], the *separation* property is introduced by $E(x, y) = 1$ implies $x = y$. Obviously, the strong separation implies the separation.

A **lattice-valued subalgebra** of an algebra $\mathcal{A} = (A, F)$, here an **Ω-valued subalgebra** of $\mathcal{A}$ is a function $\mu : A \to \Omega$ which is not constantly equal to 0, and which fulfils the following: For any operation f from F with arity greater than 0, $f : A^n \to A, n \in \mathbb{N}$, and for all $a_1, \ldots, a_n \in A$, we have that

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1, \ldots, a_n)), \tag{7}$$

$$\text{and for a nullary operation } c \in F,\ \mu(c) = 1. \tag{8}$$

How the term operations behave in the lattice valued settings is formulated in the sequel. The proof goes easily by induction on the complexity of terms ([6]).

Proposition 1: Let $\mu : A \to \Omega$ be an Ω-valued subalgebra of an algebra $\mathcal{A}$ and let $t(x_1, \ldots, x_n)$ be a term in the language of $\mathcal{A}$. If $a_1, \ldots, a_n \in A$, then the following holds:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(t^A(a_1, \ldots, a_n)). \tag{9}$$

□

An Ω-valued relation $R : A^2 \to \Omega$ on an algebra $\mathcal{A} = (A, F)$ is **compatible** with the operations in F if the following two conditions holds: for every n-ary operation $f \in F$, for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, and for every constant (nullary operation) $c \in F$

$$\bigwedge_{i=1}^{n} R(a_i, b_i) \leqslant R(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)); \tag{10}$$

$$R(c, c) = 1. \tag{11}$$

III. Ω-ALGEBRAS

A. Ω-set

The following is defined in [10], and then adopted to a fuzzy framework in [6].

An **Ω-set** is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A, fulfilling the strong separation property (6).

For an Ω-set (A, E), we denote by μ the Ω-valued function on A, defined by

$$\mu(x) := E(x, x). \tag{12}$$

We say that μ is *determined by* E. Clearly, by the strictness property, E is an Ω-valued relation on μ, namely, it is an Ω-valued equality on μ. That is why we say that in an Ω-set (A, E), E is an **Ω-valued equality**.

Recall that by the definition of a cut, for $p \in \Omega$, a p-cut of $E : A^2 \to \Omega$ is a binary relation on A given by

$$(x, y) \in E_p \text{ if and only if } E(x, y) \geqslant p.$$

Lemma 1: If (A, E) is an Ω-set and $p \in \Omega$, then the cut E_p is an equivalence relation on the corresponding cut μ_p of μ.

Proof: We prove reflexivity of E_p on μ_p: $(x, x) \in E_p$ if and only if $E(x, x) = \mu(x) \geqslant p$, if and only if $x \in \mu_p$. Similarly, one could prove symmetry and transitivity. ■

B. Ω-algebra; identities

Next we introduce a notion of a lattice-valued algebra with a lattice valued equality.

Let $\mathcal{A} = (A, F)$ be an algebra and $E : A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F. Then we say that $(\mathcal{A}, E)$ is an **Ω-algebra**. Algebra $\mathcal{A}$ is the **underlying algebra** of $(\mathcal{A}, E)$.

Next we present some cut properties of Ω-algebras. These have been proved in [6], in the framework of groups.

Proposition 2: Let $(\mathcal{A}, E)$ be an Ω-algebra. Then the following hold:

$(i\ )$ The function $\mu : A \to \Omega$ determined by E ($\mu(x) = E(x, x)$ for all $x \in A$), is an Ω-valued subalgebra of A.

$(ii~)$ *For every $p \in \Omega$, the cut μ_p of μ is a subalgebra of $\mathcal{A}$, and*

$(iii~)$ *For every $p \in \Omega$, the cut E_p of E is a congruence relation on μ_p.*

Next we define how identities hold on Ω-algebras, according to the approach in [20].

Let and $u(x_1, \ldots, x_n) \approx v(x_1, \ldots, x_n)$ (briefly $u \approx v$) be an identity in the type of an Ω-algebra $(\mathcal{A}, E)$. We assume, as usual, that variables appearing in terms u and v are from $x_1, \ldots, x_n$ Then, $(\mathcal{A}, E)$ **satisfies identity** $u \approx v$ (i.e., this identity **holds** on $(\mathcal{A}, E)$) if the following condition is fulfilled:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E(u^A(a_1, \ldots, a_n), v^A(a_1, \ldots, a_n)), \qquad (13)$$

for all $a_1, \ldots, a_n \in A$ and for the term-operations u^A and v^A on $\mathcal{A}$ corresponding to terms u and v respectively.

Hence, identity $u \approx v$ holds on $(\mathcal{A}, E)$, if the inequality (13) is fulfilled in the lattice Ω whenever variables are replaced by elements from A, and the operational symbols are replaced by the corresponding operations.

If Ω-algebra $(\mathcal{A}, E)$ satisfies an identity, then this identity need not hold on $\mathcal{A}$. On the other hand, if the supporting algebra fulfills an identity then also the corresponding Ω-algebra does.

Proposition 3: [6] *If an identity $u \approx v$ holds on an algebra $\mathcal{A}$, then it also holds on an Ω-algebra $(\mathcal{A}, E)$.*

Proof: Suppose that $x_1, x_2, \ldots, x_n$ are variables appearing in terms u, v. If $u \approx v$ holds on $\mathcal{A}$, then for any $a_1, \ldots, a_n \in A$, $u^A(a_1, \ldots, a_n) = v^A(a_1, \ldots, a_n)$, hence

$$E(u^A(a_1, \ldots, a_n), v^A(a_1, \ldots, a_n)) =$$

$$E(u^A(a_1, \ldots, a_n), u^A(a_1, \ldots, a_n)) = \mu(u(a_1, \ldots, a_n)) \geqslant$$

$$\bigwedge_{i=1}^{n} \mu(a_i).$$

■

In the following we analyze basic properties of Ω-algebras and how they are related to properties of the corresponding underlying algebras.

Let $(\mathcal{A}, E)$ be an Ω-algebra.

First we deal with nullary operations (constants), if they exist in $\mathcal{A}$. By formulas (8) and (11), if c is a constant in $\mathcal{A}$ (determined by a nullary operation in the language), then $E(c, c) = \mu(c) = 1$. The reason for this requirement included in the definition of compatibility is based on the following property.

Lemma 2: If $(\mathcal{A}, E)$ is an Ω-algebra and $c \in F$ a constant nullary fundamental operation on $\mathcal{A}$, then $E(c, c) \geqslant E(x, x)$, for every $x \in A$.

Proof: Recall that we denote by $\mu : A \to \Omega$ the Ω-valued mapping defined by $\mu(x) = E(x, x)$. By Proposition 2 (ii), for every $p \in \Omega$, we have that μ_p is a subalgebra of $\mathcal{A}$. Therefore, being of the same type as $\mathcal{A}$, μ_p should contain c. Hence, for every $p \in \Omega$, $x \in \mu_p$, if $x \in \mu_p$, then also $c \in \mu_p$. Therefore, for every $x \in A$, $E(x, x) = \mu(x) \leqslant \mu(c) = E(c, c)$. ■

Proposition 4: Let $(\mathcal{A}, E)$ be an Ω-algebra. For every term $u(x_1, \ldots, x_n)$ in the language of $\mathcal{A}$,

$$\mu(u^A(c_{i_1}, \ldots, c_{i_n})) = 1,$$

where u^A is the term operation on $\mathcal{A}$, corresponding to the term u, and $c_{i_1}, \ldots, c_{i_n}$ are constants (not necessarily different) from the set F of fundamental operations of $\mathcal{A}$.

Proof: Since μ is an Ω-valued subalgebra of $\mathcal{A}$, by (9) we have

$$\mu(u^A(c_{i_1}, \ldots, c_{i_n})) \geqslant \mu(c_{i_1}) \wedge \ldots \wedge \mu(c_{i_n}) = 1 \wedge \cdots \wedge 1 = 1.$$

■

If $u(x_1, \ldots, x_n)$ is a term in the language of an algebra $\mathcal{A}$, then a term operation $u^A(x_1, \ldots, x_n)$ on $\mathcal{A}$ is **idempotent**, if $\mathcal{A}$ satisfies the identity $u(x, \ldots, x) \approx x$.

Theorem 1: A term operation $u^A(x_1, \ldots, x_n)$ is idempotent on an Ω-algebra $(\mathcal{A}, E)$ if and only if it is idempotent on $\mathcal{A}$.

Proof: If the identity $u(x, \ldots, x) \approx x$ holds on $\mathcal{A}$, then for all $x \in A$, $u^A(x, \ldots, x) = x$, hence $E(x, u^A(x, \ldots, x)) = \mu(x)$, and by (13), the same identity holds also on $(\mathcal{A}, E)$.

Conversely, suppose that the identity $u(x, \ldots, x) \approx x$ holds on $(\mathcal{A}, E)$ i.e., let $E(x, u^A(x, \ldots, x) \geqslant \mu(x)$ for all $x \in A$. Since $E(x, u^A(x, \ldots, x)) \leqslant \mu(x)$ due to (3), we have $E(x, u^A(x, \ldots, x)) = \mu(x)$, and by the strong separation property we conclude that for all $x \in A$, we have $u^A(x, \ldots, x) = x$. ■

Our main results are characterizations of Ω-algebras in terms of subalgebras, congruence relations and classical quotient structures. Recall that for an Ω-algebra $(\mathcal{A}, E)$, the function $\mu : A \to \Omega$, $\mu(x) = E(x, x)$ is an Ω-subalgebra of $\mathcal{A}$. Further, for every $p \in \Omega$, the cut μ_p is a subalgebra of $\mathcal{A}$, and the cut relation E_p is a congruence relation on μ_p.

Theorem 2: Let $(\mathcal{A}, E)$ be an Ω-algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies all identities in $\mathcal{F}$ if and only if for every $p \in \Omega$ the quotient algebra μ_p/E_p satisfies the same identities.

Proof: Let $(\mathcal{A}, E)$ be an Ω-algebra, fulfilling the set $\mathcal{F}$ of identities. For $p \in \Omega$, consider the quotient algebra μ_p/E_p of the subalgebra μ_p of $\mathcal{A}$ over the congruence E_p on μ_p. We prove that this classical algebra satisfies all identities in $\mathcal{F}$.

Let $u(x_1, \ldots, x_n) \approx v(x_1, \ldots, x_n)$ be an identity from $\mathcal{F}$. By assumption, for all $x_1, \ldots, x_n \in A$, and for the corresponding term operations u^A, v^A, we have

$$\mu(x_1) \wedge \ldots \wedge \mu(x_n) \leqslant E(u^A(x_1, \ldots, x_n), v^A(x_1, \ldots, x_n)).$$

In particular, for $x_1, \ldots, x_n \in \mu_p$, we have

$$\mu(x_1) \wedge \ldots \wedge \mu(x_n) \geqslant p,$$

hence

$$E(u^A(x_1, \ldots, x_n), v^A(x_1, \ldots, x_n)) \geqslant p,$$

and

$$(u^A(x_1,\dots,x_n), v^A(x_1,\dots,x_n)) \in E_p,$$

since E_p is a congruence relation on the subalgebra μ_p. Therefore, these values belong to the same congruence class, in other words

$$[u^A(x_1,\dots,x_n)]_{E_p} = [v^A(x_1,\dots,x_n)]_{E_p},$$

and by the compatibility of E_p

$$u^A([x_1]_{E_p},\dots,[x_n]_{E_p}) = v^A([x_1]_{E_p},\dots,[x_n]_{E_p}).$$

Therefore, the identity $u \approx v$ holds on μ_p/E_p.

Conversely, assume that for every $p \in \Omega$, the quotient algebra μ_p/E_p satisfies every identity from $\mathcal{F}$, i.e., if $u \approx v$ is an identity from $\mathcal{F}$, then for all $x_1,\dots,x_n \in \mu_p$, we have

$$u^A([x_1]_{E_p},\dots,[x_n]_{E_p}) = v^A([x_1]_{E_p},\dots,[x_n]_{E_p}),$$

then obviously

$$[u^A(x_1,\dots,x_n)]_{E_p} = [v^A(x_1,\dots,x_n)]_{E_p}.$$

For arbitrary $x_1,\dots,x_n \in A$, we take $\mu(x_1)\wedge\dots\wedge\mu(x_n) = p$, and by the above we get

$$E(u^A(x_1,\dots,x_n), v^A(x_1,\dots,x_n)) \geqslant p = \mu(x_1)\wedge\dots\mu(x_n),$$

which proves that the identity $u \approx v$ holds on $(\mathcal{A}, E)$. ∎

Next we prove that for special Ω-equalities, the underlying algebra satisfies a set of identities if the corresponding Ω-algebra does.

We call an Ω-equality $E : A^2 \to \Omega$ **quasi-diagonal** if for all $x, y \in A$, such that $x \neq y$,

$$E(x,y) < \bigwedge_{t\in A} E(t,t).$$

Theorem 3: Let $(\mathcal{A}, E)$ be an Ω-algebra, where E is a quasi-diagonal Ω-equality. If $(\mathcal{A}, E)$ satisfies a set of identities $\mathcal{F}$, then also the underlying algebra $\mathcal{A}$ satisfies the same identities.

Proof: By the strong separation property (6), for every $x \in A$, we have $E(x,x) \neq 0$. Hence, since E is a quasi-diagonal Ω-equality, it follows that for $q = \bigwedge_{x\in A}\mu(x)$, $\mu_q = \mathcal{A}$, and $E_q = \{(x,x) \mid x \in A\} = \Delta$ i.e., E_q is the equality relation Δ on A. By Theorem 2, for every $p \in \Omega$, the quotient algebra μ_p/E_p fulfills all identities in $\mathcal{F}$. In particular, μ_q/E_q fulfills these identities. Therefore, $\mu_q/E_q = \mathcal{A}/\Delta$. Obviously, $\mathcal{A}/\Delta$ is isomorphic with $\mathcal{A}$, hence $\mathcal{A}$ satisfies all identities in $\mathcal{F}$. ∎

IV. Ω-SUBALGEBRA

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $E_1 : A \to \Omega$ a symmetric and transitive Ω-relation on A, so that the following holds: for all $x, y \in A$

$$E_1(x,y) = E(x,y) \wedge E_1(x,x) \wedge E_1(y,y) \tag{14}$$

Let also E_1 be compatible with the operations in $\mathcal{A}$. Obviously, $(\mathcal{A}, E_1)$ is an Ω-algebra and we say that it is an **Ω-subalgebra** of $(\mathcal{A}, E)$.

The proof of the following proposition is straightforward, due to the definition $\mu_1(x) = E_1(x,x)$, and since E_1 is compatible with the operations in $\mathcal{A}$.

Proposition 5: If $(\mathcal{A}, E_1)$ is an Ω-subalgebra of an Ω-algebra $(\mathcal{A}, E)$, and $\mu_1 : A \to \Omega$ is an Ω-valued function on A, defined by $\mu_1(x) := E_1(x,x)$, then μ_1 is an Ω-valued subalgebra of $\mathcal{A}$, i.e., it fulfills (7) and (8).

Next we prove that an Ω-subalgebra $(\mathcal{A}, E_1)$ of $(\mathcal{A}, E)$ fulfills all the identities that the latter does.

Theorem 4: Let Let $(\mathcal{A}, E_1)$ be an Ω-subalgebra of an Ω-algebra $(\mathcal{A}, E)$. If $(\mathcal{A}, E)$ satisfies the set Σ of identities, then also $(\mathcal{A}, E_1)$ satisfies all identities in Σ.

Proof: Let $u \approx v$ be an identity from Σ, with variables $x_1,\dots,x_n$. Then, since $u \approx v$ holds in $(\mathcal{A}, E)$, by the definition of E_1 and the fact that it is compatible with operations on $\mathcal{A}$, by the definition of μ_1, and by Proposition 1, for all $a_1,\dots,a_n \in A$, we have

$$\begin{gathered}
\bigwedge_{i=1}^{n}\mu_1(a_i) = \bigwedge_{i=1}^{n}E_1(a_i,a_i) = \bigwedge_{i=1}^{n}E(a_i,a_i) \wedge \bigwedge_{i=1}^{n}E_1(a_i,a_i) = \\
\bigwedge_{i=1}^{n}\mu(a_i) \wedge \bigwedge_{i=1}^{n}E_1(a_i,a_i) \leqslant \\
E(u^A(a_1,\dots,a_n), v^A(a_1,\dots,a_n)) \wedge \bigwedge_{i=1}^{n}E_1(a_i,a_i) = \\
E(u^A(a_1,\dots,a_n), v^A(a_1,\dots,a_n)) \wedge \bigwedge_{i=1}^{n}\mu_1(a_i) \leqslant \\
E(u^A(a_1,\dots,a_n), v^A(a_1,\dots,a_n)) \wedge \\
\mu_1(u^A(a_1,\dots,a_n)) \wedge \mu_1(v^A(a_1,\dots,a_n)) = \\
E(u^A(a_1,\dots,a_n), v^A(a_1,\dots,a_n)) \wedge \\
E_1(u^A(a_1,\dots,a_n), u^A(a_1,\dots,a_n)) \wedge \\
E_1(v^A(a_1,\dots,a_n), v^A(a_1,\dots,a_n)) = \\
E_1(u^A(a_1,\dots,a_n), v^A(a_1,\dots,a_n)).
\end{gathered}$$

∎

V. CONCLUSION

The paper introduces a new type of algebraic structures, in the framework of lattice-valued sets and a suitable fuzzy equality replacing the classical one. We deal with general properties of these algebras, using lattice-valued (fuzzy) identities. It turns out that this approach enables investigation of structures which do not satisfy identities (commutativity, associativity etc.), but their quotient cut substructures do.

The next task would be investigation of the corresponding congruences, homomorphisms and product structures.

ACKNOWLEDGMENT

Research supported by Ministry of Education and Science, Republic of Serbia, Grant No. 174013 and by the Provincial Secretariat for Science and Technological Development, Autonomous Province of Vojvodina, Grant "Ordered structures and applications".

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